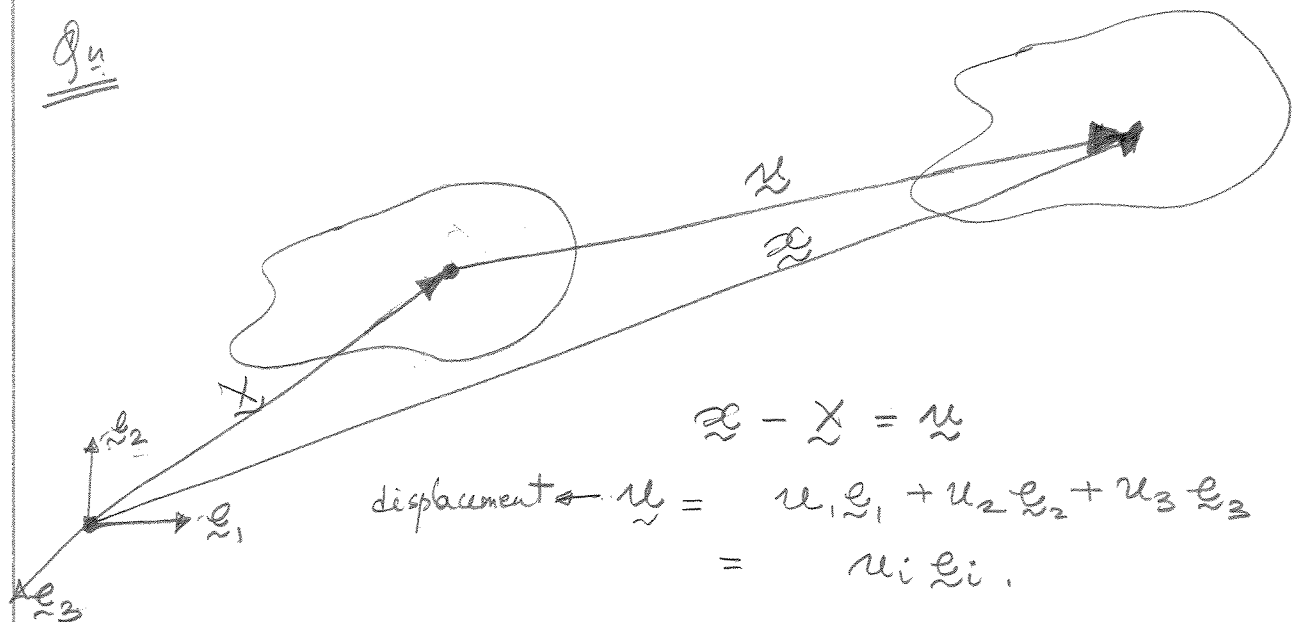
Kinematics of Compatibility

Qu



$$\underset{\sim}{x} - \underset{\sim}{X} = \underset{\sim}{u}$$

displacement $\leftarrow$ $\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3$
$= u_i \underset{\sim}{e}_i .$

displacement gradient $\dfrac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{X}} = \dfrac{\partial u_i}{\partial X_j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j .$

deformation gradient $\dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \dfrac{\partial x_i}{\partial X_j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

Qu: Given a field $A_{ij}(\underset{\sim}{X})$, under what condns on it does there exist a vector field $(\underset{\sim}{u}/\underset{\sim}{x})$ s.t. $\left(\dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}\right) \dfrac{\partial \underset{\sim}{u}}{\partial x} = \underline{A}$

or $u_{i,j} = A_{ij}.$

mention simple connectedness

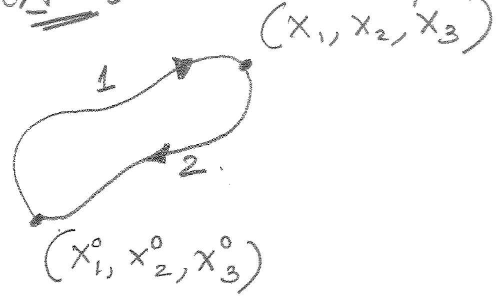
* ~~Necessity:~~ Suppose $\exists$ u_i s.t. $u_{i,j} = A_{ij}.$

Then because $u_{i,jk} = u_{i,kj}$

$\Rightarrow A_{ij,k} = A_{ik,j}. \Leftrightarrow \text{curl} \underline{A} = \underline{0}$

$$(\text{curl} \underline{A})_{ij} = \varepsilon_{jmn} A_{in,m}$$

Suppose $\text{Curl}\,\underline{A} = 0$ (Sufficiency).



define $$u_i(x_1, x_2, x_3) = \oint_{1\,(x_1^0,\,x_2^0,\,x_3^0)}^{(x_1,\,x_2,\,x_3)} A_{ij}(x_1', x_2', x_3')\,dx_j'$$

meaning of line integral

$$u_i(x_1, x_2, x_3) = \int_{s=0}^{s=1} A_{ij}\left(x_1'(s), x_2'(s), x_3'(s)\right)\frac{dx_j'}{ds}\,ds$$

have to prove that $\underset{\sim}{u}$ as defined is independent of path:

$$\oint_{C_1} A_{ij}\,dx_j + \oint_{C_2} A_{ij}\,dx_j = \int_{1+2} A_{ij}\,dx_j$$

$$= \int_{\mathcal{a}} \text{curl}\,\underline{A}\;\underset{\sim}{n}\,da$$

for any $\mathcal{a}$ whose bounding curve is $1+2$

but $\text{curl}\,\underline{A} = \underline{0}$

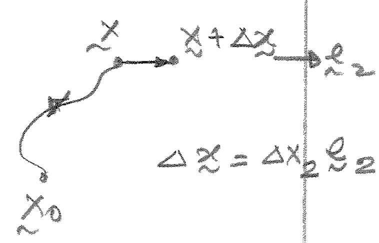
$$\therefore \oint_{C_1} A_{ij}\,dx_j = -\oint_{C_2} A_{ij}\,dx_j$$ & because 1 & 2 were chosen arbitrarily between pts. $\underset{\sim}{x}^0$ & $\underset{\sim}{x}$, we are done.

Now $\quad u_i(x_1, x_2, x_3) = \int_{\mathbf{x}_0}^{\mathbf{x}} \underline{A}\, d\mathbf{x}$

$$u_i(x_1, x_2+\Delta x_2, x_3) - u_i(x_1, x_2, x_3)$$

$$= \int_{\mathbf{x}_0}^{\mathbf{x}} \underline{A}\, d\mathbf{x} + \int_{(x_1, x_2, x_3)}^{(x_1, x_2+\Delta x_2, x_3)} \underline{A}(\mathbf{x}')\, d\mathbf{x}'$$

$$- \int_{\mathbf{x}_0}^{\mathbf{x}} \underline{A}\, d\mathbf{x}$$



$\Delta\mathbf{x} = \Delta x_2 \mathbf{e}_2$

$$= \int_{\mathbf{x}}^{\mathbf{x}+\Delta\mathbf{x}} \underline{A}(\mathbf{x}')\, d\mathbf{x}' \qquad \mathbf{x}'(s) = \mathbf{x} + s\Delta x_2 \mathbf{e}_2$$

$$= \int_{s=0}^{s=1} \underline{A}(\mathbf{x}(s)) \frac{d\mathbf{x}'}{ds}\, ds = \int_{s=0}^{s=1} \underline{A}(\mathbf{x} + s\Delta x_2 \mathbf{e}_2)\, \Delta x_2 \mathbf{e}_2\, ds$$

$$s\Delta x_2 = z$$

$$= \int_0^{\Delta x_2} \underline{A}(\mathbf{x} + z\mathbf{e}_2)\, dz\, \mathbf{e}_2 .$$

$$= \int_0^{\Delta x_2} A_{i2}(\mathbf{x} + z\mathbf{e}_2)\, dz = \int_0^{\Delta x_2} A_{i2}(x_1, x_2+z, x_3)\, dz$$

$$\therefore \lim_{\Delta x_2 \to 0} \frac{u_i(x_1, x_2+\Delta x_2, x_3) - u_i(x_1, x_2, x_3)}{\Delta x_2} = \frac{A_{i2}(x_1, x_2^*, x_3)\cancel{\Delta x_2}}{\cancel{\Delta x_2}} \quad \text{for } x_2 \le x_2^* \le x_2+\Delta x_2$$

$$= A_{i2}(x_1, x_2, x_3) \;//.$$

Strain Compatibility

Qn. Given a symmetric second order tensor e_{ik}, when is it possible to construct a u_i s.t. $\frac{1}{2}(u_{i,k} + u_{k,i}) = e_{ik}$ — (*)

Ans. Necessity: Suppose $\exists$ u_i s.t. — (*) holds.

Now $$u_{i,k} = \frac{1}{2}\left[(u_{i,k} + u_{k,i}) + (u_{i,k} - u_{k,i})\right]$$

$$= e_{ik} + \frac{1}{2}(u_{i,k} - u_{k,i}).$$

Call $w_{ik} := \frac{1}{2}(u_{i,k} - u_{k,i})$.

Now $$w_{ik,l} = \frac{1}{2}(u_{i,kl} - u_{k,il})$$

$$= \frac{1}{2}\left(u_{i,kl} + u_{l,ki} - u_{l,ki} - u_{k,il}\right)$$

$$= e_{il,k} - e_{kl,i}.$$

So the following eqns are satisfied.

$$\boxed{\begin{aligned} u_{i,k} &= e_{ik} + w_{ik} \\ w_{ik,l} &= e_{il,k} - e_{kl,i} \end{aligned}}$$

Because w_{ik} is continuously differentiable

$w_{ik,lm} - w_{ik,ml} = 0$ $\quad [w_{ik,m} = e_{im,k} - e_{km,i}]$

$\therefore$ $\boxed{e_{il,km} - e_{kl,im} - e_{im,kl} + e_{km,il} = 0}$ — (**)

Sufficiency -

Now suppose we define

$$E_{ikl} := e_{il,k} - e_{kl,i} \quad \& \ (xx) \text{ is satisfied}$$

& then $\quad w_{ik}(\underset{\sim}{X}) = \int_{\underset{\sim}{X}_0}^{\underset{\sim}{X}} E_{ikl}\, dX'_l.$

As before we have $w_{ik}(\underset{\sim}{X})$ uniquely defined (upto $w^0_{ik}(\underset{\sim}{X}_0)$) is

$$E_{ikl,m} = E_{ikm,l}$$

but (xx) guarantees this.

$\therefore$ we have a 'unique' w_{ik} field that satisfies

$$w_{ik,l} = E_{ikl} = e_{il,k} - e_{kl,i}. \quad \text{---} (xxx)$$

Now define $\quad u_i(\underset{\sim}{X}) = \int_{\underset{\sim}{X}_0} \left(e_{ik}(\underset{\sim}{X}') + w_{ik}(\underset{\sim}{X}') \right) dX'_k$

Again as before u_i is independent of path if

$$e_{ik,l} + w_{ik,l} = e_{il,k} + w_{il,k}.$$

but (XXX) implies

$$\cancel{e_{ik,l}} + \cancel{e_{il,k} - e_{kl,i}} = \cancel{e_{il,k}} + \cancel{e_{ik,l} - e_{lk,i}}$$

$\therefore$ $u_i(\underset{\sim}{X})$ exists (independent of path)

* If we have two displacement fields whose strain fields are identical, then they differ at most by an infinitesimally rigid deformation.

<u>Proof</u>: Let $u_i = u_i^1 - u_i^2$

with $e_{ij}^1 = e_{ij}^2$.

Then $e_{ij} = 0$.

$$\Rightarrow \quad u_{i,j} = e_{ij} + \omega_{ij}$$

$$u_{i,j} = \omega_{ij} = (\omega_{ij}^1 - \omega_{ij}^2)$$

$$u_{i,jk} = \omega_{ij,k}$$

but since $\omega_{ij,k} = (u^1_{[i,j]} - u^2_{[i,j]})_{,k}$

$$\omega_{ij,k} = e_{ik,j} - e_{jk,i}$$

$$= 0.$$

$$\Rightarrow \quad \omega_{ij}(\underset{\sim}{x}) = \omega_{ij}^0 \quad \text{const.}$$

$$\therefore \quad u_{i,j} = \omega_{ij}^0$$

$$\boxed{u_i(\underset{\sim}{x}) = u_i^0 + \omega_{ij}^0 x_j}$$

Thus, note that a tensor field $A_{ij}(\underset{\sim}{x})$ can have a compatible strain field

i.e. $A_{ij} + A_{ji}$ is compatible

but A_{ij} need not be compatible.

This is because $(\underline{A})_{sym}$ compatible generates a unique rotation field up to a constant skew tensor and if $(A)_{skew}$ does not match this field up to a const skew tensor, then $\underline{A}$ cannot be compatible.

For suppose $\underline{A}$ is assumed compatible & let $\underset{\sim}{u}^1$ be the corresponding displ field. s.t. $\frac{\partial \underset{\sim}{u}'}{\partial \underset{\sim}{x}} = \underline{A}$.

Let $\underset{\sim}{u}^2$ be the displacement field whose strain matches $\underline{A}_{sym}$.

$$\left(\frac{\partial \underset{\sim}{u}^2}{\partial \underset{\sim}{x}}\right)_{sym} = \underline{A}_{sym}.$$

But then $\underset{\sim}{u}' - \underset{\sim}{u}^2$ can at most differ by a rigid defmn $\Rightarrow \left(\frac{\partial \underset{\sim}{u}^2}{\partial \underset{\sim}{x}}\right)_{skew}$

$$= \underset{\sim}{\omega}^0 + \underbrace{\left(\frac{\partial \underset{\sim}{u}'}{\partial \underset{\sim}{x}}\right)_{skew}}_{= (A)_{skew}}.$$