

On the origins of the idea of the multiplicative decomposition of the deformation gradient*

Souhayl Sadik¹ and Arash Yavari^{†1,2}

¹*School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

²*The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

September 22, 2015

The objective of this short note is to trace back the major contributions that led to the multiplicative decomposition of the deformation gradient in finite plasticity, nonlinear thermoelasticity, and growth mechanics. In the 1940s, Eckart in the US and Kondo in Japan, independently paved the road to the formulation of a nonlinear theory capable of modeling anelastic phenomena. As opposed to assuming, for a given body, the existence of a global stress-free configuration (the “principle of relaxability-in-the-large” according to Eckart) that the body takes whenever it is completely relaxed, Eckart [1948] suggested an alternative framework for anelasticity based on what he called “relaxability-in-the-small”. He conceptually constructed a local stress-free “fragmented” state following a local relaxation of the reference configuration by “cutting out” a “small bit of matter” around every material point and letting it relax independently of the remainder of the body. He also asserted that such a construction should be accompanied by an elastic deformation to ensure that the body keeps its structural integrity. This is nothing but the decomposition of the deformation gradient into an anelastic relaxation, leading to the so-called “intermediate” configuration, followed by the elastic portion of the deformation gradient.

Independently of Eckart’s work, Kondo [1949] observed that due to plastic deformations, the relaxed state of a body has a non-trivial geometry that is not compatible with that of the Euclidean space. This observation first led him to construct a stress-free configuration as a Riemannian manifold in which a non-vanishing curvature is a measure of the incompatibility of the plastic deformation. Inspired by the works of Cartan [1926, 1928] on non-trivial holonomy groups, Kondo [1950a,b, 1952] extended his framework to consider the material body as a non-Riemannian space with a non-zero torsion. He used this geometric framework in the context of crystals with geometrical imperfections, e.g. dislocations, and introduced the idea of considering the stress-free state as “an amorphous aggregation” of small pieces of relaxed perfect “crystalline pieces” that he modeled as a non-Riemannian manifold. Further, he interpreted the torsion tensor as a measure of the density of dislocations and initiated the development of a geometric theory of dislocation mechanics. Soon after, further contributions to the nonlinear theory of dislocation mechanics were introduced by Kröner [1955], Kröner and Seeger [1959], and Bilby et al. [1955]. For a review of the interactions between the Japanese (led by Kondo), the British (led by Bilby), and the German (led by Kröner) schools and their contributions, see Kondo [1964]. It is worth mentioning that Sedov [1965] independently realized that a body in

*To appear in *Mathematics and Mechanics of Solids*.

†Corresponding author, e-mail: arash.yavari@ce.gatech.edu

plastic deformation can be relaxed in a stress-free intermediate configuration, which he called “a new starting position”, with a changing metric that is generally non-Euclidean.

Following the original idea of local relaxation inspired by the pioneering works cited above, the first formal introduction of the multiplicative decomposition of the deformation gradient in finite plasticity appeared in the late 1950s in the work of [Bilby et al. \[1957\]](#). [Bilby et al. \[1957\]](#) called the total deformation gradient \mathbf{F} , the elastic deformation gradient \mathbf{F}_e , and the plastic deformation gradient \mathbf{F}_p , “shape deformation”, “lattice deformation”, and “dislocation deformation”, respectively. The decomposition $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$ was explicitly written in [[Bilby et al., 1957](#), Page 41, Eq. (12)]. The same decomposition is seen in [[Kröner, 1959](#), Page 286, Eq. (4)] as well. Almost a decade later, [Lee and Liu \[1967\]](#) and [Lee \[1969\]](#) discussed the multiplicative decomposition in finite plasticity and received most of the credit for it. In nonlinear thermoelasticity, the first formal introduction of the multiplicative decomposition of the deformation gradient is due to [Stojanović et al. \[1964\]](#) and [Stojanović \[1969\]](#). In the biomechanics and growth mechanics literature, the introduction of the multiplicative decomposition is usually attributed to [Rodriguez et al. \[1994\]](#). However, it was first introduced about a decade earlier independently in Russia by [Kondaurov and Nikitin \[1987\]](#) and in Japan by [Takamizawa and Hayashi \[1987\]](#), [Takamizawa and Matsuda \[1990\]](#), and [Takamizawa \[1991\]](#).

References

- B. Bilby, R. Bullough, and E. Smith. Continuous distributions of dislocations: a new application of the methods of non-riemannian geometry. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 231, pages 263–273. The Royal Society, 1955.
- B. A. Bilby, L. R. T. Lardner, and A. N. Stroh. Continuous distributions of dislocations and the theory of plasticity. In *Actes du IXe congrès international de mécanique appliquée, (Bruxelles, 1956)*, volume 8, pages 35–44, 1957.
- É. Cartan. Les groupes d’holonomie des espaces généralisés. *Acta mathematica*, 48(1-2):1–42, 1926.
- É. Cartan. *Leçons sur la géométrie des espaces de Riemann*. Gauthier-Villars, 1928.
- C. Eckart. The thermodynamics of irreversible processes. iv. the theory of elasticity and anelasticity. *Physical Review*, 73(4):373, 1948.
- V. Kondaurov and L. Nikitin. Finite strains of viscoelastic muscle tissue. *Journal of Applied Mathematics and Mechanics*, 51(3):346–353, 1987.
- K. Kondo. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry, I. *Journal of the Society of Applied Mechanics of Japan*, 2(11):123–128, 1949.
- K. Kondo. On the dislocation, the group of holonomy and the theory of yielding. *Journal of the Society of Applied Mechanics of Japan*, 3(17):107–110, 1950a.
- K. Kondo. On the fundamental equations of the theory of yielding. *Journal of the Society of Applied Mechanics of Japan*, 3(20):184–188, 1950b.
- K. Kondo. On the geometrical and physical foundations of the theory of yielding. In *Proc. 2nd Japan Nat. Congr. Applied Mechanics*, volume 2, pages 41–47, 1952.

- K. Kondo. On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. *International Journal of Engineering Science*, 2(3):219–251, 1964.
- E. Kröner. Die inneren spannungen und der inkompatibilitätstensor in der elastizitätstheorie. *Zeitschrift für angewandte Physik*, 7(5):249–257, 1955.
- E. Kröner. Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. *Archive for Rational Mechanics and Analysis*, 4(1):273–334, 1959.
- E. Kröner and A. Seeger. Nicht-lineare elastizitätstheorie der versetzungen und eigenspannungen. *Archive for Rational Mechanics and Analysis*, 3(1):97–119, 1959.
- E. Lee and D. Liu. Finite-strain elasticplastic theory with application to plane-wave analysis. *Journal of Applied Physics*, 38(1):19–27, 1967.
- E. H. Lee. Elastic-plastic deformation at finite strains. *Journal of Applied Mechanics*, 36(1):1–6, 1969.
- E. K. Rodriguez, A. Hoger, and A. D. McCulloch. Stress-dependent finite growth in soft elastic tissues. *Journal of biomechanics*, 27(4):455–467, 1994.
- L. I. Sedov. *Introduction to the Mechanics of a Continuous Medium [First published in Russian in 1962]*. Addison-Wesley Reading, 1965.
- R. Stojanović. On the stress relation in non-linear thermoelasticity. *International Journal of Non-Linear Mechanics*, 4(3):217–233, 1969.
- R. Stojanović, S. Djurić, and L. Vujošević. On finite thermal deformations. *Archiwum Mechaniki Stosowanej*, 1(16):103–108, 1964.
- K. Takamizawa. Stress-free configuration of a thick-walled cylindrical model of the artery: an application of riemann geometry to the biomechanics of soft tissues. *Journal of applied mechanics*, 58(3):840–842, 1991.
- K. Takamizawa and K. Hayashi. Strain energy density function and uniform strain hypothesis for arterial mechanics. *Journal of biomechanics*, 20(1):7–17, 1987.
- K. Takamizawa and T. Matsuda. Kinematics for bodies undergoing residual stress and its applications to the left ventricle. *Journal of applied mechanics*, 57(2):321–329, 1990.