On Eshelby's Inclusion Problem in Nonlinear Anisotropic Elasticity*

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Abstract

The recent literature of finite eignestrains in nonlinear elastic solids is reviewed, and Eshelby's inclusion problem at finite strains is revisited. The subtleties of the analysis of combinations of finite eigenstrains for the example of combined finite radial, azimuthal, axial, and twist eigenstrains in a finite circular cylindrical bar are discussed. The stress field of a spherical inclusion with uniform pure dilatational eigenstrain in a radially-inhomogeneous spherical ball made of arbitrary incompressible isotropic solids is analyzed. The same problem for a finite circular cylindrical bar is revisited. The stress and deformation fields of an orthotropic incompressible solid circular cylinder with distributed eigentwists are analyzed.

Keywords: Eigenstrain, Anelasticity, Nonlinear Elasticity, Inclusion, Geometric Mechanics, Material Manifold.

1 Introduction

In general, only part of strain (we assume that some measure of strain is chosen, e.g., deformation gradient) is directly related to stress through the constitutive equations. The remaining part is called eigenstrain. Eigenstrain is a hybrid German-English term whose origin goes back to the pioneering paper of Hans Reissner [Reissner, 1931] (Eigenspannung means proper or self strain). The term eigenstrain was popularized by Mura [Kinoshita and Mura, 1971, Mura, 1982]. In the literature several other terms can be found that describe the same concept; *initial strain* [Kondo, 1949], *nuclei of strain* [Mindlin and Cheng, 1950], *transformation strain* [Eshelby, 1957], and *inherent strain* [Ueda et al., 1975] (see also [Jun and Korsunsky, 2010, Zhou et al., 2013]). Inclusions and their stress fields were systematically studied in the setting of linear elasticity and for infinite bodies by Eshelby [1957]. Eshelby showed that for an ellipsoidal inclusion that is embedded in an infinite linear elastic medium and has uniform eigenstrains the stress field inside the inclusion is uniform. This uniformity property does not hold for finite bodies, in general. For a spherical inclusion centered at a finite ball Li et al. [2007] showed that, in general, stress inside the inclusion is not uniform.

Eshelby's inclusion problem in nonlinear elasticity has been studied only fairly recently. The study of inclusions has been overwhelmingly restricted to linear elasticity, with the exception of some two-dimensional solutions in the case of harmonic solids [Ru and Schiavone, 1996, Ru et al., 2005, Kim and Schiavone, 2007, 2008, Kim et al., 2008]. The first three-dimensional investigation of the stress fields of inclusions in nonlinear solids was the numerical study of Diani and Parks [2000]. In the case of a spherical inclusion with pure dilatational eigenstrains in their finite element simulations they observed uniform hydrostatic stress inside

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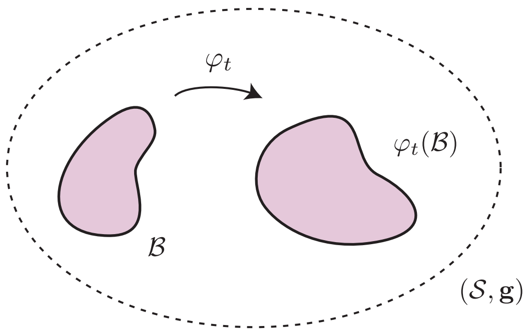
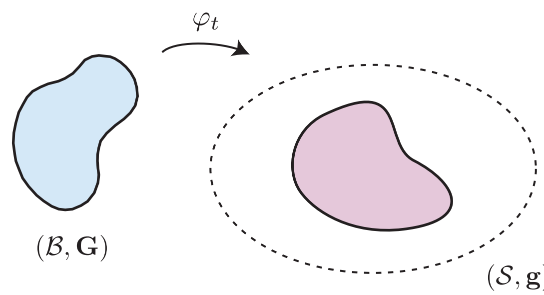
the inclusion. This was later proved analytically for incompressible isotropic solids and a class of compressible isotropic solids by Yavari and Goriely [2013]. For transversely isotropic and orthotropic solids Golgoon and Yavari [2018b] proved some similar results. The first exact solutions for the stress fields of inclusions in nonlinear elasticity were obtained in [Yavari and Goriely, 2013]. In that work a theory of distributed finite eigenstrains in nonlinear solids was formulated. The idea is to construct a global natural configuration—the material manifold—for a body with a distribution of finite eigenstrains. The natural configuration is a Riemannian manifold with a metric that explicitly depends on the eigenstrain distribution (see Fig.1). Yavari and Goriely [2015a] analyzed certain stress singularities induced by distributed eigenstrains. Yavari and Goriely [2015b] analyzed finite cylindrical bars with distributed finite eigentwists. The stress fields of finite eigenstrains in elastic wedges were studied in [Golgoon et al., 2016]. Toroidal inclusions with uniform finite pure dilatational eigenstrains in solid tori were studied in [Golgoon and Yavari, 2017]. It was shown that the stress field inside the inclusion is not uniform.



(a) Classical nonlinear elasticity

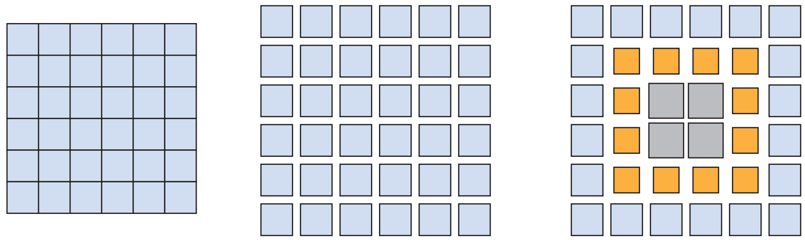


(a) Geometric nonlinear anelasticity

Figure 1: *(a) In classical nonlinear elasticity both the reference and deformed configurations are submanifolds of the Euclidean ambient space* $(\mathcal{S}, \mathbf{g})$*. (b) In anelasticity the deformed configuration is still a submanifold of the Euclidean ambient space while the reference configuration is an abstract Riemannian manifold. Anelasticity is encoded in the geometry of the material manifold* $(\mathcal{B}, \mathbf{G})$*.*

Suppose in a body $\mathcal{B}$, a subset $\mathcal{I} \subset \mathcal{B}$ has non-vanishing eigenstrains. This subset is called an inclusion (or inclusions when $\mathcal{I}$ is not a connected set). Eigenstrains change the natural configuration of the body. We model the natural configuration of the body by a Riemannian manifold $(\mathcal{B}, \mathbf{G})$, where $\mathbf{G}$ is the so-called material metric. The distances of material points in the natural configuration are calculated using $\mathbf{G}$. These distances for a body with eigenstrains, in general, do not agree with the corresponding distances calculated using the metric of the Euclidean ambient space. This discrepancy between the two geometries is the source of residual stresses (see Fig.2). In the literature, eigenstrains have been used to model a large class of anelasticity problems including swelling and cavitation [Pence and Tsai, 2005, 2006, 2007, Goriely et al., 2010, Moulton and Goriely, 2011], bulk and surface growth [Amar and Goriely, 2005, Yavari, 2010, Sozio and Yavari, 2017, 2019], thermal strains [Stojanovic et al., 1964, Ozakin and Yavari, 2010, Sadik and Yavari, 2015], and defects [Yavari and Goriely, 2012a,b,c, 2014, Sadik and Yavari, 2016, Golgoon and Yavari, 2018a].

This paper is organized as follows. In §2 we tersely review nonlinear elasticity and modeling anisotropy at finite strains. In §3 we discuss a geometric theory of finite eigenstrains. We also discuss the subtleties of analyzing combinations of eigenstrains. The problem of radially-symmetric distributions of finite pure dilatational eigenstrains in a finite spherical ball made of a nonlinear incompressible elastic solid is revisited in §4. We present a simple but significant generalization of the analysis of Yavari and Goriely [2013]; we assume that the spherical ball is radially inhomogeneous. A similar problem for finite circular cylindrical bars is revisited in §5. The analysis of Yavari and Goriely [2015b] is extended to orthotropic circular cylindrical bars in §6. Conclusions are given in §7.



(a) (b) (c)

Figure 2: *(a) Part of a stress-feee 2D body is partitioned into a set of squares. (b) Imagine that each square is cut from the body. Note that all the squares are already relaxed. In other words the squares in (b) can be put back together to reconstruct the stress-free configuration (a). (c) Assume that each square is given a pure dilatational eigenstrain λ. Every square is relaxed to another square, which is unique up to rigid body translations and rotations. The gray squares have $\lambda > 1$ while the orange ones have $\lambda < 1$. The blue squares have no eigenstrain, i.e., $\lambda = 1$. The relaxed squares cannot be put back together without elastic strains. In other words, this configuration is incompatible. The material metric $\mathbf{G}$ is defined such that the area of each square in the set shown in (a) calculated using $\mathbf{G}$ is equal to its relaxed area shown in (c).*

2 Anisotropic nonlinear elasticity

Kinematics. Motion in nonlinear elasticity and anelasticity is modeled by a time-dependent mapping between a reference configuration (or natural configuration) and the ambient space. This is written as $\varphi_t : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ are the material and the ambient space Riemannian manifolds, respectively [Marsden and Hughes, 1994] (see Fig.1). Using the material metric $\mathbf{G}$ one can measure distances in a natural stress-free configuration. This metric explicitly depends on eigenstrains. In the ambient space $\mathbf{g}$ is a fixed background metric.[1]

A line element at $X \in \mathcal{B}$ in the reference configuration is a vector on $\mathcal{B}$, i.e., an element of the tangent space $T_X\mathcal{B}$. The corresponding line element in the deformed configuration at $x = \varphi(X)$ is an element of $T_x\mathcal{S}$. The deformation gradient $\mathbf{F}$ maps the undeformed line element to its corresponding deformed line element, i.e., $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$ is the tangent map of φ_t. The transpose of $\mathbf{F}$ is denoted by $\mathbf{F}^\mathsf{T}$, and is defined as

$$\mathbf{F}^\mathsf{T}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}\,, \qquad \langle\!\langle \mathbf{W}, \mathbf{F}^\mathsf{T}\mathbf{w} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \mathbf{F}\mathbf{W}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}}, \quad \forall\, \mathbf{W} \in T_X\mathcal{B},\, \mathbf{w} \in T_{\varphi_t(X)}\mathcal{S}\,. \tag{2.1}$$

It has components, $(F^\mathsf{T})^A{}_a = G^{AB} F^b{}_B\, g_{ab}$. The right Cauchy-Green deformation tensor is another measure of strain and is defined as $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} : T_X\mathcal{B} \to T_X\mathcal{B}$, which has components $C^A{}_B = F^a{}_M F^b{}_B\, g_{ab}\, G^{AM}$. Note that $\mathbf{C}^\flat$ is the pull-back of the ambient space metric by φ_t, i.e., $\mathbf{C}^\flat = \varphi_t^*\mathbf{g}$, where $\flat$ is the flat operator. In components, $C_{AB} = F^a{}_A F^b{}_B\, g_{ab}$.

Balance laws. The balance of linear momentum in spatial form reads

$$\mathrm{div}_{\mathbf{g}}\, \boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}, \tag{2.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, and ρ, $\mathbf{b}$, and $\mathbf{a}$ are the mass density, body force, and acceleration, respectively. The balance of angular momentum is equivalent to symmetry of the Cauchy stress.

Incompressibility. The Jacobian of deformation J relates the deformed and undeformed Riemannian volume elements $dv(x,\mathbf{g}) = J dV(X,\mathbf{G})$, and is defined as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\, \det \mathbf{F}. \tag{2.3}$$

[1]The metric $\mathbf{g}$ can be time dependent for some problems of physical interest [Arroyo and DeSimone, 2009, Yavari et al., 2016].

Constitutive equations. The energy function of an inhomogeneous anisotropic hyperelastic material at a material point X has the following form

$$W = \hat{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)\,, \tag{2.4}$$

where $\boldsymbol{\zeta}_i, i = 1, \ldots, n$ are *structural tensors* that characterize the material symmetry group at the point X [Spencer, 1971, Boehler, 1979, Spencer, 1982, Liu et al., 1982, Zheng and Spencer, 1993, Lu and Papadopoulos, 2000]. Using structural tensors makes the energy function an isotropic function of its arguments. Hilbert's theorem tells us that for any finite collection of tensors, there exist a finite number of isotropic invariants forming a basis—an *integrity basis*—for the space of isotropic invariants of the collection of tensors. Therefore, if $I_j, j = 1, \ldots, m$, form an integrity basis for the set of tensors in (2.4), one has $W = W(X, I_1, ..., I_m)$. Let us define

$$W_j = \frac{\partial W}{\partial I_j}, \quad j = 1, ..., m\,. \tag{2.5}$$

Isotropic solids. For isotropic solids, the energy function has the form $W = W(X, I_1, I_2, I_3)$, where $I_1 = \operatorname{tr}\mathbf{C}$, $I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}$, and $I_3 = \det\mathbf{C}$ are the principal invariants of the right (or left) Cauchy-Green deformation tensor. The Cauchy stress for compressible and incompressible isotropic solids has the following representations

$$\begin{aligned} \sigma^{ab} &= \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab}\right]\,, \\ \sigma^{ab} &= -p g^{ab} + 2\left(W_1 b^{ab} - W_2\, c^{ab}\right)\,, \end{aligned} \tag{2.6}$$

where

$$b^{ab} = F^a{}_A F^b{}_B G^{AB}, \quad c^{ab} = (F^{-1})^M{}_m (F^{-1})^N{}_n G_{MN} g^{am} g^{bn}\,, \tag{2.7}$$

and p is the Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$.

Transversely isotropic solids. Let us assume that the unit vector $\mathbf{N}(\mathrm{X})$ identifies the material preferred direction at a point X in the reference configuration. The energy function has the form $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor that represents the transverse isotropy of the material symmetry group [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]. The energy function W depends on the following five independent invariants

$$I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\,\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,. \tag{2.8}$$

In components they read

$$I_1 = C^A{}_A\,,\ I_2 = \det(C^A{}_B)(C^{-1})^D{}_D\,,\ I_3 = \det(C^A{}_B)\,,\ I_4 = N^A N^B C_{AB}\,,\ I_5 = N^A N^B C_{BD} C^D{}_A\,. \tag{2.9}$$

For a compressible transversely isotropic solid the Cauchy stress tensor has the following representation [Golgoon and Yavari, 2018b, 2021]

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab} + W_4\, n^a n^b + W_5 (n^a b^{bc} n_c + n^b b^{ac} n_c)\right]\,, \tag{2.10}$$

where $n^a = F^a{}_A N^A$. For incompressible transversely isotropic solids

$$\sigma^{ab} = -p g^{ab} + 2\left[W_1 b^{ab} - W_2\, c^{ab} + W_4\, n^a n^b + W_5 (n^a b^{bc} n^d g_{cd} + n^b b^{ac} n^d g_{cd})\right]\,. \tag{2.11}$$

Orthotropic solids. For an orthotropic material three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ specify the orthotropic axes in the reference configuration at a point X. A choice of structural tensors is $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3 \otimes \mathbf{N}_3$. However, only two of the structural tensors are independent because $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$. The energy function has the form $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)$

[Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]. The energy function W depends on the following seven independent invariants

$$
\begin{aligned}
&I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\,\operatorname{tr}\,\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,,\\
&I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,, \quad I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2\,.
\end{aligned}
\tag{2.12}
$$

The Cauchy stress tensor has the following representation for compressible orthotropic solids [Golgoon and Yavari, 2018b, 2021]

$$
\begin{aligned}
\sigma^{ab} =& \frac{2}{\sqrt{I_3}}\Big[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab}\\
&+ W_4\, n_1^a n_1^b + W_5\left(n_1^a b^{bc} n_1^d g_{cd} + n_1^b b^{ac} n_1^d g_{cd}\right) + W_6\, n_2^a n_2^b + W_7\left(n_2^a b^{bc} n_2^d g_{cd} + n_2^b b^{ac} n_2^d g_{cd}\right)\Big]\,,
\end{aligned}
\tag{2.13}
$$

where $n_1^a = F^a{}_A N_1^A$, and $n_2^a = F^a{}_A N_2^A$. In the case of incompressible solids

$$
\begin{aligned}
\sigma^{ab} =& -p g^{ab} + 2\Big[W_1 b^{ab} - I_3 W_2\, c^{ab} + W_4\, n_1^a n_1^b + W_5\left(n_1^a b^{bc} n_1^d\, g_{cd} + n_1^b b^{ac} n_1^d\, g_{cd}\right)\\
&+ W_6\, n_2^a n_2^b + W_7\left(n_2^a b^{bc} n_2^d\, g_{cd} + n_2^b b^{ac} n_2^d\, g_{cd}\right)\Big]\,.
\end{aligned}
\tag{2.14}
$$

3 A geometric theory of finite eigenstrains

3.1 The material manifold of an elastic body with distributed finite eigenstrains

The stress-free body in the absence of eigenstrains is denoted by $\mathcal{B}$, which is a subset of the Euclidean space and has metric $\mathbf{G}_0$. Let us consider a coordinate chart $\{X_0^A\}$ for the Euclidean space. In the absence of eigenstrains the natural length of a line element $d\mathbf{X}_o$ is calculated as

$$
dS_o^2 = \langle\!\langle d\mathbf{X}_0, d\mathbf{X}_0\rangle\!\rangle_{\mathbf{G}_0} = G_{0AB}\, dX_0^A dX_0^B\,,
\tag{3.1}
$$

where $\langle\!\langle,\rangle\!\rangle_{\mathbf{G}_0}$ is the inner product induced by $\mathbf{G}_0$. The same line element with eigenstrain if allowed to relax in the Euclidean space would be relaxed to another line element $d\mathbf{X}$, which is unique up to rigid body translations and rotations. Eigenstrain can be defined as the relaxation map $\mathbf{K}$ such that $d\mathbf{X} = \mathbf{K}d\mathbf{X}_o$. Let us describe the relaxed line element using another coordinate chart $\{X^\alpha\}$ for the Euclidean space. Thus, $dX^\alpha = K^\alpha{}_A dX_o^A$. The length of the relaxed line element is calculated as

$$
dS^2 = \langle\!\langle d\mathbf{X}, d\mathbf{X}\rangle\!\rangle_{\mathbf{G}_0} = G_{0\alpha\beta}\, dX^\alpha dX^\beta = \left(K^\alpha{}_A K^\beta{}_B G_{0\alpha\beta}\right) dX_o^A dX_o^B = \langle\!\langle d\mathbf{X}_0, d\mathbf{X}_0\rangle\!\rangle_{\mathbf{K}^*\mathbf{G}_0}\,,
\tag{3.2}
$$

where $\mathbf{G} = \mathbf{K}^*\mathbf{G}_0$ is the material metric.[2] Note that $\mathbf{F}^p$ of finite plasticity is a special case of $\mathbf{K}$ [Sadik and Yavari, 2017]. Note also that, in general, $\mathbf{K}$ is incompatible, i.e., it is not the tangent of any map from $\mathcal{B}$ to itself. Incompatibility of $\mathbf{K}$ is the source of residual stresses (see Fig.2).

Let us assume that the body is made of an isotropic material in its relaxed state. The relaxed configuration is locally described by the metric $\mathbf{G}$. Therefore, $W = W(X, I_1, I_2, I_3)$, where the invariants are calculated using the metric $\mathbf{G}$. Similarly, for transversely isotropic and orthotropic solids all the invariants are calculated using the metric $\mathbf{G}$.

3.2 Combinations of radial, azimuthal, axial, and torsional eigenstrains in a circular cylindrical bar

Let us consider a circular cylindrical bar with radius R_o and length L in its undeformed configuration. For this bar we consider the radial, azimuthal, and axial eigentrains $e^{\omega_R(R)}$, $e^{\omega_\Theta(R)}$, and $e^{\omega_Z(R)}$ in the cylindrical coordinates (R, Θ, Z). We also consider an eigentwist per unit length $\psi(R)$. We next show that unlike the

[2] Note that there is a typo in Eq.(2.7) in [Yavari and Goriely, 2015b]. However, none of the calculations were affected by this typo.

problems that were considered in [Yavari and Goriely, 2013, 2015b] having the four functions $\omega_R(R)$, $\omega_\Theta(R)$, $\omega_Z(R)$, and $\psi(R)$ does not specify the material metric unambiguously. Let us denote the radial, azimuthal, axial, and twist eigenstrains by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$, and $\mathbf{K}_4$ respectively, which have the following representations in cylindrical coordinates:

$$\mathbf{K}_1 = \begin{bmatrix} e^{\omega_R(R)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\omega_\Theta(R)} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\omega_Z(R)} \end{bmatrix}, \quad \mathbf{K}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi(R) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

Note that

$$[\mathbf{K}_1, \mathbf{K}_2] = [\mathbf{K}_2, \mathbf{K}_3] = [\mathbf{K}_3, \mathbf{K}_1] = \mathbf{0}\,, \tag{3.4}$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is the commutator of the matrices $\mathbf{A}$ and $\mathbf{B}$. Eq.(3.4) implies that in the absence of eigentwists the material metric is defined unambiguously as $\mathbf{K} = \mathbf{K}_{\tau(1)}\mathbf{K}_{\tau(2)}\mathbf{K}_{\tau(3)}$, where $\{\tau(1), \tau(2), \tau(3)\}$ is any of the six permutations of $\{1, 2, 3\}$. Note that

$$[\mathbf{K}_1, \mathbf{K}_4] = \mathbf{0}, \quad [\mathbf{K}_2, \mathbf{K}_4] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{\omega_\Theta(R)} - 1 & \psi(R)(e^{\omega_\Theta(R)} - 2) \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{K}_3, \mathbf{K}_4] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\psi(R) \\ 0 & 0 & e^{\omega_Z(R)} - 1 \end{bmatrix}. \tag{3.5}$$

The twenty four permutations of $\{1, 2, 3, 4\}$ give four different total eigenstrain matrices. However, we are interested in the eigenstrains $\mathbf{K}_1\mathbf{K}_2\mathbf{K}_3$ and $\mathbf{K}_4$. The two total eigenstrains $\mathbf{K}_i = \mathbf{K}_1\mathbf{K}_2\mathbf{K}_3\mathbf{K}_4$ and $\mathbf{K}_{ii} = \mathbf{K}_4\mathbf{K}_1\mathbf{K}_2\mathbf{K}_3$ have the following corresponding material metrics

$$\begin{aligned} \mathbf{G}_i &= \begin{bmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R)}R^2 & e^{2\omega_\Theta(R)}R^2\psi(R) \\ 0 & e^{2\omega_\Theta(R)}R^2\psi(R) & e^{2\omega_Z(R)} + e^{2\omega_\Theta(R)}R^2\psi(R)^2 \end{bmatrix}, \\ \mathbf{G}_{ii} &= \begin{bmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R)}R^2 & e^{\omega_\Theta(R)+\omega_Z(R)}R^2\psi(R) \\ 0 & e^{\omega_\Theta(R)+\omega_Z(R)}R^2\psi(R) & e^{2\omega_Z(R)}(1 + R^2\psi(R)^2) \end{bmatrix}. \end{aligned} \tag{3.6}$$

Note that $\mathbf{G}_i = \mathbf{G}_{ii}$ if and only if $\omega_\Theta(R) = \omega_Z(R)$. In the special case of $\omega_R(R) = \omega_\Theta(R) = \omega_Z(R) = \omega(R)$, the material metric reads

$$\mathbf{G} = e^{2\omega(R)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & R^2\psi(R) \\ 0 & R^2\psi(R) & 1 + R^2\psi(R)^2 \end{bmatrix}. \tag{3.7}$$

4 Radially-symmetric eigenstrains in a finite spherical ball

The problem of calculating the deformation and stress fields of a finite spherical ball with a radially-symmetric distribution of finite eigenstrains was solved by Yavari and Goriely [2013] for isotropic solids and by Golgoon and Yavari [2018b] for transversely isotropic solids. In [Goodbrake et al., 2020] it was shown that the eigenstrain distributions considered in [Yavari and Goriely, 2013] are the only universal eigenstrains consistent with the universal deformations of incompressible isotropic spherical shells [Ericksen, 1954]. In this section we revisit Yavari and Goriely [2013]'s analysis for pure dilatational eigenstrains and extend their analysis to radially inhomogeneous spherical balls.

Let us consider an inhomogeneous finite ball of radius R_o made of an incompressible isotropic nonlinear elastic solid at R with an energy function $W = W(R, I_1, I_2)$.[3] Also consider a radially-symmetric distribution of pure dilatational eigenstrains. We assume that in the absence of eigenstrains the body is stress free. In the spherical coordinates (R, Θ, Φ) a line element $dS_0^2 = dR^2 + R^2 d\Theta^2 + R^2 \sin^2\Theta d\Phi^2$ in the initial stress-free

[3]Our analysis can be easily extended to a ball made of a transversely isotropic solid with radial material preferred direction [Golgoon and Yavari, 2018b]. Also note that radial deformations are still universal for radially inhomogeneous isotropic and transversely isotropic spherical balls [Golgoon and Yavari, 2021].

configuration is mapped to the line element $dS^2 = e^{2\omega(R)} dS_0^2$, for some function $\omega(R)$. This means that in the presence of eigenstrains the material metric has the following representation

$$\mathbf{G}(\mathbf{X}) = \mathbf{G}(R) = e^{2\omega(R)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2\Theta \end{bmatrix} . \tag{4.1}$$

Note that the material metric is independent of the constitutive equations of the ball. It is natural to use the spherical coordinates (r, θ, ϕ) for the Euclidean ambient space with the metric

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix} . \tag{4.2}$$

Let us consider deformations of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$. Thus, $\mathbf{F} = \mathrm{diag}(r'(R), 1, 1)$, and hence $\det \mathbf{F} = r'(R)$. For an incompressible solid $J = 1$, where

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r^2(R) r'(R)}{R^2 e^{3\omega(R)}} . \tag{4.3}$$

Therefore, assuming that $r(0) = 0$, one obtains

$$r(R) = \left[\int_0^R 3\xi^2 e^{3\omega(\xi)} d\xi \right]^{\frac{1}{3}} . \tag{4.4}$$

Using the representation (2.6) the non-zero components of the Cauchy stress are

$$\begin{aligned} \sigma^{rr}(R) &= -p(R) + 2W_1(R) \frac{e^{4\omega(R)} R^4}{r^4(R)} + 4W_2(R) \frac{e^{2\omega(R)} R^2}{r^2(R)} , \\ \sigma^{\theta\theta}(R) &= -\frac{p(R)}{r^2(R)} + 2W_1(R) \frac{1}{e^{2\omega(R)} R^2} + 2W_2(R) \left[\frac{e^{2\omega(R)} R^2}{r^4(R)} + \frac{r^2(R)}{e^{4\omega(R)} R^4} \right] , \\ \sigma^{\phi\phi}(R) &= \frac{1}{\sin^2\Theta} \sigma^{\theta\theta} , \end{aligned} \tag{4.5}$$

where

$$W_1(R) = \frac{\partial W(R, I_1, I_2)}{\partial I_1} , \quad W_2(R) = \frac{\partial W(R, I_1, I_2)}{\partial I_2} , \tag{4.6}$$

and

$$I_1 = I_1(R) = \frac{R^4 e^{4\omega(R)}}{r^4(R)} + \frac{2r^2(R) e^{-2\omega(R)}}{R^2} , \quad I_2 = I_2(R) = \frac{e^{-4\omega(R)} \left(r^6(R) + 2R^6 e^{6\omega(R)} \right)}{R^4 r^2(R)} . \tag{4.7}$$

In the absence of body forces, the only non-trivial equilibrium equation is

$$\sigma^{rr}{}_{,r} + \frac{2}{r} \sigma^{rr} - r\sigma^{\theta\theta} - r \sin^2\theta \, \sigma^{\phi\phi} = \frac{1}{r'(R)} \sigma^{rr}{}_{,R} + \frac{2}{r} \sigma^{rr} - 2r\sigma^{\theta\theta} = 0. \tag{4.8}$$

Or

$$\frac{d\,\sigma^{rr}(R)}{dR} = \frac{4W_1(R)}{R} \left[\frac{e^{\omega(R)} R}{r(R)} - \frac{e^{7\omega(R)} R^7}{r^7(R)} \right] + \frac{4W_2(R)}{R} \left[\frac{r(R)}{R e^{\omega(R)}} - \frac{e^{5\omega(R)} R^5}{r^5(R)} \right] . \tag{4.9}$$

Therefore

$$\sigma^{rr}(R) = \sigma^{rr}(R_o) - \int_R^{R_o} \left\{ \frac{4W_1(\xi)}{\xi} \left[\frac{e^{\omega(\xi)} \xi}{r(\xi)} - \frac{e^{7\omega(\xi)} \xi^7}{r^7(\xi)} \right] + \frac{4W_2(\xi)}{\xi} \left[\frac{r(\xi)}{\xi e^{\omega(\xi)}} - \frac{e^{5\omega(\xi)} \xi^5}{r^5(\xi)} \right] \right\} d\xi . \tag{4.10}$$

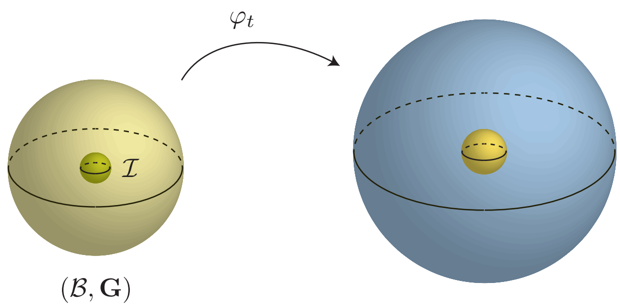


Figure 3: *A spherical inclusion $\mathcal{I}$ with uniform pure dilatational finite eigenstrain centered at a spherical ball. Left: The material manifold is a Riemannian manifold whose metric has a jump discontinuity on the boundary of the inclusion. Right: The deformed configuration.*

A spherical inclusion in a finite ball. Let us consider an inclusion of radius $R_i < R_o$ with pure dilatational eigenstrain ω_0 (see Fig.3). This corresponds to the following eigenstrain distribution in the ball

$$\omega(R) = \begin{cases} \omega_0 & 0 \leq R \leq R_i, \\ 0 & R_i < R \leq R_o. \end{cases} \tag{4.11}$$

The incompressibility constraint fully determines the kinematics of deformation as

$$r(R) = \begin{cases} e^{\omega_0} R & 0 \leq R \leq R_i, \\ \left[R^3 + \left(e^{3\omega_0} - 1\right) R_i^3\right]^{\frac{1}{3}} & R_i < R \leq R_o. \end{cases} \tag{4.12}$$

For $R < R_i$, from (4.12) and (4.9) one can easily see that $\frac{d\,\sigma^{rr}(R)}{dR} = 0$. Therefore, $\sigma^{rr}(R) = \sigma_i$, and $\sigma^{\theta\theta}(R) = \sigma^{rr}(R)/r^2(R) = \sigma_i/r^2(R)$. Denoting the physical components of the Cauchy stress by $\hat{\sigma}^{rr}, \hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{\phi\phi}$, it is seen that inside the inclusion $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = \sigma_i$. In other words, inside the inclusion stress is homogeneous and hydrostatic. This is a nonlinear analogue of Eshelby's celebrated result and a generalization of [Yavari and Goriely, 2013, Proposition 3.1] to radially-inhomogeneous spherical balls. The value of σ_i is calculated using the continuity of traction on the boundary of the inclusion and is written as

$$\sigma_i = \sigma^{rr}(R_i) = \sigma^{rr}(R_o) - \int_{R_i}^{R_o} \left\{ \frac{4W_1(\xi)}{\xi} \left[\frac{\xi}{r(\xi)} - \frac{\xi^7}{r^7(\xi)} \right] + \frac{4W_2(\xi)}{\xi} \left[\frac{r(\xi)}{\xi} - \frac{\xi^5}{r^5(\xi)} \right] \right\} d\xi \,. \tag{4.13}$$

Note that the constant σ_i explicitly depends on the energy function $W(R, I_1, I_2)$.

For a homogeneous neo-Hookean spherical ball ($\alpha = \mu/2,\ \beta = 0$) the uniform stress inside the inclusion is calculated as

$$\sigma_i = \frac{\mu}{2} \left[e^{-4\omega_0} + 4e^{-\omega_0} - \frac{5R_o^3 + 4(e^{3\omega_0} - 1)R_i^3}{[R_o^3 + (e^{3\omega_0} - 1)R_i^3]^{\frac{4}{3}}} R_o \right] . \tag{4.14}$$

Assuming that $\sigma^{rr}(R_o) = 0$, outside the inclusion ($R_i < R < R_o$) the radial stress has the following distribution

$$\sigma^{rr}(R) = \frac{2\mu R_i^3 \left(e^{3\omega_0} - 1\right) R}{[R^3 + R_i^3 \left(e^{3\omega_0} - 1\right)]^{\frac{4}{3}}} + \frac{5\mu R^4}{2\,[R^3 + R_i^3 \left(e^{3\omega_0} - 1\right)]^{\frac{4}{3}}} - \frac{\mu R_o \left[4R_i^3 \left(e^{3\omega_0} - 1\right) + 5R_o^3\right]}{2\,[R_i^3 \left(e^{3\omega_0} - 1\right) + R_o^3]^{\frac{4}{3}}} \,. \tag{4.15}$$

Fig.4 shows the distribution of radial stress for four different values of eigenstrain for $\frac{R_i}{R_o} = 0.2$ (solid curves). The corresponding solutions using linear elasticity are also shown (dashed curves). The classical linear solution is

$$\sigma_{\text{lin}}^{rr}(R) = \begin{cases} \sigma_i^{\text{lin}} = -4\mu\omega_0 \left[1 - \frac{R_i^3}{R_o^3}\right] & 0 \leq R \leq R_i, \\ -4\mu\omega_0 \left[\frac{R_i^3}{R^3} - \frac{R_i^3}{R_o^3}\right] & R_i < R \leq R_o. \end{cases} \tag{4.16}$$

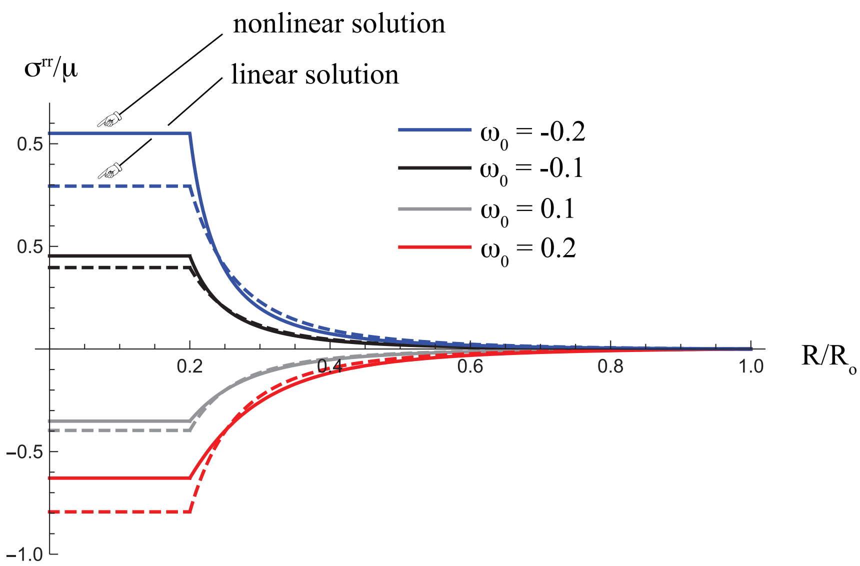


Figure 4: *Radial stress distribution inside and outside a spherical inclusion. It is assumed that* $\frac{R_i}{R_o} = 0.2$ *and the ball is made of a homogeneous incompressible neo-Hookean solid. Four different values of* ω_0 *are considered. The dotted curves are the linear elasticity solutions. It is seen that for* $\omega_0 > 0$ *the linear elasticity solution overestimates the compressive stress inside the inclusion, while for* $\omega_0 < 0$ *the linear elasticity solution underestimates the tensile stress inside the inclusion. Note that unlike the linear solution the nonlinear solution is not symmetric with respect to change in sign of* ω_0.

5 Radially-symmetric eigenstrains in a finite circular cylindrical bar

In this section we revisit another example that was analyzed by Yavari and Goriely [2013] and relax the homogeneity assumption. Let us consider a cylindrical bar with radius R_o and length L in its initial stress-free configuration in the absence of eigenstrains. We assume that this solid cylinder is radially inhomogeneous and is made of an arbitrary incompressible isotropic solid at R with energy function $W = W(R, I_1, I_2)$. Now if the cylinder has a radially-symmetric distribution of finite eigenstrains the problem is to calculate the induced stress field. We assume that radial and circumferential eigenstrains are equal (the more general case was discussed in [Yavari and Goriely, 2013]) but the axial eigenstrain can be different. In cylindrical coordinates (R, Θ, Z) the material metric has the following representation

$$\mathbf{G} = \mathbf{G}(R) = \begin{bmatrix} e^{2\omega(R)} & 0 & 0 \\ 0 & R^2 e^{2\omega(R)} & 0 \\ 0 & 0 & e^{2\omega_Z(R)} \end{bmatrix}, \tag{5.1}$$

where $\omega(R)$, and $\omega_Z(R)$ are some given functions (note that the two functions $\omega(R)$ and $\omega_Z(R)$ unambiguously specify the metric as was discussed in §3.2). The natural coordinates to describe the deformed configuration are the cylindrical coordinates (r, θ, z). We assume deformations of the form $(r, \theta, z) = (r(R), \Theta, \lambda Z)$, where λ is a constant axial stretch. Deformation gradient reads $\mathbf{F} = \operatorname{diag}(r'(R), 1, \lambda)$, and hence $\det \mathbf{F} = \lambda r'(R)$. The incompressibility constraint is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{\lambda r(R)}{R e^{2\omega(R)+\omega_Z(R)}} r'(R) = 1\,. \tag{5.2}$$

Assuming that $r(0)=0$ this gives

$$r(R)=\frac{1}{\sqrt{\lambda}}\left[\int_0^R 2\xi e^{2\omega(\xi)+\omega_Z(\xi)}d\xi\right]^{\frac{1}{2}}. \tag{5.3}$$

Using the representation (2.6) the non-zero components of the Cauchy stress are

$$\begin{aligned}\sigma^{rr}(R)=&-p(R)+2W_1(R)\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}+2W_2(R)\left[\frac{1}{\lambda^2 e^{-2\omega_Z(R)}}+\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}\right],\\ \sigma^{\theta\theta}(R)=&-\frac{p(R)}{r^2(R)}+2W_1(R)\frac{1}{e^{2\omega(R)}R^2}+2W_2(R)\left[\frac{1}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}+\frac{\lambda^2 e^{-2\omega_Z(R)}}{e^{2\omega(R)}R^2}\right],\\ \sigma^{zz}(R)=&-p(R)+2W_1(R)\lambda^2 e^{-2\omega_Z(R)}+2W_2(R)\left[\frac{e^{2\omega(R)}R^2}{r^2(R)}+\frac{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}{e^{2\omega(R)}R^2}\right],\end{aligned} \tag{5.4}$$

where

$$W_1(R)=\frac{\partial W(R,I_1,I_2)}{\partial I_1},\quad W_2(R)=\frac{\partial W(R,I_1,I_2)}{\partial I_2}, \tag{5.5}$$

and

$$\begin{aligned}I_1=I_1(R)&=\lambda^2 e^{-2\omega_Z}(R)+\frac{r^2(R)e^{-2\omega(R)}}{R^2}+\frac{R^2 e^{2\omega(R)+2\omega_Z}(R)}{\lambda^2 r^2(R)},\\ I_2=I_2(R)&=\frac{e^{2\omega_Z(R)}}{\lambda^2}+\frac{R^2 e^{2\omega(R)}}{r^2(R)}+\frac{\lambda^2 r^2(R)e^{-2(\omega(R)+\omega_Z(R))}}{R^2}.\end{aligned} \tag{5.6}$$

The only non-trivial equilibrium equation reads

$$\sigma^{rr}{}_{,r}+\frac{1}{r}\sigma^{rr}-r\sigma^{\theta\theta}=0. \tag{5.7}$$

Or

$$\frac{d}{dR}\sigma^{rr}(R)=\frac{2}{R}\left[\frac{W_1(R)e^{\omega_Z(R)}}{\lambda}+\frac{W_2(R)\lambda}{e^{\omega_Z(R)}}\right]\left[1-e^{2\omega_Z(R)}\lambda^{-2}\frac{e^{4\omega(R)}R^4}{r^4(R)}\right]. \tag{5.8}$$

Thus

$$\sigma^{rr}(R)=\sigma^{rr}(R_o)-\int_R^{R_o}\frac{2}{\xi}\left[\frac{W_1(\xi)e^{\omega_Z(\xi)}}{\lambda}+\frac{W_2(\xi)\lambda}{e^{\omega_Z(\xi)}}\right]\left[1-e^{2\omega_Z(\xi)}\lambda^{-2}\frac{e^{4\omega(\xi)}\xi^4}{r^4(\xi)}\right]d\xi. \tag{5.9}$$

This means that

$$\begin{aligned}-p(R)=\sigma^{rr}(R_o)&-2W_1(R)\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}-2W_2(R)\left[\frac{1}{\lambda^2 e^{-2\omega_Z(R)}}+\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}\right]\\ &-\int_R^{R_o}\frac{2}{\xi}\left[\frac{W_1(\xi)e^{\omega_Z(\xi)}}{\lambda}+\frac{W_2(\xi)\lambda}{e^{\omega_Z(\xi)}}\right]\left[1-e^{2\omega_Z(\xi)}\lambda^{-2}\frac{e^{4\omega(\xi)}\xi^4}{r^4(\xi)}\right]d\xi.\end{aligned} \tag{5.10}$$

Therefore

$$\begin{aligned}\sigma^{zz}(R)=\sigma^{rr}(R_o)&+2W_1(R)\lambda^2 e^{-2\omega_Z(R)}+2W_2(R)\left[\frac{e^{2\omega(R)}R^2}{r^2(R)}+\frac{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}{e^{2\omega(R)}R^2}\right]\\ &-2W_1(R)\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}-2W_2(R)\left[\frac{1}{\lambda^2 e^{-2\omega_Z(R)}}+\frac{e^{2\omega(R)}R^2}{\lambda^2 e^{-2\omega_Z(R)}r^2(R)}\right]\\ &-\int_R^{R_o}\frac{2}{\xi}\left[\frac{W_1(\xi)e^{\omega_Z(\xi)}}{\lambda}+\frac{W_2(\xi)\lambda}{e^{\omega_Z(\xi)}}\right]\left[1-e^{2\omega_Z(\xi)}\lambda^{-2}\frac{e^{4\omega(\xi)}\xi^4}{r^4(\xi)}\right]d\xi.\end{aligned} \tag{5.11}$$

A cylindrical inclusion in a finite cylindrical bar. Let us consider an inclusion with radius $R_i < R_o$ such that

$$\omega(R) = \begin{cases} \omega_1 & 0 \leq R < R_i \\ 0 & R_i < R \leq R_o \end{cases}, \qquad \omega_Z(R) = \begin{cases} \omega_2 & 0 \leq R < R_i \\ 0 & R_i < R \leq R_o \end{cases}. \tag{5.12}$$

Therefore

$$r(R) = \begin{cases} \frac{1}{\sqrt{\lambda}} e^{\frac{1}{2}(2\omega_1+\omega_2)} R & 0 \leq R \leq R_i, \\ \frac{1}{\sqrt{\lambda}} \left[R^2 + \left(e^{2\omega_1+\omega_2} - 1\right) R_i^2\right]^{\frac{1}{2}} & R_i < R \leq R_o. \end{cases} \tag{5.13}$$

For $R < R_i$, from (5.13) and (5.8) one can easily see that $\frac{d}{dR}\sigma^{rr}(R) = 0$, and hence $\sigma^{rr}(R) = \sigma_i$, where[4]

$$\sigma_i = \sigma^{rr}(R_o) - \int_{R_i}^{R_o} \frac{2}{\xi} \left[W_1(\xi)\lambda^{-1} + W_2(\xi)\lambda\right] \left[1 - \lambda^{-2} \frac{\xi^4}{r^4(\xi)}\right] d\xi\,. \tag{5.15}$$

This also implies that inside the inclusion $\sigma^{rr}(R) = r^2 \sigma^{\theta\theta}(R)$, and thus the physical component $\hat{\sigma}^{\theta\theta}(R) = \sigma_i$. From $(5.4)_1$ one can see that inside the inclusion $p(R)$ is not uniform. The axial stress reads

$$\sigma^{zz}(R) = \sigma_i + 2W_1(R)(\lambda^2 e^{-2\omega_2} - \lambda^{-1} e^{\omega_2}) + 2W_2(R)(\lambda e^{-\omega_2} - \lambda^{-2} e^{2\omega_2})\,. \tag{5.16}$$

It is seen that the axial stress is not uniform inside the inclusion unless the inclusion is homogeneous. For $R > R_i$:

$$\begin{aligned} \sigma^{zz}(R) = \sigma^{rr}(R_o) &+ 2W_1(R)\left[\lambda^2 - \frac{R^2}{\lambda^2 r^2(R)}\right] + 2W_2(R)\left[\frac{R^2}{r^2(R)} + \frac{\lambda^2 r^2(R)}{R^2} - \frac{1}{\lambda^2} - \frac{R^2}{\lambda^2 r^2(R)}\right] \\ &- \int_R^{R_o} \frac{2}{\xi}\left[W_1(\xi)\lambda^{-1} + W_2(\xi)\lambda\right]\left[1 - \lambda^{-2}\frac{\xi^4}{r^4(\xi)}\right] d\xi\,. \end{aligned} \tag{5.17}$$

Therefore, stress inside the inclusion is not homogeneous unless the inclusion is homogeneous. Even in that case stress is not necessarily hydrostatic. This is a generalization of [Yavari and Goriely, 2013, Proposition 3.5] to radially-inhomogeneous circular cylindrical bars.

The axial force F needed to maintain the deformation is calculated as

$$F = \int_0^{r_o} \sigma^{zz}(r)\, 2\pi r\, dr = 2\pi \int_0^{R_o} \sigma^{zz}(R)\, r(R) r'(R)\, dR = \frac{2\pi}{\lambda} \int_0^{R_o} R\, e^{2\omega(R)+\omega_Z(R)} \sigma^{zz}(R)\, dR. \tag{5.18}$$

Suppose that there is no axial force, i.e, $F = 0$. Thus

$$\int_0^{R_o} R\, e^{2\omega(R)+\omega_Z(R)} \sigma^{zz}(R)\, dR = e^{2\omega_1+\omega_2} \int_0^{R_i} R\, \sigma^{zz}(R)\, dR + \int_{R_i}^{R_o} R\, \sigma^{zz}(R)\, dR = 0\,. \tag{5.19}$$

For a homogeneous neo-Hookean solid ($\alpha = \mu/2,\ \beta = 0$) and for zero axial eigenstrain $\omega_2 = 0$ this gives

$$\lambda^3 = \lambda^3(\omega_1, c_0) = e^{\omega_1} \frac{\cosh\omega_1 - \left[1 - 2c_0 + c_0 \ln c_0 + c_0 \ln \frac{1+(e^{2\omega_1}-1)c_0}{e^{2\omega_1} c_0}\right] \sinh\omega_1}{1 + 2e^{\omega_1} c_0 \sinh\omega_1}\,, \tag{5.20}$$

where $c_0 = R_i^2/R_o^2$. Note that $\lambda^3(0, c_0) = 1$, and it can be shown that λ^3 is a strictly increasing function of ω_1 for $\omega_1 > 0$ and is strictly decreasing for $\omega_1 < 0$. Therefore, $\lambda(\omega_1, c_0) > 1$ for $\omega_1 \neq 0$.

[4] In the case of a homogeneous neo-Hookean solid

$$\sigma_i = \frac{\mu}{2\lambda}\left[e^{-2\omega_1} - \frac{R_o^2}{R_o^2 + (e^{2\omega_1} - 1)R_i^2} + \ln\frac{R_i^2}{R_o^2} + \ln\frac{R_o^2 + (e^{2\omega_1} - 1)R_i^2}{e^{2\omega_1} R_i^2}\right]. \tag{5.14}$$

6 Radially-symmetric eigentwists in an orthotropic circular cylindrical bar

In this section we extend Yavari and Goriely [2015b]'s analysis to orthotropic solids. Let us consider a circular cylindrical bar that has initial length L and radius R_o and is made of an incompressible orthotropic material with an energy function $W = W(R, I_1, I_2, I_4, I_5, I_6, I_7)$. We also assume that the material preferred directions are radial, azimuthal, and axial, i.e., $\mathbf{N}_1 = \hat{\mathbf{R}}$, $\mathbf{N}_2 = \hat{\mathbf{Z}}$, and $\mathbf{N}_3 = \hat{\boldsymbol{\Theta}}$, where $\hat{\mathbf{R}}$, $\hat{\mathbf{Z}}$, and $\hat{\boldsymbol{\Theta}}$ are the unit vectors in the radial, longitudinal, and circumferential directions, respectively. We assume an eigentwist distribution $\psi(R)$. In cylindrical coordinates the material metric has the following representation

$$\mathbf{G}(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & \psi(R)R^2 \\ 0 & \psi(R)R^2 & 1+\psi^2(R)R^2 \end{bmatrix}. \tag{6.1}$$

Therefore

$$\begin{aligned} \mathbf{N}_1 &= \mathbf{E}_R = \frac{\partial}{\partial R}\,, \\ \mathbf{N}_2 &= \frac{1}{\sqrt{1+\psi^2(R)R^2}}\mathbf{E}_Z = \frac{1}{\sqrt{1+\psi^2(R)R^2}}\frac{\partial}{\partial Z}\,, \\ \mathbf{N}_3 &= \frac{1}{R}\mathbf{E}_\Theta = \frac{1}{R}\frac{\partial}{\partial \Theta}\,. \end{aligned} \tag{6.2}$$

In the ambient space the cylindrical coordinates (r, θ, z) are used and metric has the representation (4.2). We consider deformations of the form:

$$(r, \theta, z) = (r(R), \Theta + \tau Z, \lambda Z)\,, \tag{6.3}$$

where τ and λ are some unknown constants to be determined. The deformation gradient reads

$$\mathbf{F} = \begin{bmatrix} r'(R) & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & \lambda \end{bmatrix}. \tag{6.4}$$

The incompressibility condition is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\det \mathbf{F} = \frac{\lambda r(R) r'(R)}{R} = 1. \tag{6.5}$$

Assuming that $r(0) = 0$, we have $r(R) = \frac{R}{\sqrt{\lambda}}$. The principal invariants read

$$\begin{aligned} I_1(R) &= \frac{2}{\lambda} + \lambda^2 + \frac{R^2}{\lambda}(\tau - \psi(R))^2\,, \quad I_2(R) = 2\lambda + \frac{1}{\lambda^2} + \frac{R^2}{\lambda^2}(\tau - \psi(R))^2\,, \\ I_4(R) &= \frac{1}{\lambda}, \quad I_5(R) = \frac{1}{\lambda^2}, \quad I_6(R) = \frac{\lambda^3 + R^2\tau^2}{\lambda[1+R^2\psi^2(R)]}\,, \\ I_7(R) &= \frac{R^2\tau\psi(R)\left[R^2\tau\psi(R) - 2\left(\lambda^3 + R^2\tau^2\right)\right] + \left(\lambda^3 + R^2\tau^2\right)^2 + R^2\tau^2}{\lambda^2[1+R^2\psi^2(R)]}\,. \end{aligned} \tag{6.6}$$

From (2.14) the Cauchy stress has the following non-zero components

$$
\begin{aligned}
\sigma^{rr}(R) &= -p(R) + \frac{2W_1(R)}{\lambda} + 2\left[\lambda + \frac{1}{\lambda^2} + \frac{R^2}{\lambda^2}(\tau-\psi(R))^2\right]W_2(R) + \frac{2W_4(R)}{\lambda} + \frac{4W_5(R)}{\lambda^2}\,,\\
\sigma^{\theta\theta}(R) &= -\frac{\lambda p(R)}{R^2} + 2\left[\frac{1}{R^2} + (\tau-\psi(R))^2\right]W_1(R) + \frac{2}{\lambda}\left[\frac{\lambda^3+1}{R^2} + (\tau-\psi(R))^2\right]W_2(R) + \frac{2\tau^2}{1+R^2\psi^2(R)}W_6(R)\\
&\quad + \frac{4\tau}{\lambda}\frac{\tau\left(1+\lambda^3+R^2\tau^2\right) + R^2\tau\psi^2(R) - \psi(R)\left(\lambda^3+2R^2\tau^2\right)}{1+R^2\psi^2(R)}W_7(R)\,,\\
\sigma^{zz}(R) &= -p(R) + 2\lambda^2 W_1(R) + 4\lambda W2(R) + \frac{2\lambda^2}{1+R^2\psi^2(R)}W_6(R) + 4\lambda\frac{\lambda^3+R^2\tau^2-R^2\tau\psi(R)}{1+R^2\psi^2(R)}W_7(R)\,,\\
\sigma^{\theta z}(R) &= 2\lambda(\tau-\psi(R))W_1(R) + 2(\tau-\psi(R))W_2(R) + \frac{2\lambda\tau}{1+R^2\psi^2(R)}W_6(R)\\
&\quad + 2\left[\tau + \frac{2\tau\left(\lambda^3+R^2\tau^2\right) - \psi(R)\left(\lambda^3+3R^2\tau^2\right)}{1+R^2\psi^2(R)}\right]W_7(R)\,.
\end{aligned}
\tag{6.7}
$$

The radial equilibrium equation (5.7) is simplified to read

$$
\begin{aligned}
\frac{d\sigma^{rr}}{dR} &= \frac{2R(\tau-\psi(R))^2}{\lambda}W_1(R) - \frac{2}{\lambda R}W_4(R) - \frac{4}{\lambda^2 R}W_5(R) + \frac{2R\tau^2}{\lambda(1+R^2\psi^2(R))}W_6(R)\\
&\quad + \frac{4R\tau}{\lambda^2}\frac{\tau\left(1+\lambda^3+R^2\tau^2\right) - \psi(R)\left(\lambda^3+2R^2\tau^2\right) + R^2\tau\psi^2(R)}{1+R^2\psi^2(R)}W_7(R)\,.
\end{aligned}
\tag{6.8}
$$

Assuming that the cylindrical boundary of the bar is traction free, one obtains

$$
\begin{aligned}
\sigma^{rr}(R) = \int_R^{R_o} \Bigg\{ &-\frac{2\xi(\tau-\psi(\xi))^2}{\lambda}W_1(R) + \frac{2}{\lambda\xi}W_4(R) + \frac{4}{\lambda^2\xi}W_5(R) - \frac{2\xi\tau^2}{\lambda(1+\xi^2\psi^2(\xi))}W_6(R)\\
&- \frac{4\xi\tau}{\lambda^2}\frac{\tau\left(1+\lambda^3+\xi^2\tau^2\right) - \psi(\xi)\left(\lambda^3+2\xi^2\tau^2\right) + \xi^2\tau\psi^2(\xi)}{1+\xi^2\psi^2(\xi)}W_7(R)\Bigg\}d\xi\,.
\end{aligned}
\tag{6.9}
$$

Thus, from $(6.7)_1$, one obtains

$$
\begin{aligned}
-p(R) &= -\frac{2W_1(R)}{\lambda} - 2\left[\lambda + \frac{1}{\lambda^2} - \frac{R^2}{\lambda^2}(\tau-\psi(R))^2\right]W_2(R) - \frac{2W_4(R)}{\lambda} - \frac{4W_5(R)}{\lambda^2}\\
&\quad + \int_R^{R_o}\Bigg\{ -\frac{2\xi(\tau-\psi(\xi))^2}{\lambda}W_1(R) + \frac{2}{\lambda\xi}W_4(R) + \frac{4}{\lambda^2\xi}W_5(R) - \frac{2\xi\tau^2}{\lambda(1+\xi^2\psi^2(\xi))}W_6(R)\\
&\quad - \frac{4\xi\tau}{\lambda^2}\frac{\tau\left(1+\lambda^3+\xi^2\tau^2\right) - \psi(\xi)\left(\lambda^3+2\xi^2\tau^2\right) + \xi^2\tau\psi^2(\xi)}{1+\xi^2\psi^2(\xi)}W_7(R)\Bigg\}d\xi\,.
\end{aligned}
\tag{6.10}
$$

Therefore, the axial stress is calculated as

$$
\begin{aligned}
\sigma^{zz}(R) &= 2\left(\lambda^2 - \frac{1}{\lambda}\right)W_1(R) + 2\left[\lambda - \frac{1}{\lambda^2} + \frac{R^2}{\lambda^2}(\tau-\psi(R))^2\right]W_2(R) - \frac{2W_4(R)}{\lambda} - \frac{4W_5(R)}{\lambda^2}\\
&\quad + \frac{2\lambda^2}{1+R^2\psi^2(R)}W_6(R) + 4\lambda\frac{\lambda^3+R^2\tau^2-R^2\tau\psi(R)}{1+R^2\psi^2(R)}W_7(R)\\
&\quad + \int_R^{R_o}\Bigg\{ -\frac{2\xi(\tau-\psi(\xi))^2}{\lambda}W_1 + \frac{2}{\lambda\xi}W_4(R) + \frac{4}{\lambda^2\xi}W_5(R) - \frac{2\xi\tau^2}{\lambda(1+\xi^2\psi^2(\xi))}W_6(R)\\
&\quad - \frac{4\xi\tau}{\lambda^2}\frac{\tau\left(1+\lambda^3+\xi^2\tau^2\right) - \psi(\xi)\left(\lambda^3+2\xi^2\tau^2\right) + \xi^2\tau\psi^2(\xi)}{1+\xi^2\psi^2(\xi)}W_7(R)\Bigg\}d\xi\,.
\end{aligned}
\tag{6.11}
$$

The axial force and torque at the two ends of the bar $Z = 0, L$ are calculated as

$$F = 2\pi \int_0^{r_o} \sigma^{zz} dr = \frac{2\pi}{\lambda} \int_0^{R_o} \sigma^{zz}(R) R dR, \qquad M = 2\pi \int_0^{r_o} \hat{\sigma}^{\theta z} r^2 dr = \frac{2\pi}{\lambda^2} \int_0^{R_o} \sigma^{\theta z}(R) R^3 dR\,, \tag{6.12}$$

where $\hat{\sigma}^{\theta z} = r\sigma^{\theta z}$ is the physical θz component of the Cauchy stress. Assuming that there are no applied force and torque at the ends of the bar one obtains the following two nonlinear algebraic equations for (λ, τ):

$$\int_0^{R_o} \sigma^{zz}(R) R dR = 0\,, \qquad \int_0^{R_o} \sigma^{\theta z}(R) R^3 dR = 0\,. \tag{6.13}$$

7 Conclusions

The nonlinear mechanics of solids with distributed finite eigenstrains was revisited. In our geometric formulation the classical reference configuration of nonlinear elasticity is replaced by a Riemannian manifold whose metric explicitly depends on the distribution of eigenetrains. The calculation of the stress field of a spherical inclusion with pure dilatational eigenstrain centered at a finite spherical ball was revisited. The analysis of Yavari and Goriely [2013] was extended to radially-inhomogeneous spherical balls. It was shown that even for this more general case stress is uniform and hydrostatic inside the inclusion. A similar extension was presented for the problem of a cylindrical inclusion in a finite circular cylindrical bar made of arbitrary incompressible isotropic solids. Yavari and Goriely [2015b]'s analysis of eigentwists in a finite circular cylindrical bar was extended to orthotropic solids. Exact solutions in nonlinear elasticity and anelasticity are quite rare. There are only a handful of such solutions for simple geometries. These exact nonlinear solutions can serve as benchmark problems for both the linear solutions and for checking the accuracy of numerical simulations.

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