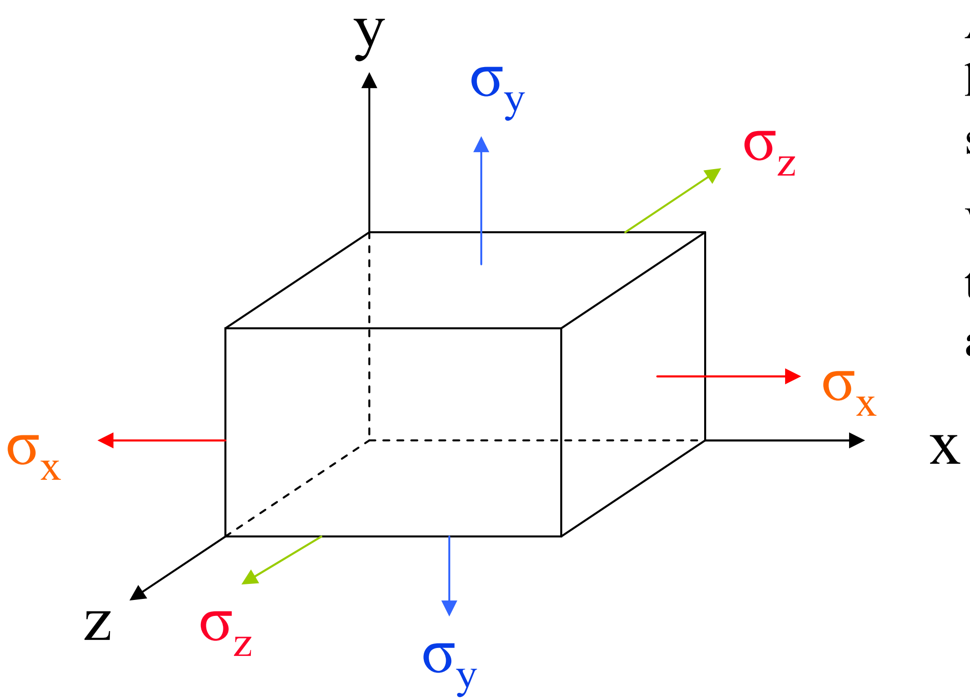
Review of Elasticity Equations

Linear, homogeneous, isotropic material behavior.

3D Isotropic Stress/Strain Law

Three-dimensional Hooke's Law: stress/strain relationships for an isotropic material



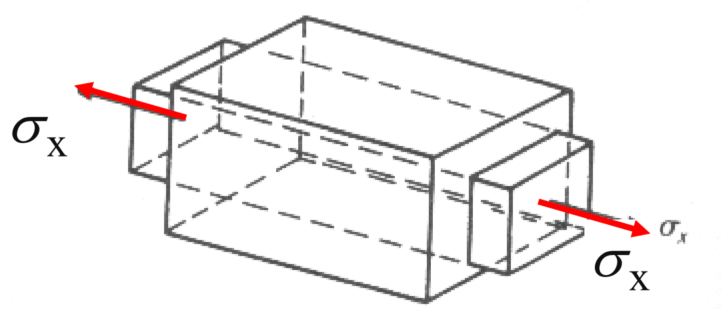
As you recall, an isotropic body can have normal stresses acting on each surface: σ_x, σ_y, σ_z

When the only normal stress is σ_x this causes a strain along the x- axis according to Hooke's Law

$$\varepsilon_x = \frac{\sigma_x}{E}$$

3D Isotropic Stress/Strain Law

Note, that a tensile stress in the x direction, produces a negative strains in the y and z directions This is called the Poisson effect.



These negative strains are computed via:

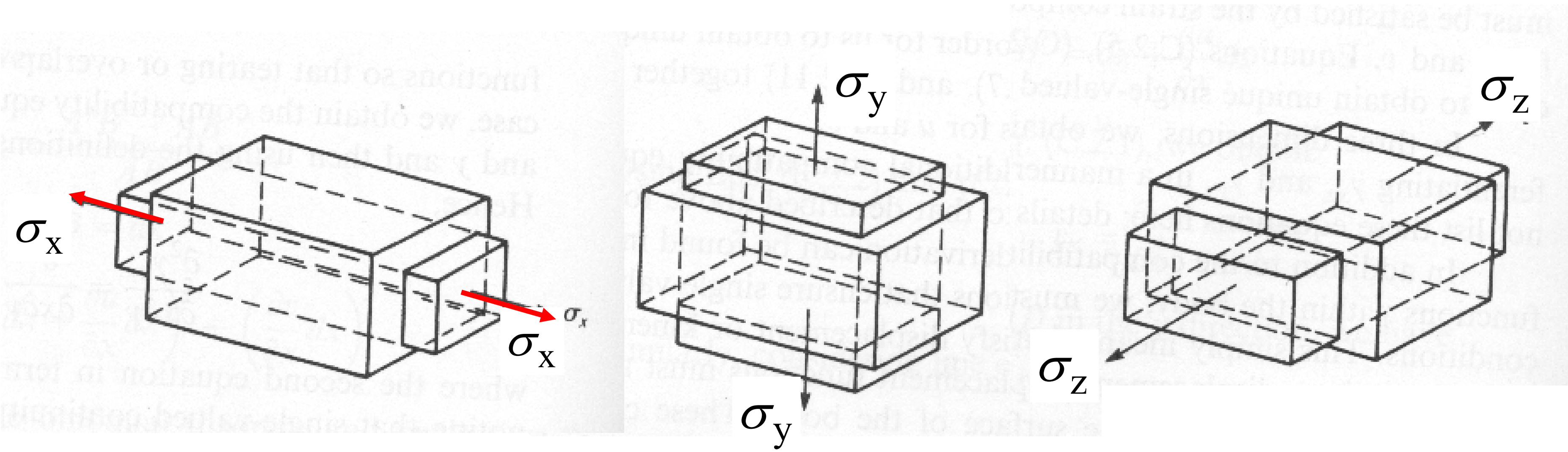
where:

$$\varepsilon_y = \varepsilon_z = -\frac{\nu \sigma_x}{E}$$

E is Young's Modulus
ν is Poisson's ratio

3D Isotropic Stress/Strain Law

Since the material is isotropic, application of normal stresses in the x, y, and z directions generates, a total normal strain in the x direction:



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

3D Isotropic Stress/Strain Law

The total normal strains in the y and z directions can be determined in a similar manner:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \nu \frac{\sigma_z}{E}$$

3D Isotropic Stress/Strain Law

Rearranging the above equations and yields 3 equations relating normal stresses and strains :

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_x(1-\nu) + \nu\varepsilon_y + \nu\varepsilon_z]$$

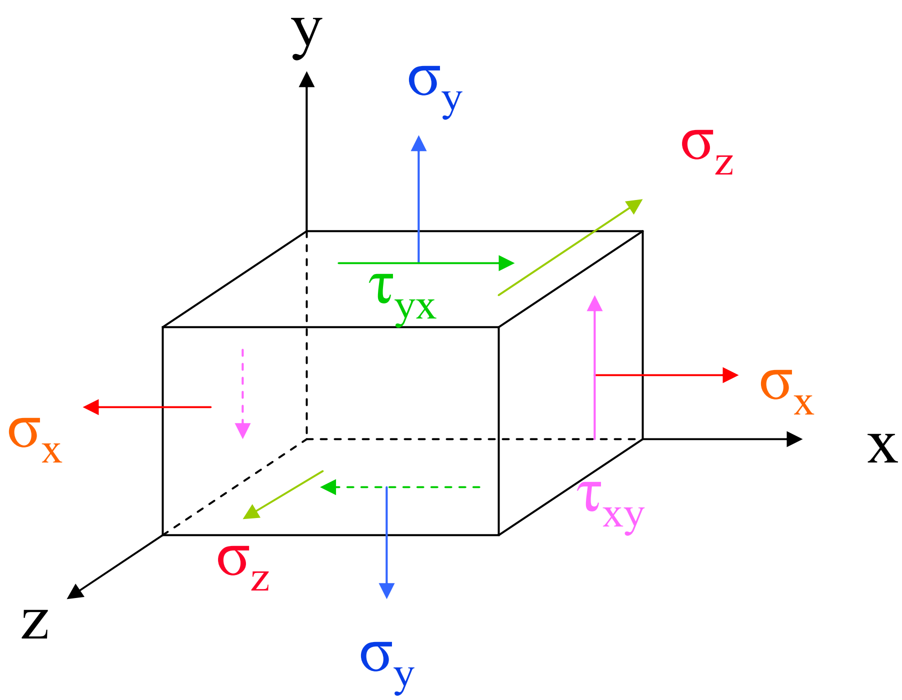
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z]$$

These equations can also be written in matrix notation: {σ}=[D]{ε}

Shear Stress/Strain Relationships

Hooke's law also applies for shear stress and strain: $\tau = G\gamma$ where G is the shear modulus, τ is a shear stress, and γ is a shear strain. For 3-D this results in a further 3 equations.



$$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{zx} = G\gamma_{zx} = 2G\varepsilon_{zx}$$

Note : $\mathbf{G = \dfrac{E}{2(1+\nu)}}$

Stress-Strain Relationships

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

3D Stress-Strain Matrix

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Note : $G=\frac{E}{2(1+\nu)}$

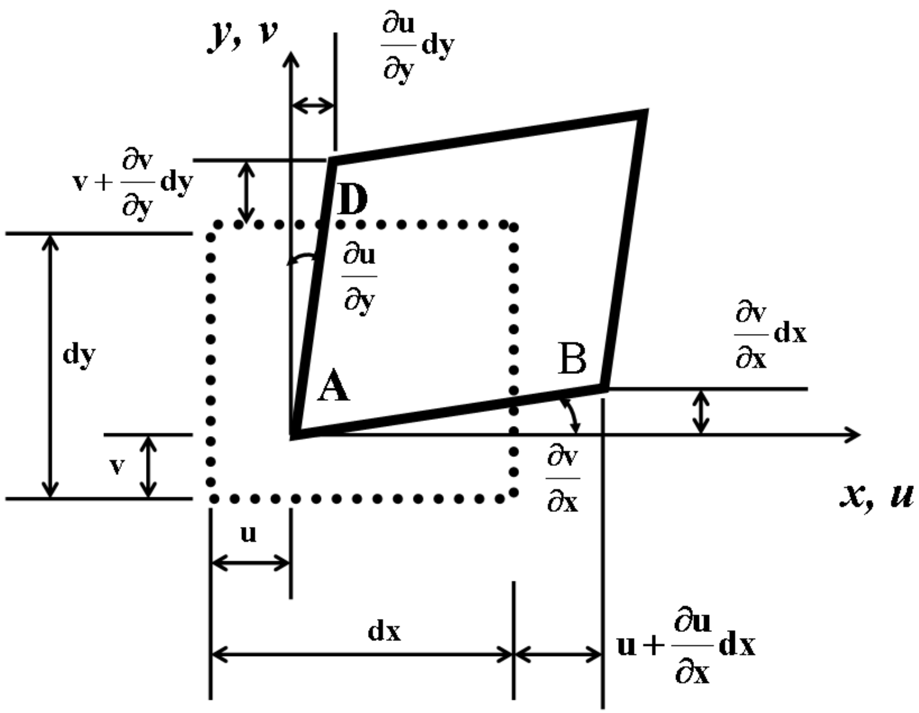
Strain-Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

(u,v,w) are the x, y and z components of displacement

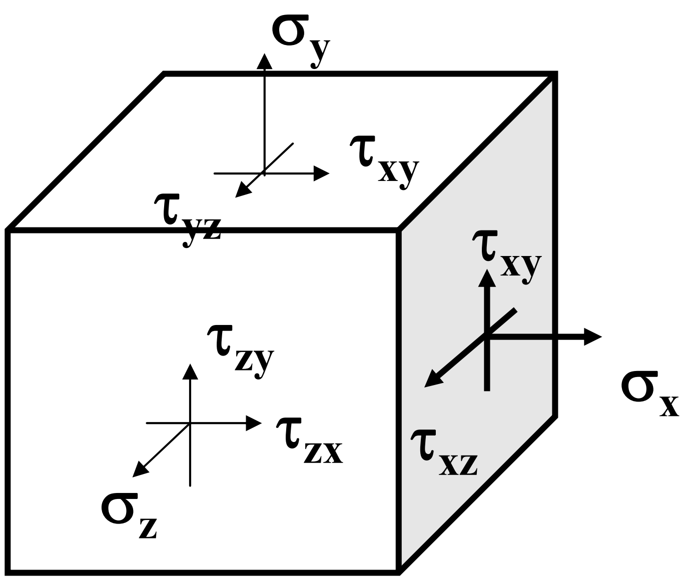


Stress Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$



Two-dimensional Elements
Plane Stress/Strain
Stiffness Equations

Two-dimensional Elements

1. Thin 2D elements .
2. Two coordinates to define position.
3. Elements connected at common nodes and/or along common edges.
4. Nodal compatibility enforced to obtain equilibrium equations
5. Two basic types
    1. Plane stress
    2. Plane Strain

Introduction to 2-D Elastic Stress Analysis

Two-dimensional stress analysis allows the engineer to determine detailed information concerning deformation, stress and strain, within a complex shaped two-dimensional elastic body.

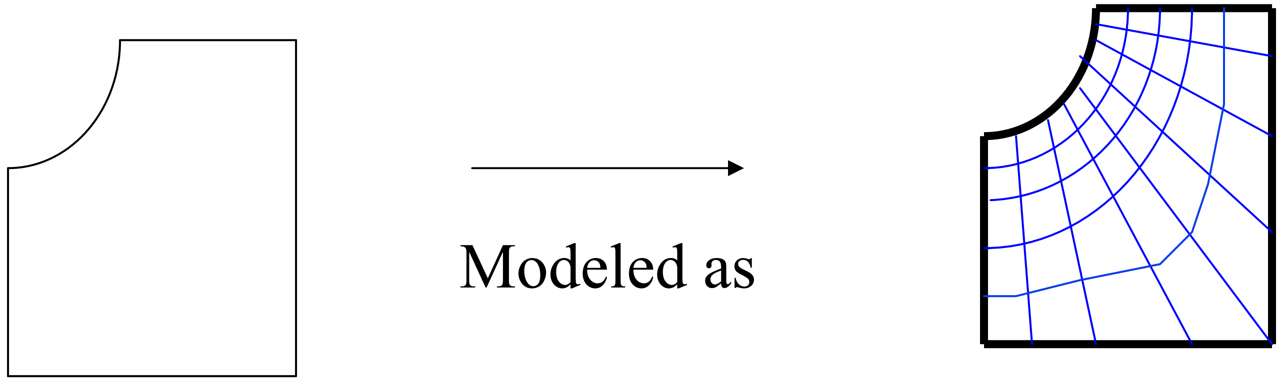
Assumptions

- Deformations and strains are very small
- Material behaves elastically – stress and strain are related by Hooke's Law.
- Hooke's Law is a matrix equation relating 3 normal stresses and one shear stress to 3 normal strains and one shear strain

$$\{\sigma\} = [D]\{\varepsilon\} \quad \text{or} \quad \{\varepsilon\} = [D]^{-1}\{\sigma\}$$

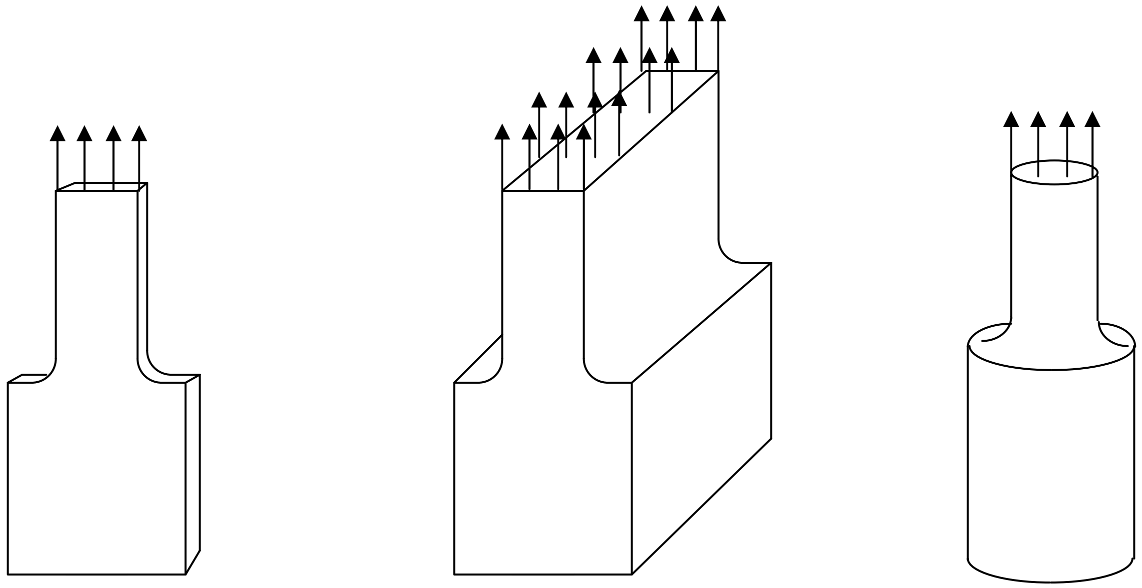
Introduction to 2-D Elastic Stress Analysis

2-D Stress analysis allows the engineer to model complex 2-D elastic bodies by discretizing the geometry with a mesh of finite elements.



Introduction to 2-D Stress Analysis

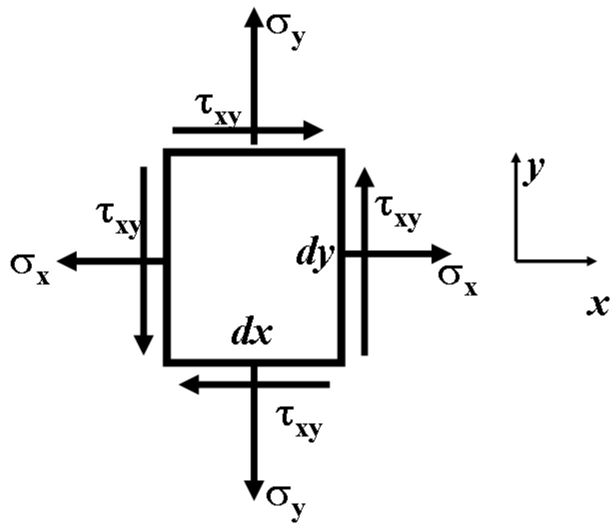
2-D planar elements are used to model complex 2-D geometries. They must connect at common nodes to form continuous structures. They are extremely important in the following analysis types:



Plane Stress Plane Strain Axisymmetric

Two-dimensional State of Stress and Strain

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$



$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

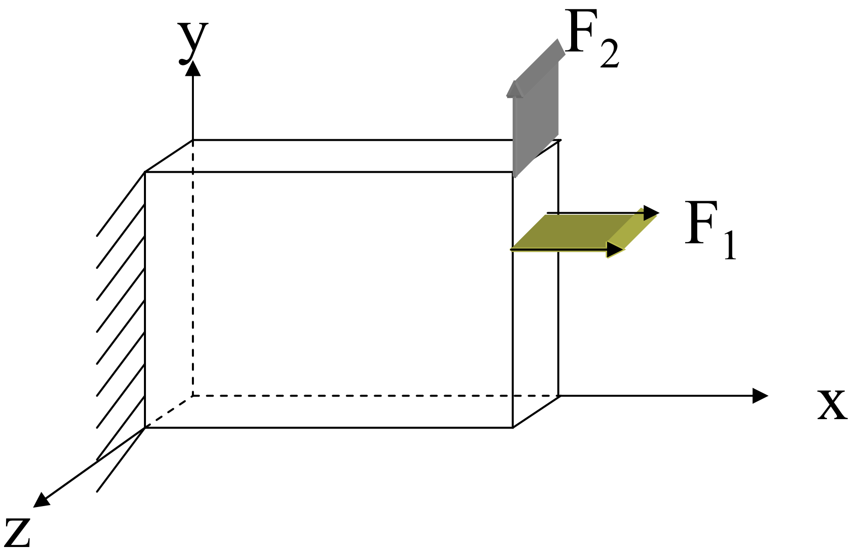
$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

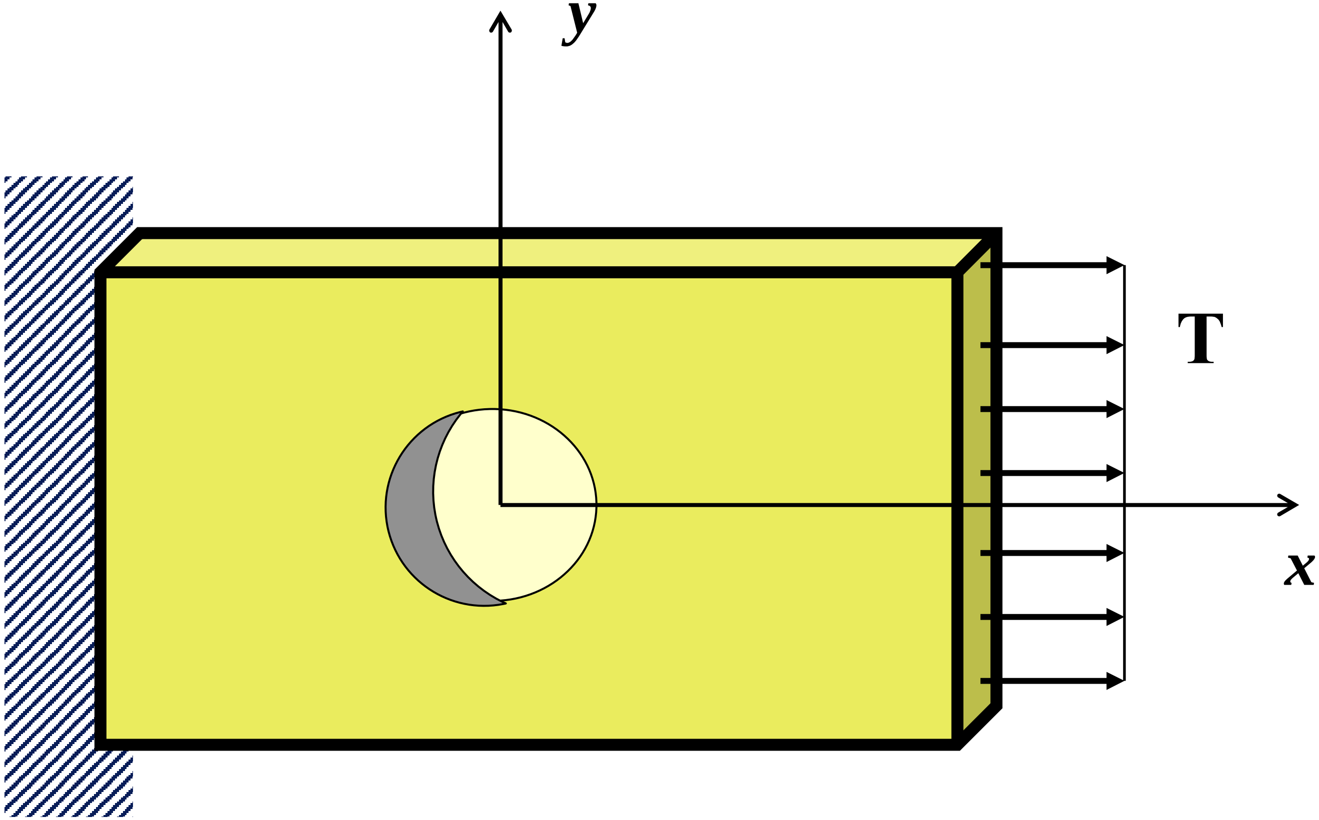
Plane Stress

Plane Stress, is defined to be a state of stress in which the normal and shear stresses perpendicular to the x-z plane are zero, the y-thickness is very small, and the constraints (Rx,Rz) and loads act only in the x-z plane and throughout the y-thickness.



- $\sigma_z = 0$
- $\tau_{xz} = 0$, $\tau_{yz} = 0$
- 'thickness', y dimension, is very small compared to x and z dimensions
- Loads act only in the x-z plane and throughout the y-thickness

Plane Stress Problem:
Plate with a Hole



Stress-Strain Relationships

For Plane-Stress: $\sigma_z = 0, \tau_{xz} = 0, \tau_{yz} = 0$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\gamma_{yz} = 0$$

$$\gamma_{zx} = 0$$

$$\varepsilon_z = \frac{-\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu-\frac{\nu^2}{1-\nu} & \nu-\frac{\nu^2}{1-\nu} & 0 \\ \nu-\frac{\nu^2}{1-\nu} & 1-\nu-\frac{\nu^2}{1-\nu} & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Stress

Stress-Strain Matrix

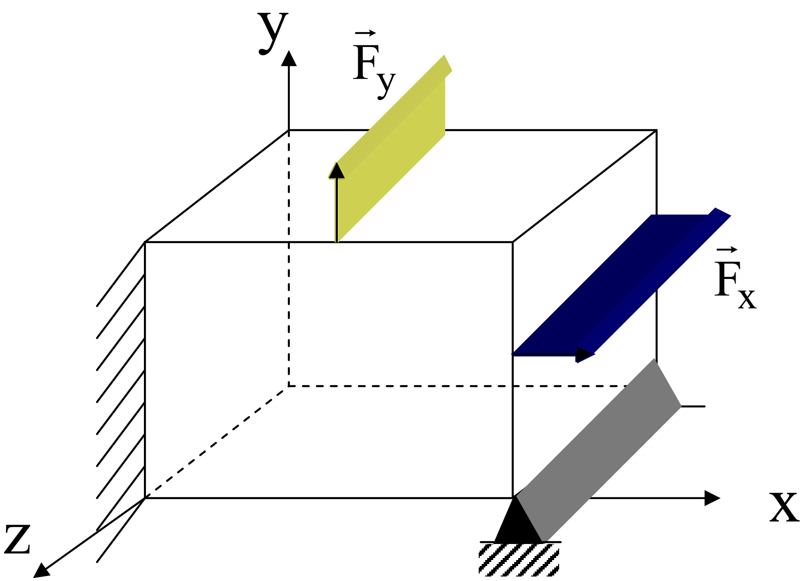
$$[D] = \frac{E}{(1-\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\} \qquad \text{or} \qquad \{\varepsilon\} = [D]^{-1}\{\sigma\}$$

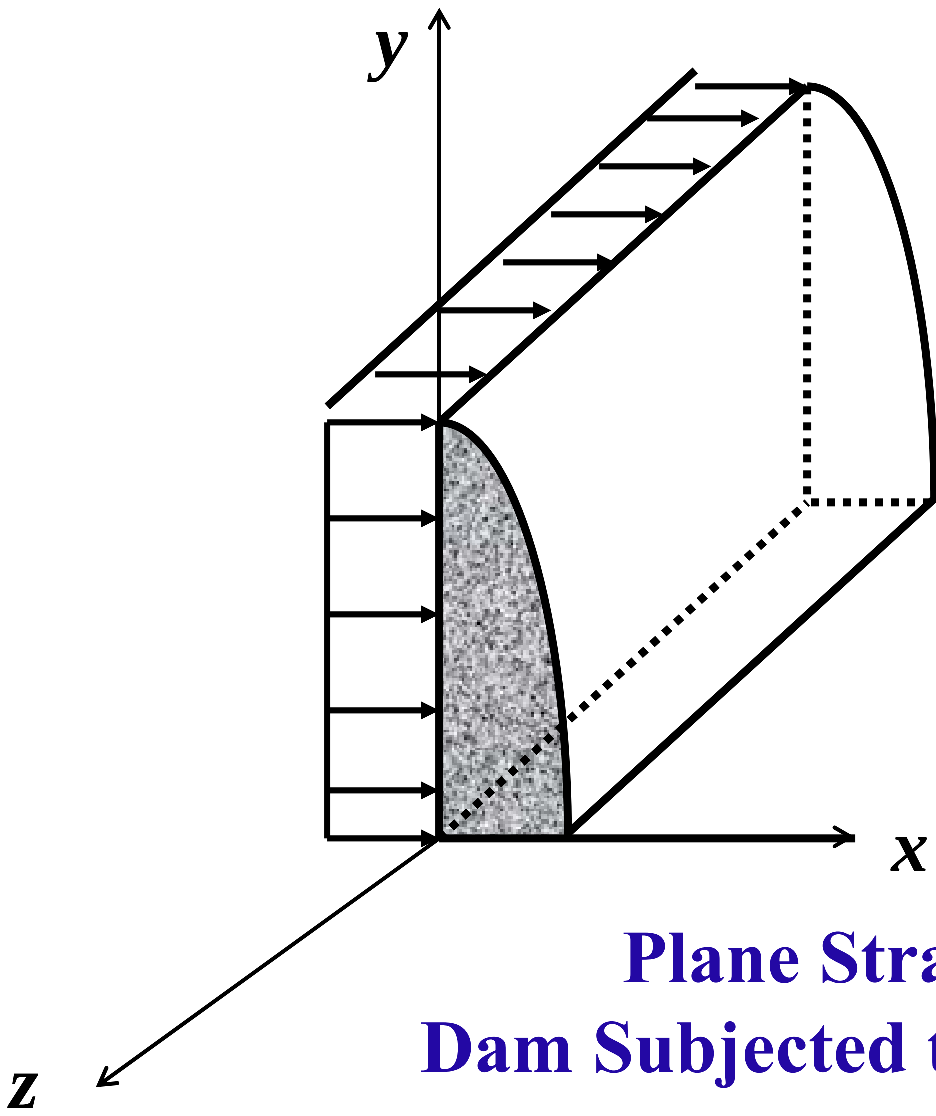
Plane Strain

Plane Strain, is defined to be a state of strain in which the normal strain in the y-direction, ε_y and the shear strains, γ_{xy} and γ_{yz} are **zero**. Note, the y-thickness of the body is very large, and constraints and loads act in x-z plane throughout thickness.

- $\varepsilon_z = 0,\ \gamma_{xz} = 0,\ \gamma_{yz} = 0$



- The 'thickness', y-dimension of the body is very large ("infinite"). Loads and constraints act only in the x-z plane through a unit y-thickness
- Forces are defined as **force per unit y-length**
- **A plane stress state, where y is a very large value, does not approximate plane strain conditions!**



**Plane Strain Problem:
Dam Subjected to Horizontal Load**

Stress-Strain Relationships

For Plane-Strain: $\varepsilon_z = 0,\ \gamma_{xz} = 0,\ \gamma_{yz} = 0$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

$$\gamma_{yz} = 0$$

$$\gamma_{zx} = 0$$

$$\varepsilon_z = 0$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Strain

Stress-StrainMatrix

$$[\mathbf{D}] = \frac{\mathbf{E}}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

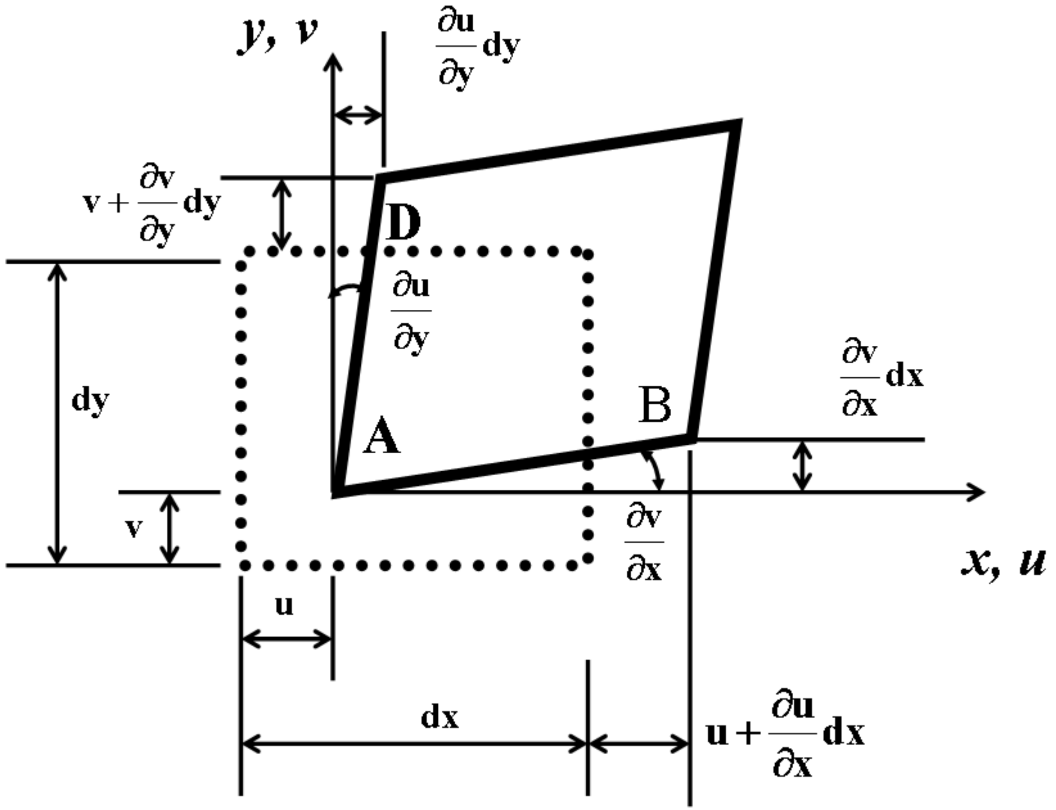
$$\{\sigma\} = [\mathrm{D}]\{\varepsilon\} \quad \text{or} \quad \{\varepsilon\} = [\mathrm{D}]^{-1}\{\sigma\}$$

Stiffness Matrix Formulations

- Stress-Strain Relationships
- Strain-Displacements Relationships
- Deformation-Displacement Relationships (Shape Functions)
- Minimum Energy Principle

Strain-Displacements Relationships

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$



Displacements and rotations of lines of an element in the x-y plane

Potential Energy Approach

$$U = \frac{1}{2} \iiint_V \{\varepsilon_x\}^T \{\sigma_x\} \, dV$$

$$U = \frac{1}{2} \iiint_V \{\varepsilon_x\}^T [D] \{\varepsilon_x\} \, dV$$

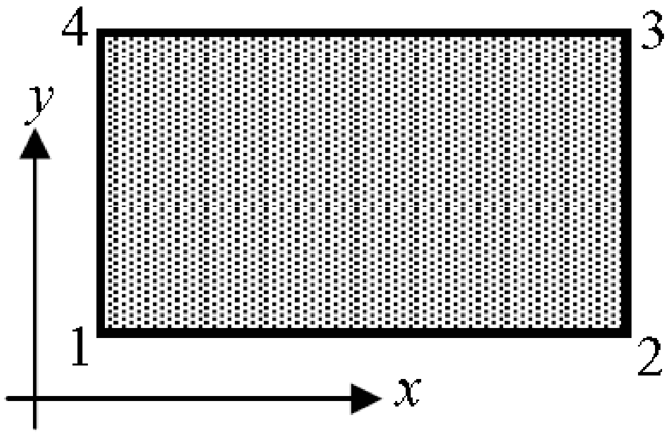
$$\iiint_V [B]^T [D] [B] \, dV \; \{d\} = \{f\}$$

$$[k] = \iiint_V [B]^T [D] [B] \, dV$$

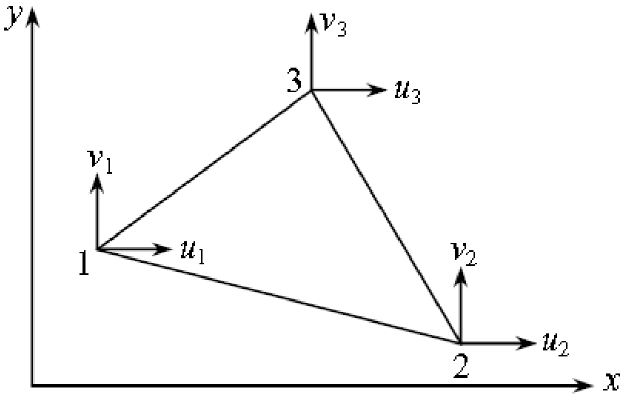
Element Type in 2D Analyses

- *Constant Strain Triangular (CST) Element (3 nodes)*
- *Linear Strain Triangular (LST) Element (3 nodes)*
- *Four Node Rectangular Element (4 nodes)*
- *Four Node Quadrilateral Element (4 nodes)*

Element Type in 2D Analyses



Four Node Rectangular Element



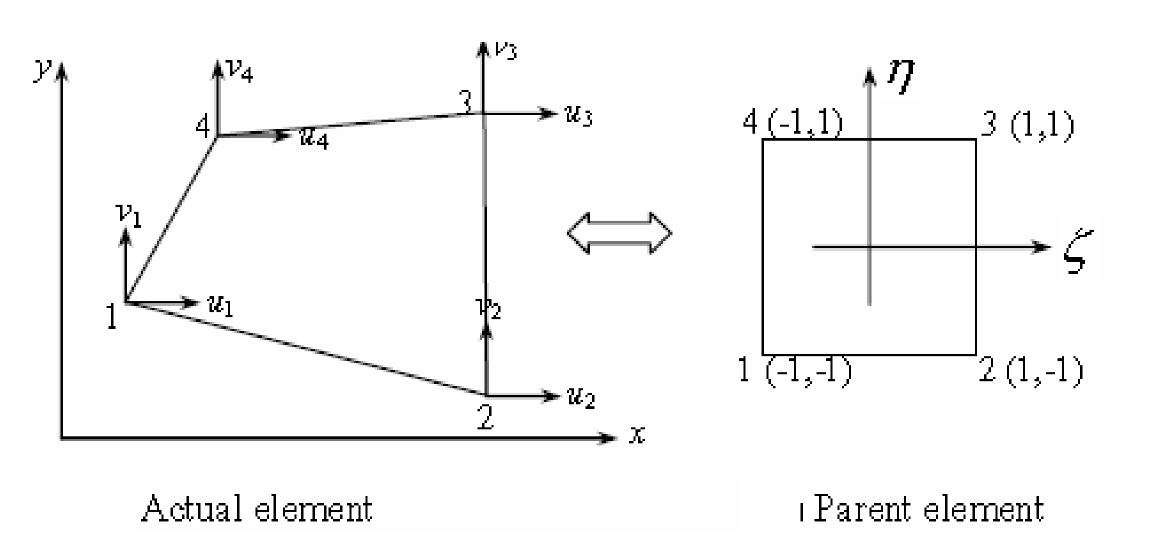
Constant Strain Triangular (CST) Element

Four Node Iso-parametric Quadrilateral Element

- - Four-node iso-parametric finite element is one of the most commonly used elements.
- - Eight unknowns: two displacements per each node.
- - Iso-parametric: the same interpolation method is used for displacement and geometry.
- - Mapping relation from physical element to reference element.
- - Numerical integration

Iso-parametric Mapping

Lagrange interpolation method (Shape functions)



$$u(x,y) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\zeta)(1-\eta)$$

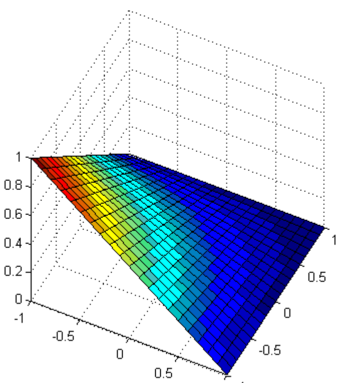
$$N_2 = \frac{1}{4}(1+\zeta)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\zeta)(1+\eta)$$

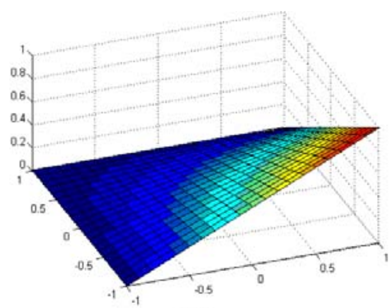
$$N_4 = \frac{1}{4}(1-\zeta)(1+\eta)$$

Linear Functions

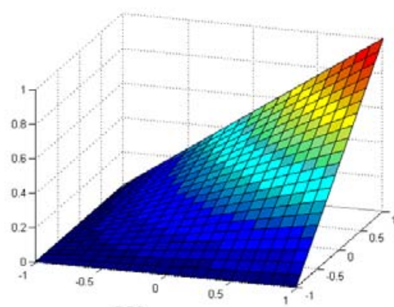
1. Ensures compatibility between elements.
2. Displacements vary linearly along any line.
3. Displacements vary linearly between nodes.
4. Edge displacements are the same for adjacent elements if nodal displacements are equal.



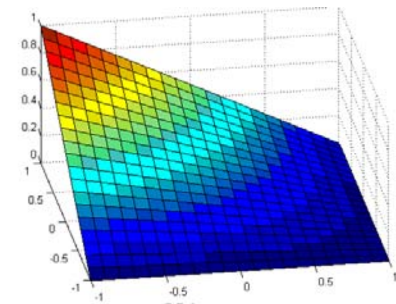
N1



N2



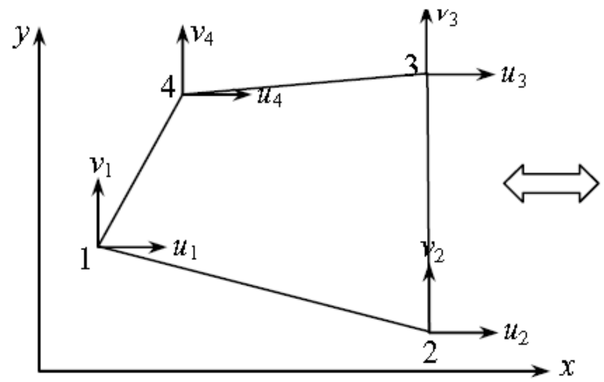
N3



N4

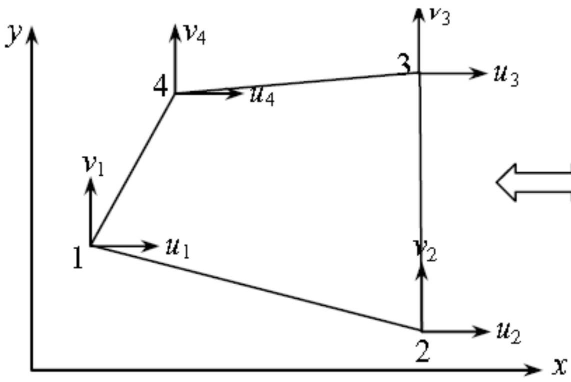
Isoparametric mapping: interpolate the geometry using shape functions

$$x = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix},$$



$$y = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

Interpolation of Displacement



$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{d}\}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Displacement-strain relationship

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \partial u / \partial x \\ \partial u / \partial y \\ \partial v / \partial x \\ \partial v / \partial y \end{Bmatrix}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 + \frac{\partial N_4}{\partial x} u_4 \qquad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 + \frac{\partial N_4}{\partial y} v_4$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_2}{\partial x} v_2 + \cdots$$

define the $\boldsymbol{B}$ matrix:

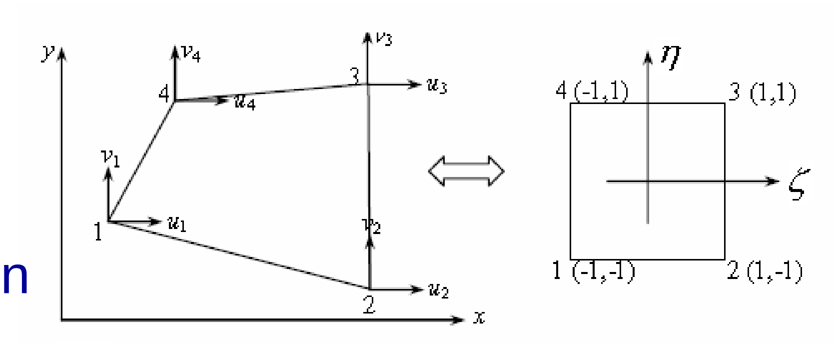
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\varepsilon = [\mathbf{B}]\{\mathbf{d}\}$$

Displacement-strain relationship

J is Jacobian matrix

Derivatives of shape functions with respect to coordinate directions are required. Since shape functions depend on ζ and η coordinates, chain rule of differentiation must be used



$$\frac{\partial N_1}{\partial \zeta}=\frac{\partial N_1}{\partial x}\frac{\partial x}{\partial \zeta}+\frac{\partial N_1}{\partial y}\frac{\partial y}{\partial \zeta}$$
$$\frac{\partial N_1}{\partial \eta}=\frac{\partial N_1}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial N_1}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial N_1}{\partial \zeta} \\ \dfrac{\partial N_1}{\partial \eta} \end{Bmatrix} = \underbrace{\boxed{\begin{bmatrix} \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}}_{\textbf{[J]}} \begin{Bmatrix} \dfrac{\partial N_1}{\partial x} \\ \dfrac{\partial N_1}{\partial y} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dfrac{\partial N_1}{\partial x} \\ \dfrac{\partial N_1}{\partial y} \end{Bmatrix} = [\boldsymbol{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_1}{\partial \zeta} \\ \dfrac{\partial N_1}{\partial \eta} \end{Bmatrix}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\frac{\partial x}{\partial \zeta}=\frac{\partial N_1}{\partial \zeta}x_1+\frac{\partial N_2}{\partial \zeta}x_2+\frac{\partial N_3}{\partial \zeta}x_3+\frac{\partial N_4}{\partial \zeta}x_4$$

$$|\mathbf{J}| = \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial \eta}$$

The Material Matrix

For elasticity, the D matrix is given by Hooke's law. For plane strain:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Similarly, for plane stress:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Finite Element Matrix Equation

$$\boldsymbol{K}^e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV = t \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \left|\boldsymbol{J}\right| d\zeta\, d\eta$$

$$\boldsymbol{K}^e = t \int_{-1}^{1}\int_{-1}^{1} \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_1}{\partial y} \\ 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \\ \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_2}{\partial y} \\ 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} \\ \cdots & \cdots & \cdots \\ \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_4}{\partial y} \\ 0 & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}}_{[\mathbf{B}]^{\mathrm{T}}} \left[\boldsymbol{D}\right] \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}}_{[\mathbf{B}]} \left|\boldsymbol{J}\right| d\zeta\, d\eta$$

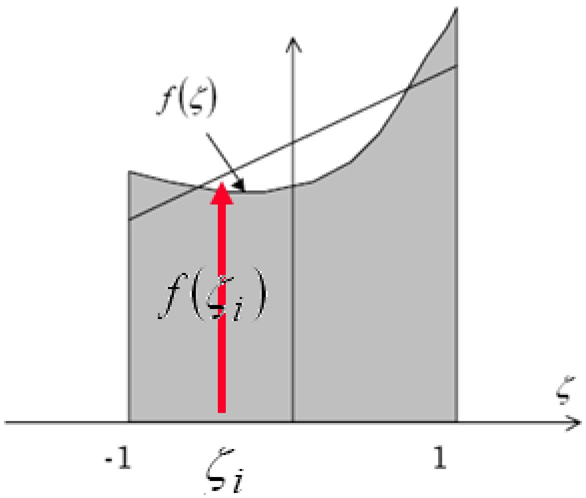
Numerical Integration

- - Numerical integration evaluates the integrals involved in the element stiffness matrix and distributed force.
- - In the finite element literature, the Gauss quadrature is usually preferred because it requires fewer function evaluations as compared to other methods.
- - In the Gauss quadrature, the integrand is evaluated at predefined points (called Gauss points). The sum of this integrand values, multiplied by appropriate weights (called Gauss weight) gives an approximation to the integral:

$$\int_{-1}^{1} f(\zeta)\,d\zeta \approx \sum_{i=1}^{N_{gp}} w_i f(\zeta_i)$$

Gauss quadrature

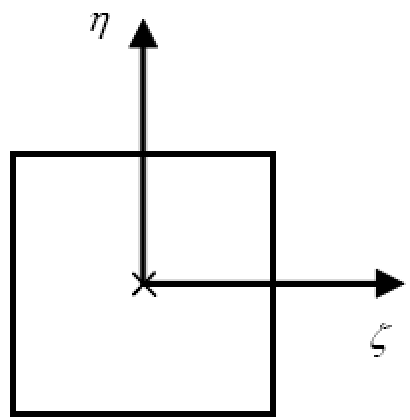
n	ζ_i	w_i
1	0.0	2.0
2	± 0.577350269189626 (or $1/\sqrt{3}$)	1.0
3	± 0.774596669241483 (or $\sqrt{0.6}$) 0.0	0.555555555555555 (or 5/9) 0.888888888888889 (or 8/9)



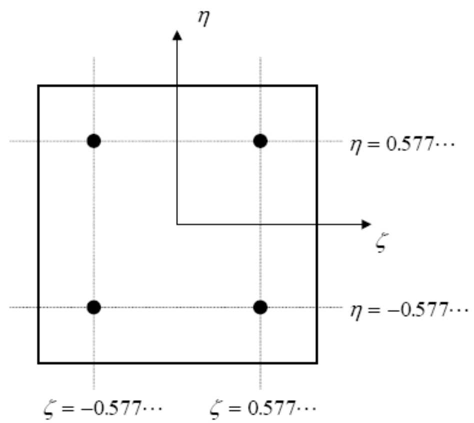
$$\int_{-1}^{1} f(\zeta)d\zeta \approx \sum_{i=1}^{N_{gp}} w_i f(\zeta_i)$$

Two-dimensional Gauss integration

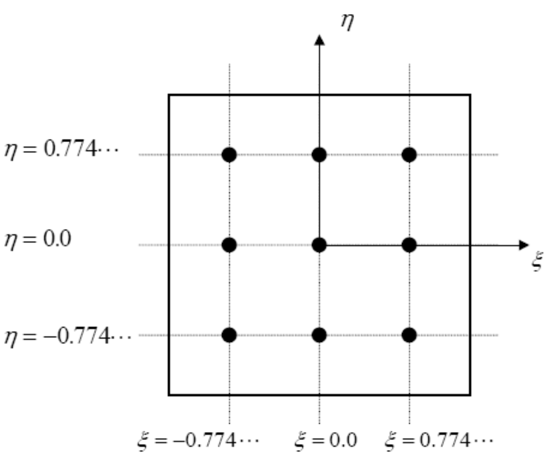
$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} f\big(g_1(\xi,\eta),g_2(\xi,\eta)\big)\left|\boldsymbol{J}\right| d\xi\, d\eta$$



1×1



Location of 2×2 Gauss points



Location of 3×3 Gauss points

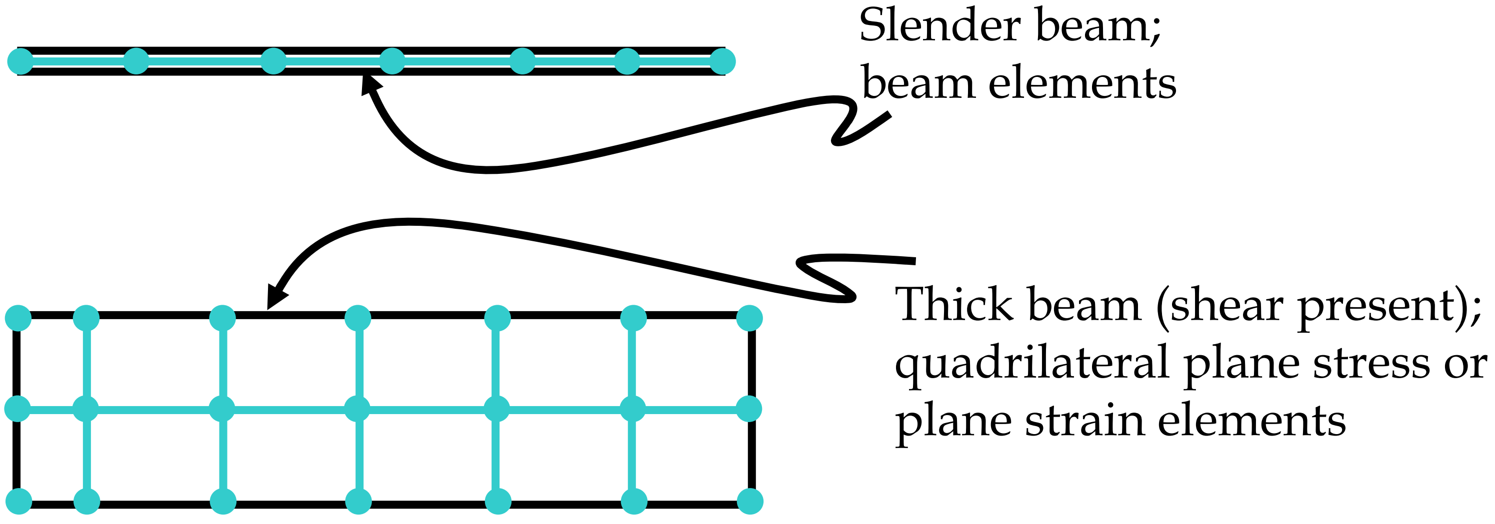
General Meshing Guidelines and Accuracy

General Considerations in Meshing

- When choosing elements and creating meshes for FEA problems users must make sure that
  - Chosen mesh size and density are optimal for the problem (to save computational time)
  - Chosen element types are appropriate for the analysis type performed (for accuracy)
  - Element shapes do not result in near singular stiffness matrices
  - Chosen elements and meshes can represent force distributions properly

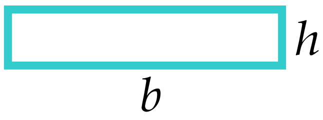
Correct Choice of Elements

- Choose element types that are appropriate for the loading and stress conditions of the problem
- Make sure that the elements chosen capture all possible significant stresses that may result from the given loading, geometry, and boundary conditions



Aspect Ratio

- For a good mesh all elements must have a low aspect ratio



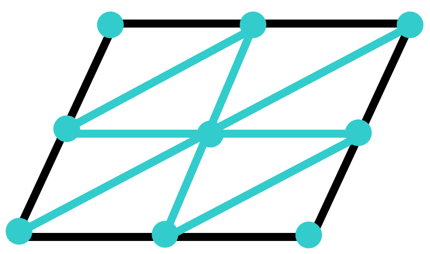
- Specifically

$$\frac{b}{h} \leq 2 - 4$$

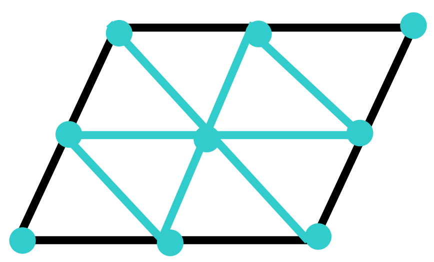
where b and h are the longest and the shortest sides of an element, respectively

Element Shape

- Angles between element sides must not approach 0° or 180°



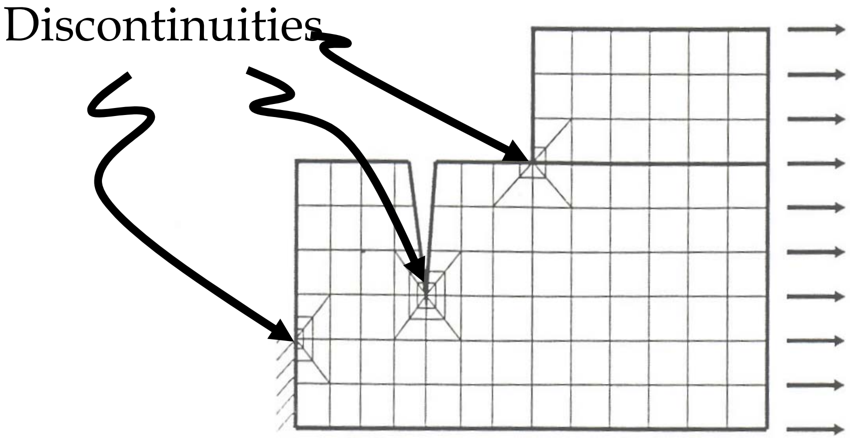
Worse



Better

Mesh Refinement

- Finer meshing must be used in regions of expected high stress gradients (usually occur at discontinuities)
- Mesh refinement must be gradual with adjacent elements of not too dissimilar size
- Mesh refinement must balance accuracy with problem size



Dissimilar Element Types

- In general different types of elements with different DOF at their nodes should not share global DOF (for example do not use a 3D beam element in conjunction with plane stress elements)

Equilibrium and Compatibility

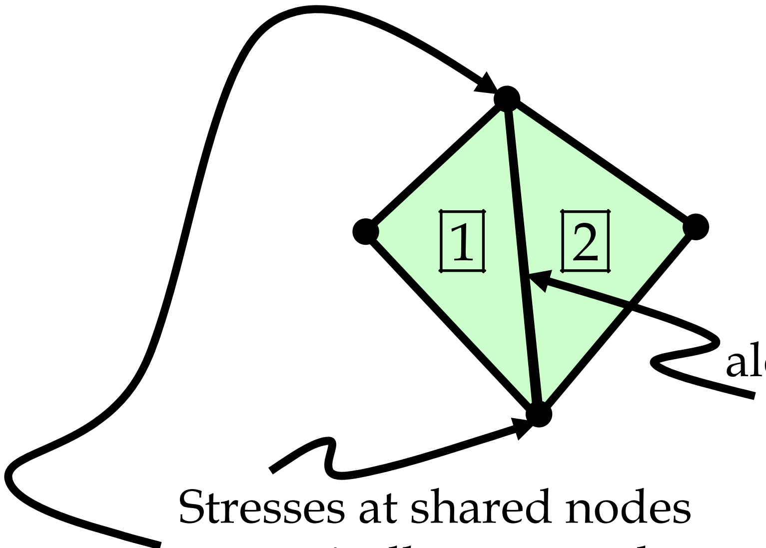
- The approximations and discretizations generated by the FE method enforce some equilibrium and compatibility conditions but not others
  - Equilibrium of nodal forces and moments is always satisfied because of

$$\mathbf{KU} = \mathbf{F}$$

  - Compatibility is guaranteed at the nodes because of the way **K** is formed; i.e. the displacements of shared nodes on two elements are the same in the global frame in which the elements are assembled

Equilibrium-Compatibility (cont'd)

- Equilibrium is usually not satisfied across inter-element boundaries; however discrepancies decline with mesh refinement



$$\sigma_x^{(1)} \neq \sigma_x^{(2)}$$

$$\sigma_y^{(1)} \neq \sigma_y^{(2)}$$

$$\tau_{xy}^{(1)} \neq \tau_{xy}^{(2)}$$

along this boundary

Stresses at shared nodes are typically averaged over the elements sharing the node

Principal Stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$