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Radial Vibrations in Short, Hollow Cylinders of Barium Titanate

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The theory of radial vibrations in short, hollow cylinders of barium titanate has been developed. This development results in an expression which relates the radial coupling coefficient to the measurement of the resonant and antiresonant frequencies of the first harmonic of the radial mode of vibration. The important parameter in this development was found to be the ratio of outside diameter to inside diameter, and once this ratio was given, the expression relating coupling coefficient to resonance measurement took a particularly simple form. An interesting result of theory was that for any value of the ratio of diameters, many of the higher harmonics are forbidden in this mode of vibration.

Experiments were devised to test some of the theoretical conclusions and were found to be in excellent agreement with the theory.

INTRODUCTION

FREQUENTLY in the use of piezoelectric and electrostrictive materials, one is concerned with a method of measuring the coupling coefficient of the element for some mode of vibration. Normally the shape of the material is that of long rods or thin plates or disks. The coupling coefficient of a crystal for shapes such as these can accurately be determined by measurement of resonant and antiresonant frequencies of the first harmonic. The mathematics for the necessary calculations for these shapes has been previously published.

Occasionally, however, a use arises for a peculiar shape and, with it, a need for a method of determining the coupling coefficient of materials of this shape. This is particularly true since the advent of electrostrictive ceramics such as barium titanate. Recently the need has arisen for a method of determining the coupling coefficient of a hollow cylinder of electrostrictive material, whose length is small compared to its outside diameter. This report concerns itself with the mathematics which allows calculation of coupling coefficients for such shapes for the radial mode of vibration. Electrostrictive equations will be used rather than piezoelectric since anyone working with such a shape will probably be working with one of the electrostrictive ceramics. However, it can easily be shown that the results of the electrostrictive case will carry over to the piezoelectric case.

RADIAL VIBRATIONS

The configuration with which we will be concerned is shown in Fig. 1. The thickness l is small compared with the outside radius a. There will be no restriction on the inside radius b. Radial vibrations in a solid disk which has been treated by Mason[1] will become a limiting case of the present treatment.

For radial vibrations, it is best to transform the usual electrostrictive equations into cylindrical coordinates.[1] They then take the following form[2]:

$$S_{rr}=s^D{}_{1111}T_{rr}+s^D{}_{1122}(T_{\theta\theta}+T_{zz})+[Q_{1111}\delta_r{}^2+Q_{1122}(\delta_\theta{}^2+\delta_z{}^2)]$$

$$S_{\theta\theta}=s^D{}_{1122}(T_{rr}+T_{zz})+s^D{}_{1111}T_{\theta\theta}+[Q_{1122}(\delta_r{}^2+\delta_z{}^2)+Q_{1111}\delta_\theta{}^2]$$

$$S_{zz}=s^D{}_{1111}T_{zz}+s^D{}_{1122}(T_{rr}+T_{\theta\theta})+[Q_{1111}\delta_z{}^2+Q_{1122}(\delta_r{}^2+\delta_\theta{}^2)]$$

$$S_{rz}=(s^D{}_{1111}-s^D{}_{1122})T_{rz}+(Q_{1111}-Q_{1122})\delta_r\delta_z$$

$$S_{r\theta}=(s^D{}_{1111}-s^D{}_{1122})T_{r\theta}+(Q_{1111}-Q_{1122})\delta_r\delta_\theta$$

$$S_{\theta z}=(s^D{}_{1111}-s^D{}_{1122})T_{\theta z}+(Q_{1111}-Q_{1122})\delta_\theta\delta_z$$

$$E_r=4\pi\beta_{11}{}^T\delta_r-2\{Q_{1111}[\delta_rT_{rr}+\delta_\theta T_{r\theta}+\delta_zT_{rz}]+Q_{1122}[\delta_r(T_{\theta\theta}+T_{zz})-(\delta_\theta T_{r\theta}+\delta_zT_{rz})]\}$$

$$E_\theta=4\pi\beta_{11}{}^T\delta_\theta-2\{Q_{1111}[\delta_\theta T_{\theta\theta}+\delta_rT_{r\theta}+\delta_zT_{\theta z}]+Q_{1122}[\delta_\theta(T_{rr}+T_{zz})-(\delta_rT_{r\theta}+\delta_zT_{\theta z})]\}$$

$$E_z=4\pi\beta_{11}{}^T\delta_z-2\{Q_{1111}[\delta_zT_{zz}+\delta_rT_{rz}+\delta_\theta T_{\theta z}]+Q_{1122}[\delta_z(T_{rr}+T_{\theta\theta})-(\delta_rT_{rz}+\delta_\theta T_{\theta z})]\}.$$

In these equations E_r, E_θ, and E_z are the component of the electric field in the r, θ, and z directions, δ_r, δ_θ, and δ_z are the components of the electric displacement divided by 4π, S_{ij} and T_{ij} are the ijth components of the strain tensor and stress tensor, respectively, $s_{ijkl}{}^D$ are the elastic compliance constants measured at constant electric displacement, $\beta_{mn}{}^T$ the dielectric impermeability constants (inverse of dielectric constants) measured at constant stress, and Q_{ijno} are the electrostrictive constants.

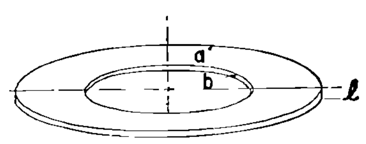

FIG. 1. Hollow cylinder whose length is short compared to outside diameter.

[1] W. P. Mason, *Piezoelectric Crystals and Their Application to Ultrasonics* (D. Van Nostrand Company, Inc., New York, 1950).

[2] These equations differ from those appearing in the first edition of Mason's book in that a correction term to the impermeability constant has been dropped after verbal communications with Mason.

In solving the equations of motion, it is also necessary to know the strains in terms of the mechanical displacements in the r, θ, and z directions. Denoting these displacements by u_r, u_θ, and u_z the strains are:

$$S_{rr}=\frac{\partial u_r}{\partial r}$$

$$S_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}$$

$$S_{zz}=\frac{\partial u_z}{\partial z}$$

$$S_{r\theta}=\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$S_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}$$

$$S_{\theta z}=\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}.$$

We will assume that the thickness is so small that the change of stress in the z direction is negligible. Since the stresses are zero at the surface, we can set

$$T_{zz}=T_{rz}=T_{\theta z}=0.$$

Furthermore, since we shall consider only motion that is entirely radial, $T_{r\theta}=0$ and also $u_\theta=u_z=0$. We will consider the case in which the field is applied only in the z direction so that $\delta_r=\delta_\theta=0$. The electrostrictive equations now become:

$$S_{rr}=s^D{}_{1111}T_{rr}+s^D{}_{1122}T_{\theta\theta}+Q_{1122}\delta_z{}^2$$

$$S_{\theta\theta}=s^D{}_{1122}T_{rr}+s^D{}_{1111}T_{\theta\theta}+Q_{1122}\delta_z{}^2$$

$$E_z=4\pi\beta_{11}{}^T\delta_z-2Q_{1122}\delta_z(T_{rr}+T_{\theta\theta}).$$

In the case of electrostrictive ceramics the electric displacement may be represented by $\delta_z=\delta_{z0}+\delta_z e^{j\omega t}$ where δ_{z0} is the remanent electric displacement caused by polarization and δ_z the alternating component. Solving the above equations simultaneously, the alternating component of the stress and displacement are given by the equations

$$T_{rr}=\left(\frac{Y_0{}^E}{1-\sigma^2}\right)(S_{rr}+\sigma S_{\theta\theta})-\frac{2Q_{1122}\delta_{z0}Y_0{}^E E_z}{4\pi\beta_{11}{}^T(1-\sigma)}$$

$$T_{\theta\theta}=\left(\frac{Y_0{}^E}{1-\sigma^2}\right)(S_{\theta\theta}+\sigma S_{rr})-\frac{2Q_{1122}\delta_{z0}Y_0{}^E E_z}{4\pi\beta_{11}{}^T(1-\sigma)}$$

$$\delta_z=\frac{E_z}{4\pi\beta_{11}{}^T}+\frac{2Q_{1122}\delta_{z0}}{4\pi\beta_{11}{}^T}(T_{rr}+T_{\theta\theta})$$

where

$-s^E{}_{1122}/s^E{}_{1111}=\sigma$ is the Poisson's ratio

and

$1/s^E{}_{1111}=Y_0{}^E$ is Young's modulus in which

$$s^E{}_{1122}=\frac{s^D{}_{1122}}{1-\dfrac{s^D{}_{1111}}{s^D{}_{1122}}\dfrac{Q^2{}_{1122}\delta^2{}_{z0}}{\pi\beta^T{}_{11}s^E{}_{1111}}}$$

and

$$s^E{}_{1111}=\frac{s^D{}_{1111}}{1-\dfrac{Q^2{}_{1122}\delta^2{}_{z0}}{\pi\beta^T{}_{11}s^E{}_{1111}}}.$$

The only remaining equation which is needed is the force equation which becomes for the described conditions

$$\rho\ddot{u}_r=\frac{\partial T_{rr}}{\partial r}+\frac{(T_{rr}-T_{\theta\theta})}{r}.$$

Since now

$$S_{rr}=\frac{\partial u_r}{\partial r}\quad\text{and}\quad S_{\theta\theta}=\frac{u_r}{r}$$

the equation of motion becomes

$$\frac{Y_0{}^E}{1-\sigma^2}\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]=\rho\frac{\partial^2 u_r}{\partial t^2}=-\omega^2\rho u_r$$

for simple harmonic motion.

This is a Bessel's equation of the first order which has the solution

$$u_r=\alpha J_1\left(\frac{\omega r}{v}\right)+\beta K_1\left(\frac{\omega r}{v}\right);\quad v^2=\frac{Y_0{}^E}{(1-\sigma^2)\rho},$$

where $J_1(\omega r/v)$ and $K_1(\omega r/v)$ are Bessel functions of the first and second kind. The boundary conditions are that the stress $T_{rr}=0$ when $r=a$ and when $r=b$, a and b being the outside and inside radii.

$$T_{rr}=\frac{Y_0{}^E}{1-\sigma^2}\left\{\alpha\left[\frac{\omega}{v}J_0\left(\frac{\omega r}{v}\right)-\frac{(1-\sigma)}{r}J_1\left(\frac{\omega r}{v}\right)\right]+\beta\left[\frac{\omega}{v}K_0\left(\frac{\omega r}{v}\right)-\frac{(1-\sigma)}{r}K_1\left(\frac{\omega r}{v}\right)\right]\right\}-\frac{Q_{1122}\delta_{z0}Y_0{}^E E_z}{2\pi\beta_{11}{}^T(1-\sigma)}.$$

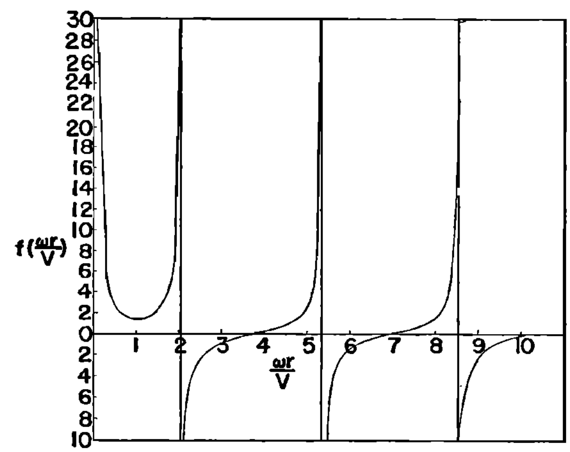

FIG. 2. Plot of function $f(\omega r/v)$ which occurs in resonance condition.

Inserting the boundary conditions and solving for α and β we get

$$\alpha=\frac{Q_{1122}\delta_{z0}E_z(1+\sigma)}{2\pi\beta_{11}{}^T}\times\left\{\frac{\left[\frac{\omega}{v}K_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}K_1\left(\frac{\omega a}{v}\right)\right]-\left[\frac{\omega}{v}K_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}K_1\left(\frac{\omega b}{v}\right)\right]}{\left[\frac{\omega}{v}K_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}J_1\left(\frac{\omega b}{v}\right)\right]-\left[\frac{\omega}{v}K_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}J_1\left(\frac{\omega a}{v}\right)\right]}\right\}$$

$$\beta=\frac{Q_{1122}\delta_{z0}E_z(1+\sigma)}{2\pi\beta_{11}{}^T}\times\left\{\frac{\left[\frac{\omega}{v}J_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}J_1\left(\frac{\omega b}{v}\right)\right]-\left[\frac{\omega}{v}J_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}J_1\left(\frac{\omega a}{v}\right)\right]}{\left[\frac{\omega}{v}K_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}J_1\left(\frac{\omega b}{v}\right)\right]-\left[\frac{\omega}{v}K_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}J_1\left(\frac{\omega a}{v}\right)\right]}\right\}.$$

This gives

$$T_{rr}=\left(\frac{Y_0{}^E}{1-\sigma^2}\right)\left\{\alpha\left[\frac{\omega}{v}J_0\left(\frac{\omega r}{v}\right)-\frac{(1-\sigma)}{r}J_1\left(\frac{\omega r}{v}\right)\right]+\beta\left[\frac{\omega}{v}K_0\left(\frac{\omega r}{v}\right)-\frac{(1-\sigma)}{r}K_1\left(\frac{\omega r}{v}\right)\right]\right\}-\frac{Q_{1122}\delta_{z0}Y_0{}^EE_z}{2\pi\beta_{11}{}^T(1-\sigma)}$$

$$T_{\theta\theta}=\left(\frac{Y_0{}^E}{1-\sigma^2}\right)\left\{\alpha\left[\frac{\sigma\omega}{v}J_0\left(\frac{\omega r}{v}\right)+\frac{(1-\sigma)}{r}J_1\left(\frac{\omega r}{v}\right)\right]+\beta\left[\frac{\sigma\omega}{v}K_0\left(\frac{\omega r}{v}\right)+\frac{(1-\sigma)}{r}K_1\left(\frac{\omega r}{v}\right)\right]\right\}-\frac{Q_{1122}\delta_{z0}Y_0{}^EE_z}{2\pi\beta_{11}{}^T(1-\sigma)}.$$

Substituting these values we find that

$$\delta_z=\frac{E_z}{4\pi\beta^T{}_{11}}-\frac{2Q^2{}_{1122}\delta^2{}_{z0}Y^E{}_0E_z}{\pi\beta^T{}_{11}(1-\sigma)}+\frac{Q_{1122}\delta_{z0}Y^E{}_0}{2\pi\beta^T{}_{11}(1-\sigma)}\left[\alpha\frac{\omega}{v}J_0\left(\frac{\omega r}{v}\right)+\beta\frac{\omega}{v}K_0\left(\frac{\omega r}{v}\right)\right].$$

The next step is to obtain an expression for the electrical admittance. The admittance is equal to the current into the element divided by the voltage across it. But for simple harmonic motion the current is $i=dQ/dt=j\omega Q$ where Q is the surface charge. This gives for admittance $1/Z=i/E_zl=i\omega Q/E_zl$. We need now to find an expression for the surface charge Q.

Since the value of δ_z at the surface is equal to the surface charge density we can find Q by performing the integration

$$Q=\int_0^{2\pi}d\theta\int_b^a\delta_zrdr.$$

Evaluating this integral and making the substitution

$$\frac{1}{4\pi\beta^T{}_{11}}\left[1-\frac{2Q^2{}_{1122}\delta^2{}_{z0}Y^E{}_0}{\pi\beta^T{}_{11}(1-\sigma)}\right]=\frac{1}{4\pi\beta^{RC}{}_{11}}$$

where $\beta_{11}{}^{RC}$ is the radially clamped impermeability constant, we have

$$Q=\frac{E_z(a^2-b^2)}{4\beta^{RC}{}_{11}}+\frac{Q_{1122}\delta_{z0}Y^E{}_0}{\beta^T{}_{11}(1-\sigma)}\left\{\alpha\left[aJ_1\left(\frac{\omega a}{v}\right)-bJ_1\left(\frac{\omega b}{v}\right)\right]+\beta\left[aK_1\left(\frac{\omega a}{v}\right)-bK_1\left(\frac{\omega b}{v}\right)\right]\right\}.$$

The radial coupling coefficient can be expressed as

$$k^2=\frac{2Q^2{}_{1122}\delta_{z0}Y^E{}_0}{\pi\beta^T{}_{11}(1-\sigma)}.$$

Using this and the two expressions for the constants α and β we arrive at the formidable expression

$$\frac{1}{Z}=\frac{j\omega(a^2-b^2)}{4\beta_{11}{}^{RC}l}\left\{1+\frac{k^2}{1-k^2}\frac{(1+\sigma)}{a^2-b^2}\right.$$

$$\left.\times\frac{\left\{\begin{array}{l}\left[aJ_1\left(\frac{\omega a}{v}\right)-bJ_1\left(\frac{\omega b}{v}\right)\right]\left\{\left[\frac{\omega}{v}K_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}K_1\left(\frac{\omega a}{v}\right)\right]-\left[\frac{\omega}{v}K_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}K_1\left(\frac{\omega b}{v}\right)\right]\right\}\\ \quad+\left[aK_1\left(\frac{\omega a}{v}\right)-bK_1\left(\frac{\omega b}{v}\right)\right]\left\{\left[\frac{\omega}{v}J_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}J_1\left(\frac{\omega b}{v}\right)\right]-\left[\frac{\omega}{v}J_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}J_1\left(\frac{\omega a}{v}\right)\right]\right\}\end{array}\right\}}{\begin{array}{l}\left[\frac{\omega}{v}K_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}J_1\left(\frac{\omega b}{v}\right)\right]\\ \quad-\left[\frac{\omega}{v}K_0\left(\frac{\omega b}{v}\right)-\frac{(1-\sigma)}{b}K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v}J_0\left(\frac{\omega a}{v}\right)-\frac{(1-\sigma)}{a}J_1\left(\frac{\omega a}{v}\right)\right]\end{array}}\right\}.$$

The resonant frequency occurs when the admittance becomes infinite or when

$$\frac{\frac{\omega a}{v}K_0\left(\frac{\omega a}{v}\right)-(1-\sigma)K_1\left(\frac{\omega a}{v}\right)}{\frac{\omega a}{v}J_0\left(\frac{\omega a}{v}\right)-(1-\sigma)J_1\left(\frac{\omega a}{v}\right)}=\frac{\frac{\omega b}{v}K_0\left(\frac{\omega b}{v}\right)-(1-\sigma)K_1\left(\frac{\omega b}{v}\right)}{\frac{\omega b}{v}J_0\left(\frac{\omega b}{v}\right)-(1-\sigma)J_1\left(\frac{\omega b}{v}\right)}.$$

This means that the function

$$f\left(\frac{\omega r}{v}\right)=\frac{\frac{\omega r}{v}K_0\left(\frac{\omega r}{v}\right)-(1-\sigma)K_1\left(\frac{\omega r}{v}\right)}{\frac{\omega r}{v}J_0\left(\frac{\omega r}{v}\right)-(1-\sigma)J_1\left(\frac{\omega r}{v}\right)}$$

must have two roots for any possible value of the function. One root corresponds to $A=\omega_r a/v$ where ω_r is the resonant frequency, and the other is $B=\omega_r b/v$. A plot of this function is given in Fig. 2. In this plot, the Poisson ratio of barium titanate $\sigma=0.30$ is assumed. The first U-shaped part of the curve corresponds to the first harmonic. It is noticed that for every value of $f(\omega r/v)$ there are two values of $\omega r/v$ which satisfy this value of $f(\omega r/v)$. It is also noticed that for any ratio of the crystal radii $a/b=A/B$ there is one and only one value of the function which will have the two roots A and B. This means that if this portion of the curve is plotted carefully, one can find the two roots A and B which correspond to each value of $f(\omega r/v)$ and these values of A and B may be plotted against the ratio of the radii a/b. Figures 3 and 4 show such plot. Thus given the values of a and b the resonant frequency is uniquely determined.

The other part of the curve in Fig. 2 corresponds to higher harmonics. It is particularly interesting to note

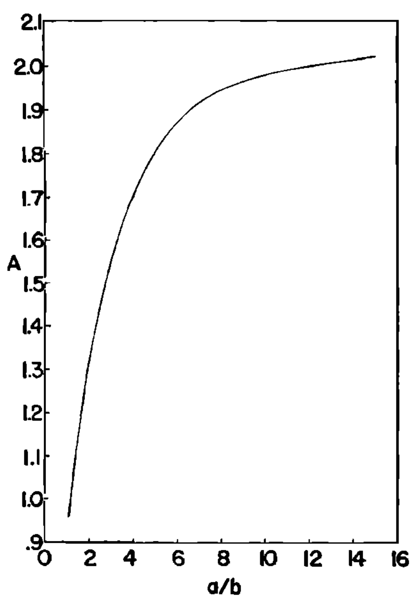

FIG. 3. Plot of $A=\omega_r a/v$ as a function of the ratio a/b.

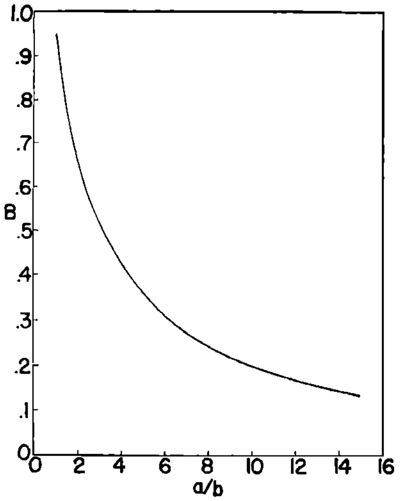

FIG. 4. Plot of $B=\omega_r b/v$ as a function of the ratio a/b.

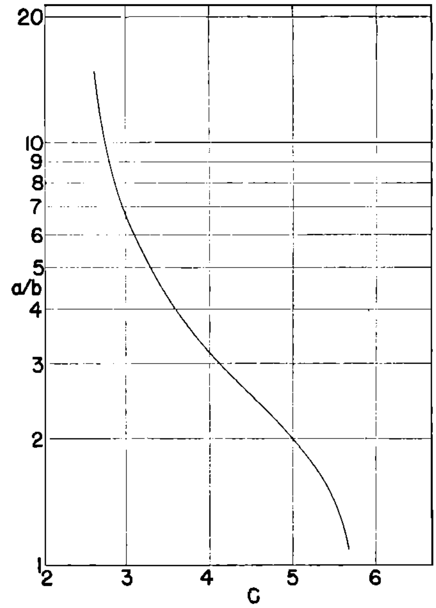

FIG. 5. Plot of the constant C which occurs in the equation $k^2/(1-k^2)=C(\Delta f/f_r)$ as a function of the ratio a/b

that whereas there is a first harmonic resonance for any ratio of a/b, for any higher harmonic there are only certain ratios of a to b which will allow this harmonic to exist. Thus for any crystal of this shape, many of the higher harmonics are forbidden. This of course does not apply in the limiting case of $b=0$ in which case all higher harmonics are permissible.

There remains the problem of determining what happens at antiresonance. This occurs when the expression in brackets in the admittance equation reduces to zero. The frequency separation between resonance and antiresonance can be obtained by developing the Bessel functions in Taylor series about the roots A and B. This gives

$$J_0\left(\frac{\omega a}{v}\right)=J_0(A)-AJ_1(A)\frac{\Delta f}{f_r}+\cdots$$

$$K_0\left(\frac{\omega a}{v}\right)=K_0(A)-AK_1(A)\frac{\Delta f}{f_r}+\cdots$$

$$J_1\left(\frac{\omega a}{v}\right)=J_1(A)+AJ_0(A)\frac{\Delta f}{f_r}-J_1(A)\frac{\Delta f}{f_r}+\cdots$$

$$K_1\left(\frac{\omega a}{v}\right)=K_1(A)+AK_0(A)\frac{\Delta f}{f_r}-K_1(A)\frac{\Delta f}{f_r}+\cdots$$

where f_r is resonant frequency. Similar expressions can be derived for $J_0(\omega b/v)$, $K_0(\omega b/v)$, $J_1(\omega b/v)$ and $K_1(\omega b/v)$ about the root B. Also we have $\omega a/v=A+A(\Delta f/f_r)$ and $\omega b/v=B+B(\Delta f/f_r)$. Inserting these values into the bracket expression, we get to a first approximation the second formidable expression

$$\frac{\Delta f}{f_r}\,\frac{\left\{\begin{aligned}&2[K_0(B)J_0(A)-K_0(A)J_0(B)]+B[K_0(A)J_1(B)-K_1(B)J_0(A)]+A[K_1(A)J_0(B)-K_0(B)J_1(A)]\\&+(1-\sigma)\left(2[K_0(A)J_0(B)-K_0(B)J_0(A)]+\frac{A^2+B^2}{AB}[K_1(B)J_1(A)-K_1(A)J_1(B)]\right)\\&+(1-\sigma^2)\left(\frac{1}{B}[K_1(B)J_0(A)-K_0(A)J_1(B)]+\frac{1}{A}[K_0(B)J_1(A)-K_1(A)J_0(B)]\right.\\&\left.\qquad+\frac{2}{AB}[K_1(A)J_1(B)-K_1(B)J_1(A)]\right)\end{aligned}\right\}}{\left\{\begin{aligned}&A[K_0(A)J_1(A)-K_1(A)J_0(A)]+B[K_0(B)J_1(B)-K_1(B)J_0(B)]+B[K_1(B)J_0(A)-K_0(A)J_1(B)]\\&+A[K_1(A)J_0(B)-K_0(B)J_1(A)]+\frac{A^2-B^2}{AB}(1-\sigma)[K_1(B)J_1(A)-K_1(A)J_1(B)]\end{aligned}\right\}}$$

$$=\frac{k^2}{1-k^2}\,\frac{1+\sigma}{A^2-B^2}.$$

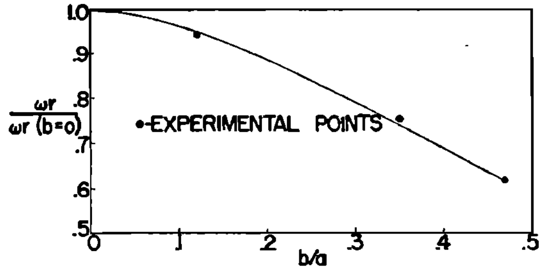

FIG. 6. Plot of the ratio of the resonant frequency of a hollow cylinder to the resonant frequency of a solid disk as a function of the ratio b/a.

If the inner and outer radii and the Poisson's ratio is known this equation can be put into the form

$$\frac{k^2}{1-k^2}=C\frac{\Delta f}{f_r}$$

where C is a constant. For barium titanate the value of this constant has been calculated for various ratios of a/b and is plotted in Fig. 5.

EXPERIMENTAL

The results of the preceding section were checked by the following experiments. Three barium titanate elements were available which had had holes cut in their center. These elements were 0.125 in. thick and had an outside diameter of 1.047 in. The hole diameters were 0.126 in., 0.367 in., and 0.492 in. Unfortunately no resonance measurements were made on these elements before the holes were cut. However, several elements of the same batch were available for measurements, and these elements had radial coupling coefficients of 0.26±0.01. Using the results of the preceding section, the radial coupling coefficients of the three test samples were 0.25, 0.26, and 0.26.

The preceding section also predicts that, when compared with the resonant frequency of the solid disk, the resonant frequency of an element with a hole in the middle should decrease with increasing hole diameter if the outside diameter is kept constant. The ratio of the resonant frequency of a ring to the resonant frequency of the solid disk is plotted against the ratio a/b as the solid line in Fig. 6. The ratios of the resonant frequencies of the three experimental elements to the average of the resonant frequencies of other crystals of the same batch are plotted as experimental points. The agreement is within experimental error.

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Propagation of Elastic Waves in Cylindrical Shells, Including the Effects of Transverse Shear and Rotatory Inertia*

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Two systems of equations of motion, designated as (I) and (II), for thin elastic cylindrical shells are derived which include the effects of both transverse shear deformation and rotatory inertia. The character of the two systems of equations is such that, upon the neglect of transverse shear deformation and rotatory inertia, Eqs. (I) will reduce to those known as Love's first approximation, while (II), which enjoys a considerable mathematical simplicity as compared to (I), will correspond to those given by Donnell. Both systems of Eqs. (I) and (II) are employed in a study for propagation of axisymmetric waves in an infinite cylindrical shell. The agreement between the predictions of the two systems of equations, in all modes of motion, for phase velocities of propagated waves in the complete range of wavelengths is found to be excellent. The results, with reference to the nature of the modes of motion according to both (I) and (II), are further examined and the relative merit of the present paper to the work of other authors is discussed.

INTRODUCTION

IN a recent paper[1] dealing with the general theory of thin isotropic elastic shells (where the effects of transverse normal stress, transverse shear deformation, and rotatory inertia are also discussed), a set of stress-displacement relations is deduced which is entirely consistent with the assumptions for the stresses and displacements in a thin shell; these results were obtained by means of a variational theorem due to E. Reissner.[2] In the sequel, using the basic equations of reference 1, two systems of equations of motion for elastic cylindrical shells will be considered. These systems of equations of motion, which include the effects of both transverse shear deformation and rotatory inertia, will be designated as Eqs. (I) and (II); Eqs. (II) enjoy some mathematical simplicity as compared to (I) and will be referred to as an "approximate"

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[1] P. M. Naghdi, ONR Project NR-064-48, Contract Nonr-1224(01), Engineering Research Institute, University of Michigan, Tech. Repts. No. 1 and 2 (January and March, 1955) (to be published).

[2] Eric Reissner, J. Math. Phys. 29, 90–95 (1950).