

Toughness

The square-root singular field is invalid in the fracture process zone, where inelasticity prevails, and is also invalid on the size scale of the external boundary, where the boundary conditions prevail. A careful interpretation of the square-root singular field clarifies why the Linear Elastic Fracture Mechanics works.

The form of the singular field around the tip of a crack is universal, independent of the shape of the body and the distribution of the load. The following idealizations are made:

- The crack is idealized as two flat planes.
- The front of the crack is idealized as a straight line.
- The body is modeled by using the linear elastic theory.
- The material is taken to be isotropic.
- The body deforms under the plane-strain conditions.
- The loading is mode I.

Within these idealizations, the field of stress around the front takes the form

$$\sigma_{ij}(r, \theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta).$$

The dependence on r is square-root singular. The functions $f_{ij}(\theta)$ are known functions. Consequently, the form of the field around the front is independent of the shape of the body and the distribution of the load. The amplitude of the field is represented by the stress intensity factor K , which does depend on the shape of the body and the distribution of the load.

Relate K to the external boundary conditions. For a given body containing a crack, one *assumes* that the crack tip is a mathematical point, the crack a mathematical cut, and the body obeys the linear elasticity theory. The linearly elastic boundary-value problem determines K . Linearity and dimensional considerations dictate that the stress intensity factor should take the form

$$K = Y\sigma\sqrt{L},$$

where L is a representative length, σ is a stress characterizing the magnitude of the load, and Y is a dimensionless number depending on the shape of the body and the distribution of the load. For example, for the Griffith crack, length $2a$, in an infinite body under a stress σ , the stress intensity factor is

$$K = \sigma\sqrt{\pi a}.$$

The crack-tip field is invalid at the crack tip. K -annulus. Consider the Griffith crack again, assuming that the body is made of glass. Sketch the distribution of the stress $\sigma_{yy}(x, 0)$. Discuss the field in three regions:

- Very close to the crack tip (less than 1 nm for the glass), atomic bonds are being stretched, and the linear elasticity theory is invalid. The stress is finite, below the theoretic strength of the glass. This region is known as the fracture process zone. The K -field is wrong in the fracture process zone. Denote the radius of the process zone by r_p .
- On the scale comparable and larger than the length of the crack a , the field is determined by the external boundary conditions. The K -field is wrong in this region also.
- In an annulus, $r_p < r < a$, the K -field is valid. We call this annulus the K -annulus, or K -dominant zone.

For a representative size of the crack, $a = 1\mu\text{m}$, the K -field is valid in the annulus $1\text{nm} < r < 1\mu\text{m}$, over several orders of magnitude in length.

The Linear Elastic Fracture Mechanics (LEFM). Imagine two bodies of the same material, each containing a crack under an opening load. The two bodies may have different

sizes and different shapes, and lengths of the cracks in the two bodies and the loads on the two bodies may also be different. Regardless of all these differences, the field in the K -annulus depends on the boundary conditions through the stress intensity factor. That is, K is the only messenger between the external boundary conditions and the fracture process.

If K is the same for the two bodies, the fracture processes at the tips of the two cracks must be identical. The doctrine of the linear elastic fracture mechanics has been so summarized:

The same K , the same fracture process.

Fracture criterion phrased in terms of K . Within the fracture mechanics, the simplest way to account for the fracture process is to stipulate that the crack extends when the stress intensity factor K reaches a critical value, K_c . This critical value K_c is known as the toughness.

The stress intensity factor is a *loading parameter*. The toughness is a *material parameter*. For a given material, K_c is determined by a fracture test. Representative values of toughness:

- Glass: $K_c = 1 \text{ MPa}\sqrt{\text{m}}$.
- Steel: $K_c = 100 \text{ MPa}\sqrt{\text{m}}$.
- Epoxy: $K_c = 1 \text{ MPa}\sqrt{\text{m}}$.

In applications, we use two handbooks. One contains the elasticity solutions of many cracked *geometries*, and the other contains the experimentally determined values of toughness for many *materials*. The crack does not grow if

$$K < K_c,$$

or

$$Y\sigma\sqrt{L} < K_c.$$

The K -based fracture criterion is equivalent to the G -based one. Irwin's formula relates the two loading parameters K and G as

$$G = \frac{K^2}{E}.$$

Consequently, the two material properties, the toughness K_c and the fracture energy Γ , are similarly related:

$$\Gamma = \frac{K_c^2}{E}.$$

The condition under which the K -annulus exists: small-scale yielding condition. The above discussion identifies two size scales:

- (1) **The size of the fracture process zone**, in which the linear elastic theory is invalid. For a glass, the linear elastic theory is valid all the way to a few atomic distances, so the fracture process zone encloses the bond-breaking atoms. For a steel, the linear elastic theory is wrong in the region where steel deforms plastically, so that the fracture process zone encloses the plastic zone around the crack tip. Representative size of the fracture process zone is nanometers for a glass, and millimeters for a steel.
- (2) **The size representative of the external boundary conditions.** For the Griffith crack, this size is the half length of the crack. For a channel crack in a brittle film bonded to a substrate, this size is the thickness of the film.

For the K -annulus to exist, the small-scale yielding condition must be satisfied:

The fracture process zone « the external boundary zone

Size of plastic zone. The Linear Elastic Fracture Mechanics (LEFM) requires the existence of the K-annulus, which in turn, requires the small scale yielding condition. We now estimate the fracture process zone size.

The elastic stress field scales as

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}}.$$

Let the yield strength be σ_Y . Approximately, the plastic zone size is given by

$$r_p \approx \frac{1}{2\pi} \left(\frac{K_c}{\sigma_Y} \right)^2.$$

The plastic zone size is expressed in terms of the material parameters. The plastic zone size itself is also a material parameter.

Glass: $\sigma_Y \sim 1 \text{ GPa}$ and $K_c \sim 1 \text{ MPa}\sqrt{\text{m}}$.

$$r_p \approx \frac{1}{2\pi} \left(\frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left(\frac{1 \text{ } \sigma^6}{1 \text{ } \sigma^0} \right)^2 \sim 1 \text{ } \sigma^9 \text{ m}.$$

High strength steel: $\sigma_Y \sim 1.4 \text{ GPa}$ and $K_c \sim 6.5 \text{ MPa}\sqrt{\text{m}}$.

$$r_p \approx \frac{1}{2\pi} \left(\frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left(\frac{6.5 \times 1 \text{ } \sigma^6}{1.4 \times 1 \text{ } \sigma^0} \right)^2 \sim 0.3 \text{ m m}.$$

Tough steel: $\sigma_Y \sim 0.35 \text{ GPa}$ and $K_c \sim 18 \text{ MPa}\sqrt{\text{m}}$.

$$r_p \approx \frac{1}{2\pi} \left(\frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left(\frac{18 \times 1 \text{ } \sigma^6}{0.35 \times 1 \text{ } \sigma^0} \right)^2 \sim 4 \text{ cm}.$$

Effect of specimen thickness. Plane strain fracture toughness. A thin sheet deforms under the plane-stress conditions, and the stress state is biaxial. A thick body deforms under the plane-strain conditions, and the stress state is triaxial. The difference is important for plastically deformable materials. For example, a thin sheet may fracture by necking, but a block may fracture by void growth. The toughness is a function of the specimen thickness. The standard test requires that the fracture toughness be measured under the plane-strain conditions.

The small-scale yielding condition in practice. “Standard Test Method for Plane-Strain Fracture Toughness of Metallic Materials,” *ASTM Annual Book of Standards*, Part 10, American Society for Testing and Materials, Philadelphia, E23-72, pp. 273-289 (1980).

To measure the plane strain fracture toughness, the Standard requires that

$$a > 2.5 \left(\frac{K_c}{\sigma_Y} \right)^2$$

$$B > 2.5 \left(\frac{K_c}{\sigma_Y} \right)^2$$

$$W > 5.0 \left(\frac{K_c}{\sigma_Y} \right)^2$$

where a is the length of the crack, B the thickness of the specimen, and W the width of the specimen.

LEFM is inapplicable for high toughness, low strength materials, because these materials have very large plastic zone size.

The effect of temperature on toughness. Steel. At low temperature, steel fractures by cleavage, and has low toughness. At high temperature, steel fractures by hole growth, and has high toughness. Ductile-to-brittle transition temperature (DBT).

The effect of loading rate on toughness.

Low loading rate: ductile.

High loading rate: brittle.

The ductile-to-brittle transition temperature (DBT) increases with the loading rate.

Avoid cracking in the matrix around an inclusion. T.C. Lu, J. Yang, Z. Suo, A.G. Evans, R. Hecht, and R. Mehrabian, Matrix cracking in intermetallic composites caused by thermal expansion mismatch. Acta Metall. Mater. 39, 1883-1890 (1991). You may wish to take a look at analytical details and micrographs of cracks around inclusions (<http://www.seas.harvard.edu/suo/papers/o13.pdf>).

A fiber of radius R is embedded in an infinite matrix. For simplicity, we assume that the fiber and the matrix are both elastic, having similar Young's modulus E and Poisson's ratio ν . The two materials are bonded at an elevated temperature. When they are cooled to the room temperature, the two materials develop stresses due to the mismatch in thermal expansion. Set the representative stress by

$$\sigma = E\Delta\alpha\Delta T / (1 - \nu),$$

where $\Delta\alpha$ is the difference in the coefficients of thermal expansion of the two materials, and ΔT the change in the temperature. This stress, as shown by the elasticity solution, corresponds to the magnitude of the longitudinal stress or twice that of the transverse stress in the cylinder.

When the cylinder is under compression, the matrix is under a tensile hoop stress, which decays far away from the interface. To evaluate if this stress field will cause the matrix to fracture, we place a crack in the matrix, and ask if this crack will grow. It is sensible to place the crack near the interface and in the radial direction, where the stress is tensile and has a large magnitude.

The choice of the length of the crack requires some thought. When the length of the crack approaches zero, the stress intensity factor vanishes. When the length of the crack approaches infinity, the stress intensity factor also vanishes. Sketch the trend of the stress intensity factor as a function of the crack length. This elastic boundary value problem has been solved analytically, giving the stress intensity factor:

$$\frac{K}{\sigma\sqrt{R}} = \left(\frac{\pi}{8}\right)^{1/2} \left(\frac{a}{R}\right) \left(1 + \frac{a}{R}\right)^{-3/2}.$$

If we can estimate the flaw size due to a certain fabrication process, we can set the crack size to this flaw size, and then calculate the stress intensity factor.

Even if we know nothing about the flaw, we may proceed as follows. The stress intensity factor attains a maximum, $K_{\max} = 0.24\sigma\sqrt{R}$, at $a/R = 0.5$. No matrix flaw can grow provided this maximum stress intensity factor is below the fracture toughness, namely,

$$0.24 \frac{E\Delta\alpha\Delta T}{1 - \nu} \sqrt{R} < K_c.$$

This condition invokes well defined quantities, and does not rely on any knowledge of the flaw. In particular, this condition predicts that a critical radius of the fiber exists, below which no flaw in the matrix can grow. Inserting representative values, $E = 1 \text{ } 0^{11} \text{ Pa}$, $\nu = 0.3$, $\Delta\alpha = 1 \text{ } 0^{-5} \text{ K}^{-1}$, $\Delta T = 1 \text{ } 0 \text{ } 0 \text{ } \text{K}$, $K_c = 1 \text{ } 0^6 \text{ Pa}\sqrt{\text{m}}$, we estimate that $R_c \approx 1 \text{ } 0 \text{ } \mu\text{m}$.