Vibration

Reference. J.P. Den Hartog, *Mechanical Vibrations*, Dover Publications, New York. This exceptional book, written by a Timoshenko Medalist, is available on amazon.com at $6.

When a bar is pulled, the stress field in the bar is uniform. When the bar is bent, the stress field is a function of position. When the bar vibrates, the stress is a function of position and time – that is, the stress in the bar is a time-dependent field. We will see how the three elements of solid mechanics play together in this new context. Before considering *structures*, we'll first recall a familiar problem: system of one degree of freedom.

A system of one degree of freedom

Mass, spring, dashpot. Model a system by a mass, connected to a spring and the dashpot in parallel. The mass is on a frictionless ground. The input to the system is an external force on the mass as a function of time, $F(t)$. The output of the system is the displacement of the mass as a function of time, $x(t)$.

Free-body diagram. Let m be the mass. Choose the origin such that the spring exerts no force when $x = 0$. Measure the displacement from this position. The sign convention: pick one direction as the positive direction for the displacement. When the displacement is $x(t)$, the velocity is $\dot{x}(t)$, and the acceleration is $\ddot{x}(t)$. The spring exerts a force $-kx$, and the dashpot exerts a force $-c\dot{x}$. The spring constant k and the damping constant c are measured experimentally.

Newton's second law, Force = (Mass)$\times$(Acceleration), gives the equation of motion

$$m\ddot{x} + c\dot{x} + kx = F \ .$$

This is an ordinary differential equation (ODE) for the function $x(t)$. In writing the ODE, we put all the terms containing the unknown function $x(t)$ on the left-hand side, and all the other terms on the right-hand side.

Free vibration ($F = 0$)**, and no damping** ($c = 0$)**.** After vibration is initiated, no external force is applied. With no damping, the mass will vibrate forever. The equation of motion is

$$m\ddot{x} + kx = 0 .$$

This is a homogeneous ODE. The general solution to this equation has the form

$$x(t) = A\sin\omega t + B\cos\omega t \ ,$$

where A, B and ω are constants to be determined. Inserting the solution to the ODE, we find that

$$\omega = \sqrt{k / m} \ .$$

The displacement repeats itself after a period of time equal to $2\pi / \omega$. Per unit time the mass vibrates this many cycles:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ .$$

We call ω the circular frequency, and f the frequency. Both are also called the natural frequency. This frequency is "natural" in that it is set by parameters of the system (i.e., the mass and the spring constant), rather than the external force.

Determine the constants A and B by the initial displacement and velocity. For example, suppose at time zero the mass has a known displacement $x(0)$ and is released at zero velocity, the subsequent displacement is

$$x(t) = x(0)\cos\omega t \ .$$

Daily experience suggests that to sustain a vibration, one needs to apply a periodic force. The above free vibration lasts forever because our model is idealized: the model has no damping. In practice, we can make damping very small by a careful design. So the model is an idealization of a system when the damping is very small.

Forced vibration, no damping. Now consider the external force as a function of time, $F(t)$. According to Fourier, any periodic function is a sum of many sine and cosine functions. Consequently, we will only consider an external force of form

$$F(t) = F_0 \sin \Omega t .$$

The frequency of the force, Ω, being determined by an external source, is unrelated to the natural frequency ω.

The equation of motion now becomes

$$m\ddot{x} + kx = F_0 \sin \Omega t .$$

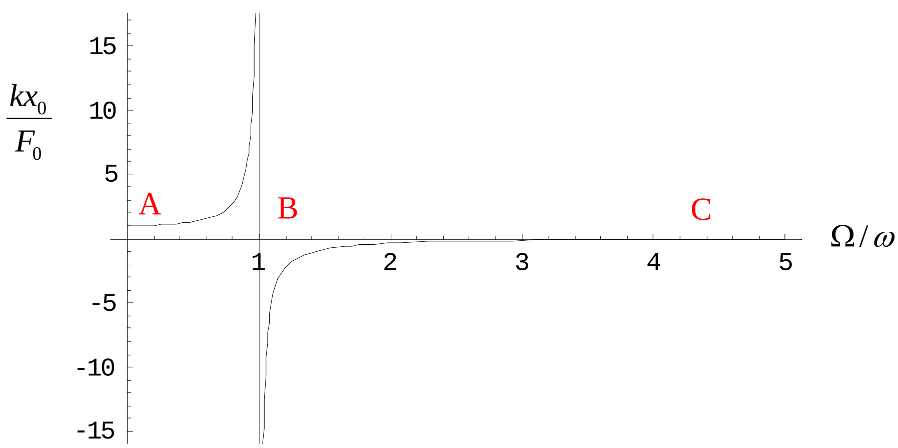
This is an inhomogeneous ODE. It governs displacement as a function of time, $x(t)$. To find *one* particular solution, try the form $x(t) = C \sin \Omega t$, which gives

$$x(t) = \frac{F_0 / k}{1 - (\Omega / \omega)^2} \sin \Omega t .$$

The full solution to the inhomogeneous, linear ODE is the sum of *all* homogeneous solution and *one* particular solution:

$$x(t) = A \sin \omega t + B \cos \omega + \frac{F_0 / k}{1 - (\Omega / \omega)^2} \sin \Omega t$$

In general, the driving frequency Ω and the natural frequency ω are unrelated. Consequently, this motion is *not* periodic. With damping, the homogenous solution will die out, but the particular solution will persist. Consequently, the particular solution is the steady response of the system to the periodic force. The steady response has the same frequency as the external force, Ω.



Three behaviors of forced vibration. Let us focus on the steady solution:

$$x(t) = \frac{F_0 / k}{1 - (\Omega / \omega)^2} \sin \Omega t$$

Plot the amplitude of the displacement as a function of the driving frequency. Depending on the ratio of the driving frequency to the natural frequency, Ω / ω, we classify three behaviors as follows.

A . $\Omega/\omega \ll 1$. The frequency of the external force is low compared to the natural frequency. The mass oscillates in phase with the external force. The behavior is similar to the mass under a static load, $x_0 \approx F_0 / k$.

B. $\Omega/\omega \sim 1$. The frequency of the external force is comparable to the natural frequency. The system is said to **resonate** with the external force. For this reason, the natural frequency, ω, is also called the **resonant** frequency. At **resonance**, the mass oscillates with a large amplitude. (The amplitude is finite when damping is included.) The load pushes at right time in the right direction.

C. $\Omega/\omega \gg 1$. The frequency of the external force is high compared to the natural frequency. The mass oscillates anti-phase with the external force. As the frequency of the external force increases, the displacement amplitude diminishes: the system is too slow to respond to the external force. This phenomenon is the basis for **vibration isolation**.

Push the swing at the right time. Here is an example of resonance. The length of the string is l. The acceleration of gravity is g. The natural frequency of the swing is

$$\omega = \sqrt{g / l} .$$

When l = 2 m, g = 9.8 m/s^2, the period is $T = 2\pi / \omega = 2.8$ s. In formulating the equation of motion, we normally neglect damping, so that the mathematical solution says that an undamped swing will persist forever. In reality, the swing loses energy by motion of air, by the friction at various joints, etc. To sustain the swing, one needs to push from time to time. If you push at a frequency Ω close to the natural frequency ω of the swing, in the right direction, the swing amplitude can even increase.

Vibration isolation. A vibrating machine is modeled by a mass m and a harmonic force, $F_0 \sin \Omega t$. For example, if the machine contains a rotating part, such as a fan, the harmonic force is due to mass imbalance. When the machine is placed directly on the ground, the force $F_0 \sin \Omega t$ is transmitted to the ground. Of course, if you can, you should balance the fan to reduce the harmonic force itself.

Now suppose that you cannot balance the fan, and has to accept the harmonic force as given, what can you do to reduce the vibration transmitted to the ground? The solution is to put the machine on a compliant spring, and then on the ground. In this case, the force transmitted to the ground is the same as the force in the spring, kx. Thus, the ratio of the force transmitted to the ground to the force generated by the machine is

$$\frac{\text{Force transmitted to the ground}}{\text{Force generated by the machine}} = \left| \frac{kx_0}{F_0} \right| = \frac{1}{\left| 1 - (\Omega / \omega)^2 \right|} .$$

To make this ratio small, we need to make $\Omega / \omega \to \infty$. Recall that the natural frequency is $\omega = \sqrt{k / m}$. Use a complaint spring to reduce the natural frequency. To isolate the ground from vibration, you don't need damping.

Free vibration with damping. After vibration is initiated, no external force is applied. The vibration decays over time. The effect of damping is modeled by a dashpot. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0 .$$

This is a homogeneous ODE with constant coefficients.

***Q* is for quality factor.** The three parameters of the system, m, c and k, form a dimensionless group:

$$Q = \frac{\sqrt{mk}}{c} .$$

The three coefficients, m, c and k, are all positive, so that $0 < Q < +\infty$. This dimensionless ratio is known as the quality factor. It quantifies the importance of the damping relative to that of inertia and stiffness. The system is over-damped when $Q \to 0$, and undamped when $Q = \infty$. We will be mainly interested in system where damping is slight, so that the quality factor far exceeds 1, e.g. in the range $Q = 10 - 10^9$.

Solution to the ODE. Recall that the natural frequency for the undamped spring-mass system is $\omega = \sqrt{k/m}$. The above ODE can be rewritten as

$$\ddot{x} + \frac{\omega}{Q}\dot{x} + \omega^2 x = 0$$

For any homogeneous ODE with constant coefficients, the solution is of the form

$$x(t) = \exp(\rho t).$$

where ρ, known as the characteristic number, is to be determined. Substituting this solution into the ODE, we obtain that

$$\rho^2 + \frac{\omega}{Q}\rho + \omega^2 = 0 .$$

This is a quadratic algebraic equation for ρ. The two roots are

$$\rho = -\frac{\omega}{2Q} \pm \omega\sqrt{\frac{1}{4Q^2} - 1} .$$

We are interested in the situation where the damping is small, so that the mass vibrates back and forward many times. That is, we assume that the quality factor is a large number. Denote

$$q = \omega\sqrt{1 - \frac{1}{4Q^2}} .$$

This is a real, positive quantity. The two characteristic roots are

$$\rho = -\frac{\omega}{2Q} \pm iq$$

where $i = \sqrt{-1}$. Recall the Euler equation:

$$\exp(iq) = \cos q + i \sin q .$$

The general solution to the ODE is

$$x(t) = \exp\left(-\frac{\omega t}{2Q}\right)(A\sin qt + B\cos qt).$$

$$Q = \frac{\sqrt{mk}}{c}, \quad \omega = \sqrt{\frac{k}{m}}, \quad q = \omega\sqrt{1 - \frac{1}{4Q^2}}$$

This solution gives the displacement as a function of time. The constants A and B are to be determined by the initial conditions, namely the displacement and the velocity at time zero.

Significant features of damped free vibration. We now examine features of this solution that are important in application.

- The above solution represents a damped wave, with frequency q. For slight damping, $Q >> 1$, the frequency of the damped system is close to that of the undamped system, $q \approx \omega = \sqrt{k/m}$.

- The vibration amplitude decays with time. In one cycle, the time goes from t to $t+2\pi/q$, the amplitude of the vibration diminishes from $\exp\left(-\frac{\omega}{2Q}t\right)$ to $\exp\left[-\frac{\omega}{2Q}\left(t+\frac{2\pi}{q}\right)\right]$. Thus,

$$\text{the ratio of two consecutive amplitudes} \approx \exp\left(-\frac{\pi}{Q}\right).$$

This ratio is the same for any two consecutive maxima, independent of the amplitude of vibration at the time. One can measure the ratio of the consecutive amplitudes, and thereby determines the quality factor.

- For the damped system to vibrate Q cycles, the time needed is

$$\Delta t = 2\pi Q / q \approx 2\pi Q / \omega .$$

Consequently, the amplitude of the vibration diminishes by a factor $\exp(-\pi)$. Roughly speaking, Q is the number of cycles for the vibration to die out.

Forced vibration, with damping. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \Omega t$$

We have discussed the homogeneous solution. We now need to find one particular solution. To keep algebra clean, recall the definitions

$$\omega = \sqrt{\frac{k}{m}} \text{ and } Q = \frac{\sqrt{mk}}{c}.$$

The equation of motion becomes

$$\frac{\ddot{x}}{\omega^2} + \frac{\dot{x}}{Q\omega} + x = \frac{F_0}{k}\sin \Omega t$$

Now we use a standard trick to simplify the algebra. Write

$$F(t) = F_0 \exp(i\Omega t)$$

Of course we known the external force is just the real part of this function. Write

$$x(t) = x_0 \exp(i\Omega t).$$

Plug into the equation of motion, and we have

$$x_0 = \frac{F_0 / k}{1 - \left(\frac{\Omega}{\omega}\right)^2 + i\frac{\Omega}{Q\omega}}$$

Note that x_0 is a complex number. The displacement of the mass should be $x = \mathrm{Re}\left[x_0 \exp(i\Omega t)\right]$.

A useful interpretation of the solution. Write

$$x_0 = |x_0| \exp(-i\phi),$$

where the amplitude of the displacement is

$$\boxed{|x_0| = \frac{F_0 / k}{\sqrt{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left(\frac{\Omega}{Q\omega}\right)^2}}},$$

and the phase shift of the displacement relative to the external force, ϕ, is given by

$$\boxed{\tan\phi = \frac{\dfrac{\Omega}{Q\omega}}{1-(\Omega/\omega)^2}}$$

Plot the displacement amplitude as a function of the forcing frequency. At the resonance, $\Omega/\omega \approx 1$, we have $|x_0| = QF_0/k$, namely, the vibration amplitude is Q times of the static displacement. This provides another experimental way to determine the quality factor.

The quality factor determines the "sharpness" of the resonance. A large quality factor corresponds to a sharp resonance. This is important to all resonance-based devices.

To discuss the phase factor, let's consider three limiting cases.

- Very stiff system. Neglect the mass and the viscous terms in the equation of motion, we have $kx = F_0 \sin(\Omega t)$. Consequently, the displacement x is in phase with the external force.
- Very viscous system. The equation of motion is $c\dot{x} = F_0 \sin(\Omega t)$. The displacement is $x = -\frac{F_0}{c\Omega}\cos(\Omega t) = \frac{F_0}{c\Omega}\sin\left(\Omega t - \frac{\pi}{2}\right)$. That is, the displacement lags behind the external force by a phase factor $\pi/2$.
- Very massive system. The equation of motion is $m\ddot{x} = F_0 \sin(\Omega t)$. The displacement is $x = -\frac{F_0}{m\Omega^2}\sin(\Omega t) = \frac{F_0}{m\Omega^2}\sin(\Omega t - \pi)$. That is, the displacement lags behind the external force by a phase factor π.

Plot the phase factor as a function of the forcing frequency for several values of the quality factor. The phase factor ranges between 0 and π. The trend is mainly a competition between the three forces: elastic, viscous, and inertial. When $\Omega/\omega \ll 1$, the elastic force prevails, and the displacement is in-phase with the external force. When $\Omega/\omega \gg 1$, the inertial force prevails, and the displacement is π out-of-phase with the external force. When $\Omega/\omega \approx 1$, the viscous force prevails, and the displacement is $\pi/2$ out-of-phase with the external force.

Vibration isolation with a spring and a dashpot in parallel. The external force is $F = F_0 \exp(i\Omega t)$. The displacement of the mass is $x = x_0 \exp(i\Omega t)$, where x_0 is the same as above. The force transmitted to the ground via the spring and the dashpot is

$$kx + c\dot{x} = (k + i\Omega c)x_0 \exp(i\Omega t).$$

Consequently, we obtain that

$$\frac{\text{Force transmitted to the ground}}{\text{Force generated by the machine}} = \sqrt{\frac{1+\left(\dfrac{\Omega}{Q\omega}\right)^2}{\left[1-\left(\dfrac{\Omega}{\omega}\right)^2\right]^2+\left(\dfrac{\Omega}{Q\omega}\right)^2}}.$$

Beam as a spring. Consider a cantilever and a lump of mass at its end, just like a diver on a spring board. For the time being, we assume that the mass of the cantilever is negligible compared to the lumped mass.

We can obtain the spring constant as follows. Apply a static force F at the free end of the beam, and the beam end deflects by the displacement Δ. According to the beam theory, the force-deflection relation is

$$F = \left(\frac{3EI}{L^3}\right)\Delta .$$

Consequently, the spring constant of the cantilever is

$$k = \frac{3EI}{L^3} .$$

For beams of other end conditions, the spring constant takes the same form, but with difference numerical factors. Consult standard textbooks. The beam is stiff when Young's modulus is high, the moment of cross section is high, and the beam is short.

If the lumped mass is M, the natural frequency is

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{3EI}{L^3 M}} .$$

Frequency meter. A cantilever provides elasticity. A lump of mass m provides inertia. Two designs: (1) several beams with different masses, and (2) a single beam with adjustable position of the mass. Place the device on a vibrating machine, and watch for resonance.

Determine Young's modulus by measuring the resonant frequency. You need to measure Young's modulus of a material. Make a beam out of the material. Make a device so that you can change the forcing frequency. Resonance experiment allows you to determine the natural frequency, and provides a measurement of Young's modulus.

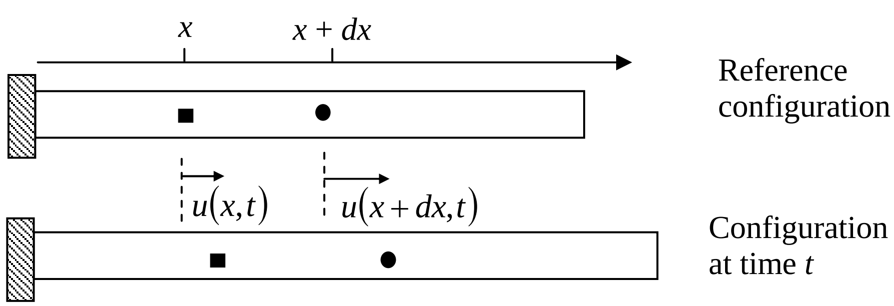
The advantage of the beam, as compared to other forms of springs, is that it can be made very small by using the microfabrication technology.

- Atomic force microscope (AFM).
- Micro-electro-mechanical systems (MEMS).
- Accelerometer in an automobile air bag.

Longitudinal vibration of a rod

When an elastic rod vibrates, each material particle provides a degree of freedom. Thus, the rod has infinite degrees of freedom. We will use the longitudinal vibration of a rod to illustrate a fundamental phenomenon in structural vibration: **normal modes**.

Consider a rod, cross-sectional area A, length L, mass density ρ, and Young's modulus E. The rod is constrained to move along its axial direction, clamped at one end, and free to move at the other end. We neglect damping.



The displacement is a time-dependent field. Take the unstressed rod as the reference state. Label each material particle by its coordinate x in the reference state. To visualize the motion of the material particles, place markers on the rod. In the figure, two markers indicate two material particles, x and $x + dx$. When the rod is stressed, the markers move to new positions. The distance by which each marker moves is the displacement of the material particle. Denote the displacement of the material particle x at time t by $u(x,t)$.

Three ingredients of solid mechanics. We now translate the three ingredients of solid mechanics into equations.

Strain-displacement relation. Denote the strain of the material particle x at time t by $\varepsilon(x,t)$. Look at a small piece of the rod between x and $x + dx$. At time t, the displacement of material particle x is $u(x,t)$, and the displacement of material particle $x + dx$ is $u(x+dx,t)$. The strain of this piece of the rod is

$$\varepsilon(x,t)=\frac{\text{elongation}}{\text{originallengh}}=\frac{u(x+dx,t)-u(x,t)}{dx}.$$

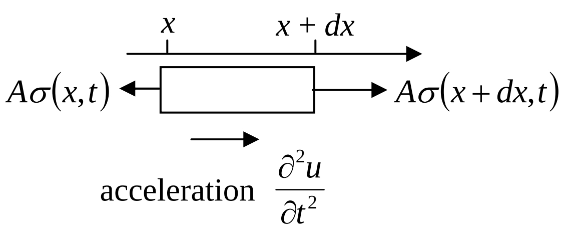
We obtain the relation between the strain field and the displacement field:

$$\boxed{\varepsilon=\frac{\partial u}{\partial x}}$$

The partial derivative is taken at a fixed time.

Material law. Denote the stress of the material particle x at time t by $\sigma(x,t)$. We assume that the rod is made of an elastic material. That is, for *every* material particle and at *any* time, Hooke's law relates the stress to the strain

$$\boxed{\sigma=E\varepsilon}$$



Newton's second law. Draw the free body diagram of the small piece of the rod between x and $x + dx$. At time t, the stress at cross-section x gives a force $A\sigma(x,t)$ to the left, and the stress at cross section $x + dx$ gives a force $A\sigma(x+dx,t)$ to the right. The piece of the rod has mass $\rho A dx$, and acceleration $\partial^2 u/\partial t^2$. Apply Newton's law, Force = (Mass)(Acceleration), to this piece of the rod. We obtain that

$$A\sigma(x+dx,t)-A\sigma(x,t)=(\rho A dx)(\partial^2 u/\partial t^2),$$

or

$$\boxed{\frac{\partial\sigma}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}}$$

Put the three ingredients together. A combination of the three boxed equations gives

$$E\frac{\partial^2 u}{\partial x^2}=\rho\frac{\partial^2 u}{\partial t^2}.$$

This is a partial differential equation that governs the displacement field $u(x,t)$. It is known as the **equation of motion**.

Separate spatial coordinates from time. As a first dynamic phenomenon, consider free vibration of the rod. In a normal mode, each material particle x vibrates with its individual

amplitude $U(x)$, but all material particles vibrate at the same **natural circular frequency** ω (radian per unit time). The displacement field of such a normal mode takes the form

$$u(x,t) = U(x)\sin(\omega t).$$

The two variables *x* and *t* are separated.

We next calculate the amplitude function $U(x)$ and the natural frequency ω. Substituting the normal mode into the equation of motion, we obtain that

$$\frac{E}{\rho}\frac{d^2U}{dx^2} = -\omega^2 U .$$

This ODE for the amplitude function $U(x)$ is homogeneous, and has constant coefficients. The general solution to the ODE is

$$U(x) = A\sin\left(\sqrt{\frac{\rho}{E}}\omega x\right) + B\cos\left(\sqrt{\frac{\rho}{E}}\omega x\right),$$

where *A* and *B* are arbitrary constants. The time-dependent displacement field is

$$u(x,t) = \left[A\sin\left(\sqrt{\frac{\rho}{E}}\omega x\right) + B\cos\left(\sqrt{\frac{\rho}{E}}\omega x\right)\right]\sin\omega t$$

The three numbers, *A*, *B*, ω are yet to be determined.

Normal modes. The above solution satisfies the equation of motion. We now examine the **boundary conditions**. The rod is constrained at the left end, so that the displacement vanishes at $x = 0$ for all time:

$$u(0,t) = 0 .$$

The rod moves freely at the right end, so that the stress vanishes at the right $x = L$ for all time. Recall that $\sigma = E\varepsilon = E\partial u / \partial x$. The stress-free boundary condition means that

$$\left.\frac{\partial u}{\partial x}\right|_{x=L, all\, time} = 0 .$$

Apply the two boundary conditions, and we obtain that

$$B = 0 ,$$

$$A\sqrt{\frac{\rho}{E}}\omega\cos\left(\sqrt{\frac{\rho}{E}}\omega L\right) = 0 .$$

Possible solutions are as follows. First, *A* = 0. This solution makes the displacement vanish at all time. Second, $\omega = 0$. This solution makes the natural frequency vanish, so that the rod is static. The third possibility is of most interest to us:

$$\cos\left(\sqrt{\frac{\rho}{E}}\omega L\right) = 0 .$$

This requires that

$$\sqrt{\frac{\rho}{E}}\omega L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}...$$

or

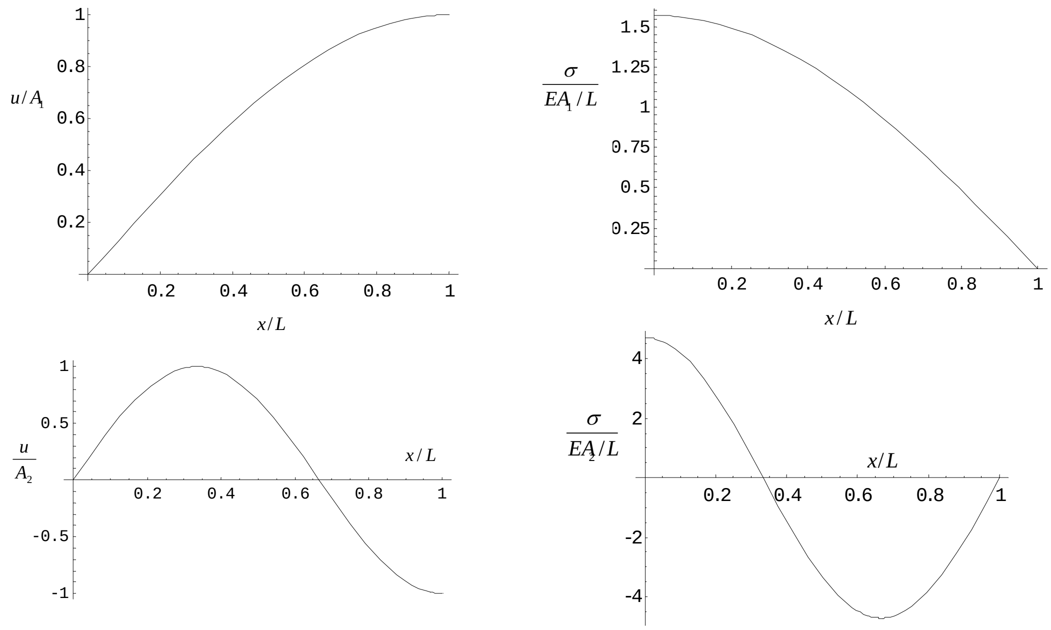
$$\omega_1 = \frac{\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}, \ \omega_2 = \frac{3\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}, \ \omega_3 = \frac{5\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2} ...$$

The rod has infinite many normal modes:

1st mode: $\omega_1 = \frac{\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $u(x,t) = A_1 \sin\left(\frac{\pi x}{2L}\right)\sin\omega_1 t$, $\sigma(x,t) = A_1 E\frac{\pi}{2L}\cos\left(\frac{\pi x}{2L}\right)\sin\omega_1 t$

2nd mode: $\omega_2 = \frac{3\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $u(x,t) = A_2 \sin\left(\frac{3\pi x}{2L}\right)\sin\omega_2 t$, $\sigma(x,t) = A_2 E \frac{3\pi}{2L}\cos\left(\frac{3\pi x}{2L}\right)\sin\omega_2 t$

and so on so forth. When excited, the motion of the rod is a superposition of all the normal modes.



The first mode is known as the **fundamental mode**. It has the lowest frequency:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{4L}\sqrt{\frac{E}{\rho}}.$$

For steel, $E = 210\text{GPa}, \rho = 7800\text{kg}$. A rod of length L = 1 m has a frequency about 1 kHz. The audible range of an average human being is between 20 Hz to 20 kHz.

Visualize engenmode. Please take a look at a video that shows the eigenmodes of a plate (http://imechanica.org/node/2004).

Finite element method for the dynamics of an elastic solid

Weak statement of momentum balance. In three-dimensional elasticity, momentum balance leads to three PDEs

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

in the body, and the three stress-traction relations

$$\sigma_{ij} n_j = t_i$$

on the surface of the body.

The momentum balance holds true if

$$\int \sigma_{ij} \frac{\partial w_i}{\partial x_j} dV = \int \left(b_i - \rho \partial^2 u_i / \partial t^2\right)^T w_i dV + \int t_i w_i dA$$

holds true for every test function $\mathbf{w}(\mathbf{x})$. It might help you memorize the above by regarding $-\rho\partial^2 \mathbf{u} / \partial t^2$ as the "inertia force"

Finite element method. Divide a body into many finite elements. Interpolate the displacement field in an element as

$$\mathbf{u} = \mathbf{N}\mathbf{q},$$

where $\mathbf{u}$ is the time-dependent displacement field inside the element, and $\mathbf{q}$ is the time-dependent nodal displacement column. The shape function matrix $\mathbf{N}$ is the same as that for static problem. The strain column is

$$\varepsilon = \mathbf{B}\mathbf{q}.$$

The stress column is

$$\sigma = \mathbf{D}\mathbf{B}\mathbf{q}.$$

These steps are the same as in static problems.

Insert these interpolations into the weak statement. Compared to the static problem, the only new term is inertia term:

$$\int \left(\rho \partial^2 \mathbf{u} / \partial t^2\right)^T \delta \mathbf{u} dV = \sum \ddot{\mathbf{q}}^T \mathbf{m} \delta \mathbf{q}.$$

The sum is carried over all elements. The mass matrix for each element is

$$\mathbf{m} = \int \rho \mathbf{N}^T \mathbf{N} dV.$$

Let $\mathbf{Q}$ be the column of displacements of all the nodes in the body. The global stiffness matrix and the global force column are assembled as before. The global mass matrix is assembled in a similar way. The weak statement takes the form

$$\left(\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} - \mathbf{F}\right)^T \delta\mathbf{Q} = 0,$$

which must holds true for every variation in the displacement column, $\delta\mathbf{Q}$. We obtain that

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{F}.$$

This is a set of ODEs for the displacement column $\mathbf{Q}(t)$.

Normal mode analysis. Consider free vibration, where $\mathbf{F} = 0$. In a normal mode, all nodes vibrate at a single frequency, and each node vibrates with its individual amplitude. That is, a normal mode takes the form

$$\mathbf{Q}(t) = \mathbf{U} \sin \omega t,$$

where $\mathbf{U}$ is the column of the amplitude of the nodal displacements, and ω is a natural frequency. Insert this expression into $\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{0}$, and we obtain that

$$\mathbf{K}\mathbf{U} = \omega^2 \mathbf{M}\mathbf{U}.$$

This is a generalized eigenvalue problem. Both the mass matrix and the stiffness matrix are positive definite. A n-DOF system has n distinct normal modes.

For example, the frequency equation for a 2DOF system is

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

or

$$\begin{bmatrix} K_{11}-\omega^2 M_{11} & K_{12}-\omega^2 M_{12} \\ K_{21}-\omega^2 M_{21} & K_{22}-\omega^2 M_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To obtain nonzero amplitude column $[U_1, U_2]$, the determinant of the matrix must vanish

$$\det \begin{bmatrix} K_{11}-\omega^2 M_{11} & K_{12}-\omega^2 M_{12} \\ K_{21}-\omega^2 M_{21} & K_{22}-\omega^2 M_{22} \end{bmatrix} = 0 .$$

This is quadratic equation for ω^2, and has two solutions, each corresponding to a normal mode.

Approximate the rod as a 1-DOF system. The computer readily assemble the stiffness matrix and mass matrix, and performs normal mode analysis. To gain some empathy for the computer, we now simulate the computer, and determine the fundamental frequency by using the PVW.

We use the single element to represent the rod. The displacement of the node on the left vanishes. The displacement of the node on the right, $q(t)$, is the degree of freedom. Interpolate the displacement inside the rod by a linear function

$$u(x,t) = \frac{x}{L} q(t).$$

From the exact solution, we know the amplitude function is sinusoidal, not linear. Thus, our assumption is wrong, resulting an approximate frequency.

Calculate the strain in the rod by $\varepsilon = \partial u / \partial x$, giving

$$\varepsilon = q / L .$$

Using Hooke's law, we obtain the stress

$$\sigma = Eq / L .$$

The variation in the displacement is

$$\delta u = \frac{x}{L} \delta q .$$

The variation in the strain is

$$\delta \varepsilon = \frac{1}{L} \delta q .$$

Inserting the above into the weak statement, we obtain that

$$\int_0^L \left(\frac{Eq}{L} \right) \left(\frac{\delta q}{L} \right) A dx = - \int_0^L \left(\rho \frac{x \ddot{q}}{L} \right) \left(\frac{x \delta q}{L} \right) A dx .$$

Evaluate the integrals, and we obtain that

$$\left(\frac{1}{3} \rho L^2 \ddot{q} + Eq \right) \delta q = 0 .$$

The PVW requires that this equation hold true for every variation δq, so that

$$\frac{1}{3} \rho L^2 \ddot{q} + Eq = 0 .$$

This ODE governs the function $q(t)$. The solution to this ODE is sinusoidal,

$$q(t) = \text{constant} \times \sin \omega t ,$$

with the frequency

$$\omega = \frac{\sqrt{3}}{L}\sqrt{\frac{E}{\rho}}.$$

We make the following comments.

- This approximate frequency takes the same form as the exact frequency, except for the numerical factor: $\sqrt{3}$ vs. $\pi / 2$. The approximate frequency is somewhat larger than the exact frequency. This trend is understood as follows. In obtaining the approximate solution, we have constrained the displacement field to a small family (i.e., the linear distribution). The constraint makes the rod appear to be more rigid, increasing the frequency.
- By approximating the rod with a single degree of freedom, we can only find one normal mode. If we want to find higher modes, we must divide the rod into more elements. View a particular normal mode as a standing wave of some wavelength. To resolve this normal mode, the element size should be smaller than a fraction of the wavelength.
- If we divide the rod into many elements, the resulting displacement distribution for the fundamental mode will approach to the sinusoidal function, and the frequency will approach to the exact value.

In a homework problem, you will appreciate these comments by dividing the rod into two linear elements.

Properties of the normal modes (I.M. Gel'fand, *Lectures on Linear Algebra*, Dover Publications)**.** What can normal modes do for us? To answer this question, we need to learn a few more facts about the normal modes.

The normal mode analysis leads to an eigenvalue problem:

$$\mathbf{KU} = \lambda \mathbf{MU}.$$

The natural frequency ω corresponds to the eigenvalue, $\lambda = \omega^2$. The amplitude column $\mathbf{U}$ corresponds to the eigenvector. The eigenvalues are roots to

$$\det[\mathbf{K} - \lambda \mathbf{M}] = 0.$$

This is a polynomial of degree *n* for a system of *n* degrees of freedom. Let the eigenvalues be

$$\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n,$$

and their associated eigenvectors be

$$\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, ..., \mathbf{U}_n.$$

To avoid a certain subtle point, we assume that the *n* eigenvalues are distinct.

Because the two matrices **M** and **K** are symmetric and positive-definite, this eigenvalue problem has several specific properties.

All eigenvalues are real and positive numbers. Let's say that an eigenvalue λ might a complex number, so that its associated eigenvector **U** would be a complex column. Denote the complex-conjugate of **U** by $\overline{\mathbf{U}}$. Multiply $\overline{\mathbf{U}}^T$ and $\mathbf{KU} = \lambda \mathbf{MU}$, giving

$$\overline{\mathbf{U}}^T \mathbf{KU} = \lambda \overline{\mathbf{U}}^T \mathbf{MU}.$$

The matrix **M** is real and symmetric, so that

$$\overline{\mathbf{U}}^T \mathbf{MU} = M_{11}\overline{U}_1 U_1 + M_{22}\overline{U}_2 U_2 + ... + M_{12}\left(\overline{U}_1 U_2 + U_1 \overline{U}_2\right) + ...$$

Copnsequently, $\overline{\mathbf{U}}^T \mathbf{MU}$ is a real number. Similarly, $\overline{\mathbf{U}}^T \mathbf{KU}$ is a real number. This proves that λ is a real number. The eigenvector **U** must also be real.

Because **M** and **K** are positive definite, $\overline{\mathbf{U}}^T \mathbf{KU}$ and $\overline{\mathbf{U}}^T \mathbf{MU}$ are positive numbers. Consequently, λ is also positive.

Eigenvectors associated with different eigenvalues are orthogonal to one another. Let λ_i and λ_j be two different eigenvalues, and $\mathbf{U}_i$ and $\mathbf{U}_j$ be their associated eigenvectors. Orthogonality here means that

$$\mathbf{U}_i^T\mathbf{M}\mathbf{U}_j = 0 .$$

If this equation holds, we also have

$$\mathbf{U}_i^T\mathbf{K}\mathbf{U}_j = \lambda_j \mathbf{U}_i^T\mathbf{M}\mathbf{U}_j = 0 .$$

To prove the orthoganality, multiply $\mathbf{KU}_i = \lambda_i \mathbf{MU}_i$ by $\mathbf{U}_j^T$, giving

$$\mathbf{U}_j^T\mathbf{K}\mathbf{U}_i = \lambda_i \mathbf{U}_j^T\mathbf{M}\mathbf{U}_i .$$

Similarly, multiply $\mathbf{KU}_j = \lambda_j \mathbf{MU}_j$ by $\mathbf{U}_i^T$, giving

$$\mathbf{U}_i^T\mathbf{K}\mathbf{U}_j = \lambda_j \mathbf{U}_i^T\mathbf{M}\mathbf{U}_j .$$

Because $\mathbf{M}$ and $\mathbf{K}$ are symmetric and real, $\mathbf{U}_j^T\mathbf{K}\mathbf{U}_i^T = \mathbf{U}_i^T\mathbf{K}\mathbf{U}_j^T$ and $\mathbf{U}_j^T\mathbf{M}\mathbf{U}_i^T = \mathbf{U}_i^T\mathbf{M}\mathbf{U}_j^T$. The difference of the above two equations gives that

$$0 = (\lambda_i - \lambda_j)\mathbf{U}_j^T\mathbf{M}\mathbf{U}_i$$

This proves that $\mathbf{U}_i^T\mathbf{M}\mathbf{U}_j = 0$.

Normalize each eigenvector. Each eigenvector is determined up to a scalar. We can choose the scalar so that the eigenvector is normalized, namely,

$$\mathbf{U}_i^T\mathbf{M}\mathbf{U}_i = 1 ,$$

and

$$\mathbf{U}_i^T\mathbf{K}\mathbf{U}_i = \lambda_i \mathbf{U}_i^T\mathbf{M}\mathbf{U}_i = \lambda_i .$$

Forced vibration. Excite a system by a periodic force column,

$$\mathbf{F}(t) = \mathbf{F}_0 \sin \Omega t ,$$

where $\mathbf{F}_0$ is the amplitude of the force column, and Ω is the **forcing frequency**. The equation of motion becomes

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{F}_0 \sin \Omega t$$

We want to determine $\mathbf{Q}(t)$. Write $\mathbf{Q}(t)$ as a liner superposition of the eigenvectors:

$$\mathbf{Q}(t) = a_1(t)\mathbf{U}_1 + a_2(t)\mathbf{U}_2 + ... + a_n(t)\mathbf{U}_n ,$$

Multiplying the equation of motion by a particular eigenvector, $\mathbf{U}_i$, we obtain that

$$\ddot{a}_i + \omega_i^2 a_i = b_i \sin \Omega t ,$$

where $b_i = \mathbf{U}_i^T\mathbf{F}_0$. This equation is identical to the equation of motion of a 1-DOF system. Thus, the *n*-DOF system is a superposition of *n* normal modes, each acting like a 1-DOF system.

This is an inhomogeneous ODE. The full solution is the sum of *all* homogeneous solution and *one* particular solution.

$$a_i(t) = A \sin \omega_i t + B \cos \omega_i t + \frac{b_i}{\omega_i^2 - \Omega^2} \sin \Omega t .$$

With damping (which is neglected here), the homogenous solution will die out, but the particular solution will persist. The particular solution is

$$a_i(t) = \frac{b_i}{\omega_i^2 - \Omega^2} \sin \Omega t \ .$$

Initial value problem. Normal modes are useful if you only care about a few modes, say to avoid resonance or perform vibration control. If you are interested in dynamic disturbance of some wavelength much smaller than the overall structure size, such as impact and wave propagation, you may want to evolve the displacement field in time. Given the initial displacements and velocities, as well as the external forces, $\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{KQ} = \mathbf{F}$ is a set of ODE that evolves the nodal displacements over time. This is a standard numerical problem. ABAQUS provides this option.