

# Vibrations of Cantilever Beams: Deflection, Frequency, and Research Uses

For: Dr. Negahban

EngrM 325H

Scott Whitney

April 23, 1999

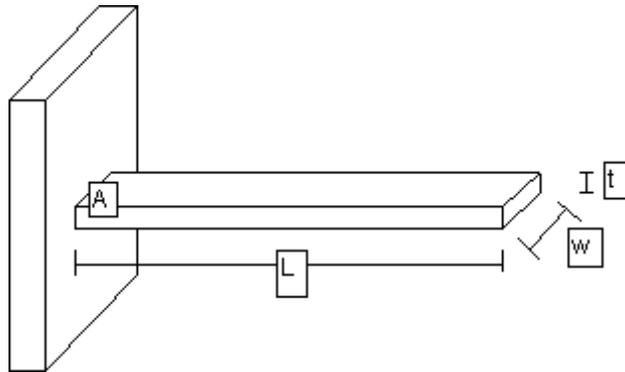
## Introduction

Measurements of thin film properties are difficult when compared to bulk materials. One method for finding the modulus of elasticity of a thin film is from frequency analysis of a cantilever beam. A straight, horizontal cantilever beam under a vertical load will deform into a curve. When this force is removed, the beam will return to its original shape; however, its inertia will keep the beam in motion. Thus, the beam will vibrate at its characteristic frequencies. If a thin film is sputtered onto the beam, the flexural rigidity will be altered.

This change causes the frequency of vibrations to shift. If the frequency shift is measured, the film's elastic modulus can be calculated.

## Thin Cantilever Beam Setup

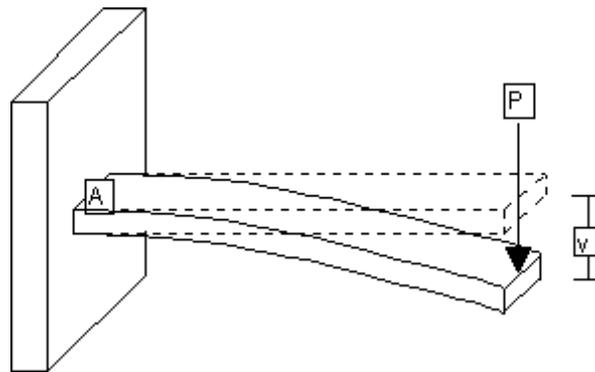
Beams studied in this paper are long, thin, cantilever beams. Figure 1, below, shows such a beam. One end of the beam is fixed, while the other end is free. The origin of the coordinate axis is at the fixed end, point A. A typical beam, used in this study, is  $L = 30$  mm long,  $w = 5$  mm wide, and  $t = 0.5$  mm thick. The beam material must be chosen so that its stiffness differs from the thin film's stiffness so that the frequency shift is significant. This study involved zinc oxide films (about  $5 \mu\text{m}$  thick), so a good substrate is aluminum.



**Figure 1: Typical cantilever beam studied**

## Deflection of Cantilever Beam

If the free end of a cantilever beam is subjected to a point load,  $P$ , the beam will deflect into a curve. See Figure 2 below. The larger the load, the greater the deflection,  $v(x)$ .



**Figure 2: Cantilever beam deflection under load at fixed end**

Assuming the beam undergoes small deflections, is in the linearly elastic region, and has a uniform cross-section, the following equations can be used (Gere, p. 602).

The curvature of the beam,  $\kappa$ , is equal to the second derivative of the deflection

$$\kappa = \frac{\partial^2 v}{\partial x^2}$$

The curvature can also be related to the bending moment,  $M$ , and the flexural rigidity,  $EI$ ,

$$\kappa = \frac{M}{EI}$$

where  $E$  is the elastic modulus of the beam and  $I$  is the moment of inertia. The bending moment in a beam can be related to the shear force,  $V$ , and the lateral load,  $q$ , on the beam. Thus,

$$M = EI \frac{\partial^2 v}{\partial x^2} \quad V = EI \frac{\partial^3 v}{\partial x^3} \quad q = -EI \frac{\partial^4 v}{\partial x^4} \quad (1a,b,c)$$

For the load shown in Figure 2, the distributed load, shear force, and bending moment are:

$$q(x) = 0 \quad V(x) = P \quad M(x) = -PL\left(1 - \frac{x}{L}\right)$$

Thus, the solution to Equation (1a) is

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \int_{x=0}^x M(x) dx = -\frac{PL}{EI} \left( x - \frac{x^2}{2L} \right) \\ v(x) &= \int_{x=0}^x \frac{\partial^2 v}{\partial x^2} dx = -\frac{PL}{EI} \left( \frac{x^2}{2} - \frac{x^3}{6L} \right) \end{aligned} \quad (2a)$$

At the free end of the beam, the displacement is:

$$v(L) = -\frac{PL^3}{3EI} \quad (2b)$$

## Vibrations of Beams

### -Determination of Equations

When the force,  $P$ , is removed from a displaced beam, the beam will return to its original shape. However, inertia of the beam will cause the beam to vibrate around that initial location. Assuming the elastic modulus, inertia, and cross sectional area ( $A$ ) are constant along the beam length, the equation for that vibration is (Volterra, p. 310)

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} = -\lambda_m \frac{\partial^2 v(x,t)}{\partial t^2} \quad (3)$$

where  $\lambda_m = \rho A$  is the linear mass density of the beam. Equation (3) is best solved by separation of variables (Atkins, p. A29). Assume that the displacement can be separated into two parts, one depends on position and another on time.

$$v(x,t) = X(x)f(t) \quad (4)$$

where  $X$  is independent of time, and  $f$  is independent of position. Then equation (3) becomes

$$EI \frac{\partial^4 (X(x)f(t))}{\partial x^4} = -\lambda_m \frac{\partial^2 (X(x)f(t))}{\partial t^2}$$

Dividing equation (3) by  $\lambda_m X(x)f(t)$ :

$$\frac{EI}{\lambda_m X(x)} \frac{\partial^4 X(x)}{\partial x^4} = - \frac{1}{f(t)} \frac{\partial^2 f(t)}{\partial t^2} \quad (5)$$

Since the left side of equation (5) does not change as  $t$  varies, the right side must be a constant. Similarly since the right half of equation (5) does not change as  $x$  varies, the left half must be a constant. Because each side is constant, equation (4) is valid and the method of separation of variables may be used. Let this constant be denoted  $\tilde{\omega}_n^2$ . It can be shown that  $\tilde{\omega}_n^2$  is a real quantity, and that  $\tilde{\omega}_n$  are natural frequencies of the beam.

Equation (5) can now be written as two differential equations (Volterra, p. 311),

$$\frac{\partial^4 X}{\partial x^4} - k_n^4 X = 0 \quad \frac{\partial^2 f}{\partial t^2} + \tilde{\omega}_n^2 f = 0 \quad (6a,b)$$

where

$$k_n^4 = \frac{\tilde{\omega}_n^2 \lambda_m}{EI} \quad (6c)$$

In order to solve equation (6a), the following boundary conditions for a cantilever beam are needed

$$v(0,t) = X(0) = 0 \quad \frac{\partial v(0,t)}{\partial x} = \frac{\partial X(0)}{\partial x} = 0$$

$$\frac{\partial^2 v(L,t)}{\partial x^2} = \frac{\partial^2 X(L)}{\partial x^2} = 0 \quad \frac{\partial^3 v(L,t)}{\partial x^3} = \frac{\partial^3 X(L)}{\partial x^3} = 0$$

These boundary conditions come from the supports of a cantilever beam. The fixed end must have zero displacement and zero slope due to the clamp. The free end cannot have a bending moment or a shearing force.

The general solution to equation (6a) is a linear combination of trigonometric equations (Volterra, p.312)

$$X(x) = C_1 \left[ \cos(k_n x) + \cosh(k_n x) \right] + C_2 \left[ \cos(k_n x) - \cosh(k_n x) \right]$$

$$+ C_3 \left[ \sin(k_n x) + \sinh(k_n x) \right] + C_4 \left[ \sin(k_n x) - \sinh(k_n x) \right] \quad (7)$$

Using the first boundary condition,  $C_1$  may be found

$$X(0) = 0 = C_1 [2] \quad \Rightarrow \quad C_1 = 0$$

Taking the first derivative of equation (7) and using the second boundary condition,  $C_3$  may be found

$$\frac{\partial X(x)}{\partial x} = C_2 \left[ -\sin(k_n x) - \sinh(k_n x) \right] + C_3 \left[ \cos(k_n x) + \cosh(k_n x) \right] + C_4 \left[ \cos(k_n x) - \cosh(k_n x) \right]$$

$$\frac{\partial X(0)}{\partial x} = 0 = C_3 [2] \quad \Rightarrow \quad C_3 = 0$$

The higher derivatives of equation (7), and remaining boundary conditions give

$$\frac{\partial^3 X(L)}{\partial x^3} = C_2 \left[ -\cos(k_n L) - \cosh(k_n L) \right] + C_4 \left[ -\sin(k_n L) - \sinh(k_n L) \right] = 0 \quad (8a)$$

$$\frac{\partial^3 X(L)}{\partial x^3} = C_2 \left[ \sin(k_n L) - \sinh(k_n L) \right] + C_4 \left[ -\cos(k_n L) - \cosh(k_n L) \right] = 0 \quad (8b)$$

Equations (8a) and (8b) can be combined to give

$$C_4 = C_2 \frac{-\cos(k_n L) - \cosh(k_n L)}{\sin(k_n L) + \sinh(k_n L)} \quad (9)$$

Thus for a cantilever beam, equation (7) reduces to

$$X_n(x) = C_2 \left\{ \left[ \cos(k_n x) - \cosh(k_n x) \right] + \left[ \frac{-\cos(k_n L) - \cosh(k_n L)}{\sin(k_n L) - \sinh(k_n L)} \right] \left[ \sin(k_n x) - \sinh(k_n x) \right] \right\} \quad (10)$$

According to Volterra, p. 312, the constants  $C_n$  are arbitrary. However, in order for the dynamic solution for the displacement to be equal to the static solution (at time  $t=0$ ),  $C_2$  must be equal to  $1/2$ . With this value for  $C_2$ ,  $X_n(0)=0$  and  $X_n(L)=1$ .

Plugging equation (9) into either (8a) or (8b) will lead to the frequency equation for a cantilever beam,

$$\cos(k_n L) \cosh(k_n L) = -1 \quad (11)$$

The frequency equation can be solved for the constants,  $k_n L$ ; the first six are shown below in Figure 3 (note,  $k_n=0$  is ignored since it implies that the bar is at rest because  $\hat{\omega}_n=0$ ). These constants along with equation (6c) can be used to find the natural frequencies of a cantilever beam.

n	$k_n L$	$A_n$ (m)	Freq (Hz)
1	1.875104069	-2.689E-04	329.7
2	4.694091133	6.795E-06	2066.2
3	7.854757438	-8.667E-07	5785.5
4	10.99554073	2.257E-07	11337.2
5	14.13716839	-8.259E-08	18741.3
6	17.27875953	3.701E-08	27996.2

**Figure 3: Constants for a cantilever beam vibrations. Note, since  $\cosh(x)$  is large when  $x$  is large,  $k_n L$  needs to be found with high precision.**

For each frequency, there exists a characteristic vibration (Volterra, p. 319)

$$x_n(x,t) = X_n(x) \left[ A_n \cos(\hat{\omega}_n t) + B_n \sin(\hat{\omega}_n t) \right] \quad (12)$$

where  $A_n$  depends on the initial position at time  $t=0$ , and  $B_n$  depends on the initial velocity. In this study, the beam starts its vibration when displaced and at rest. Thus  $B_n=0$  and

$$A_n = \frac{2}{L} \int_0^L x(x,t=0) X_n(x) dx \quad (13)$$

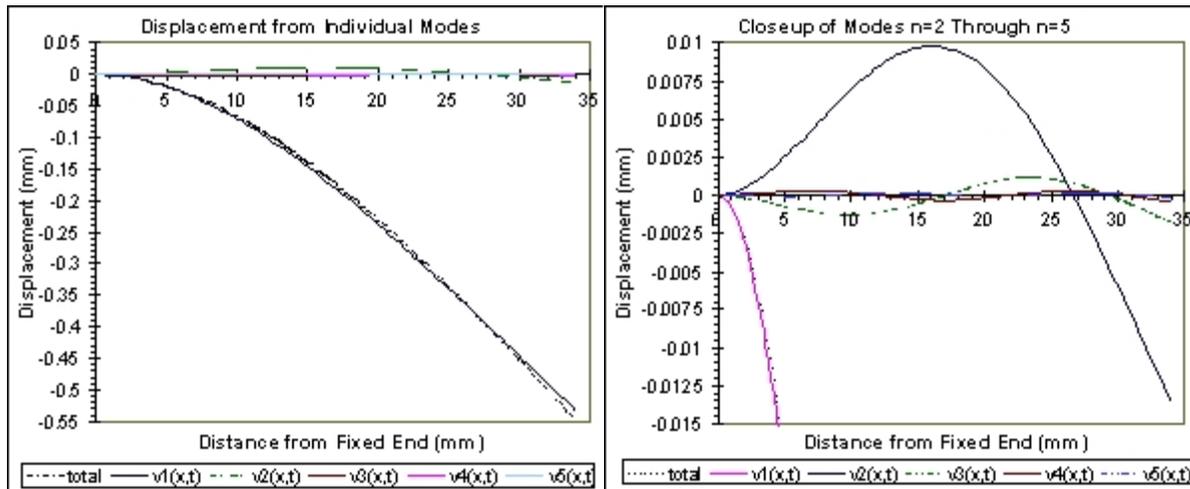
The initial displacement,  $v(x,0)$ , was found above, equation (2a). Equation (13) can be solved analytically by

a computer math program,

$$A_n = \left[ \frac{-4PL}{EIA_n k_n^4 (\sin(k_n L) e^{k_n L} + e^{2k_n L} - 1)} \right] * \left[ \begin{array}{l} 3 \sin(k_n L) (e^{2k_n L} + 1) - 2(k_n L)^3 e^{k_n L} \\ + \cos(k_n L) (3 - (k_n L)^3 (e^{2k_n L} + 1) - 3e^{2k_n L}) \end{array} \right]$$

### -Examples of Vibration Modes

Typical values of  $A_n$  and frequency (for beams used in this study) are shown in Figure 3, above. Notice that only the first few modes of vibration have significantly large values for the constant  $A_n$ . Thus, the higher order vibration modes can be ignored. To get the total displacement of the cantilever beam, simply add all the displacements found in equation (12) for each mode. Below, Figure 4 shows the displacement caused by each mode at  $t=0$ ; also included is the total initial displacement of the beam. From these displacements it is apparent that even the third characteristic mode has little effect on the total displacement of the beam.



**Figure 4: Initial displacement caused by each mode.**

As time progresses, each mode will vibrate around the zero displacement line with the frequency listed in Figure 3, above. Figure 5, below, shows this vibration for the first two modes, higher modes act similarly. The total beam motion is complex; each characteristic mode vibrates with a different size, shape, and frequency.

The following files can be downloaded to view an animation of the first few modes and total vibration of a cantilever beam. To view these, the Lotus Screencam Player must be downloaded: [Scplayer.exe](#)

- Total motion: [325Htot.scm](#)
- 1st mode: [325h1st.scm](#)
- 2nd mode: [325h2nd.scm](#)
- 3rd mode: [325h3rd.scm](#)
- 4th mode: [325h4th.scm](#)

Figure 6, below, shows a reading from a piezoelectric sensor located near the fixed end of the beam; this sensor shows the interaction of each vibration mode. Also shown is the vibration predicted from equation (12) at that location. The actual and predicted signals are nearly identical; the major difference between them is damping (both viscous and Kelvin-Voigt) of the higher order oscillations in the actual signal.

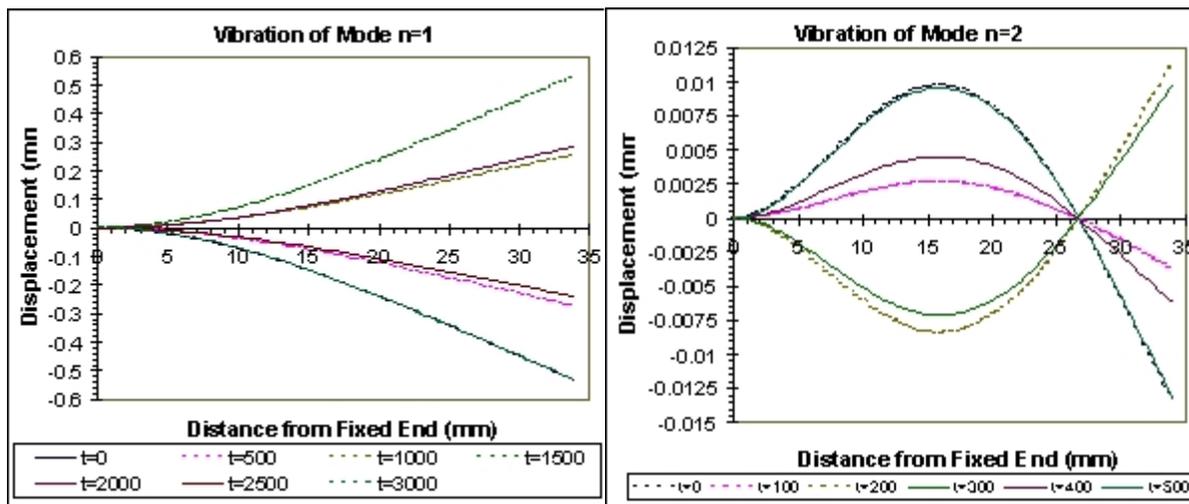


Figure 5: Vibrations of first two characteristic modes; other modes vibrate similarly. Time,  $t$ , is in microseconds.

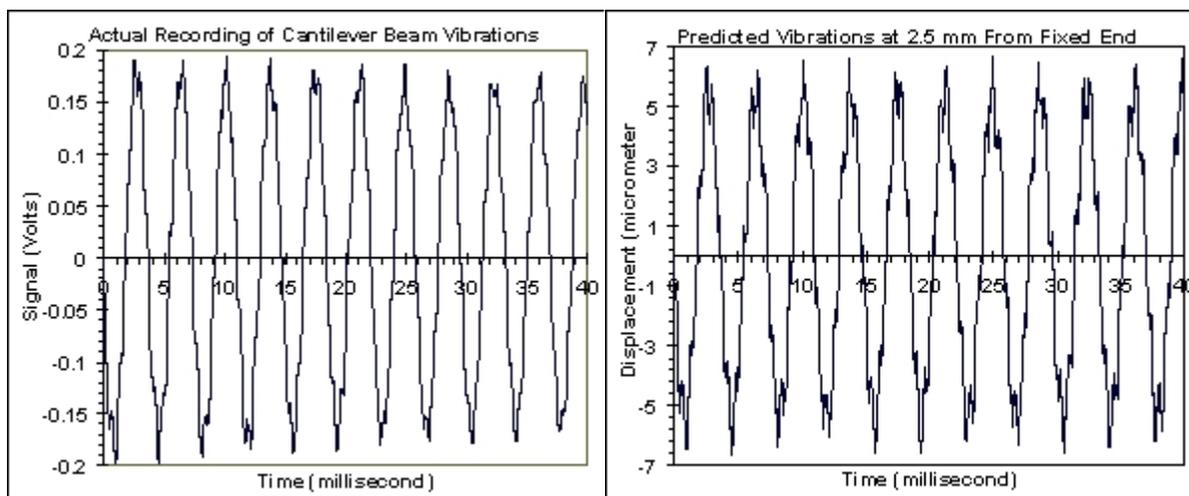
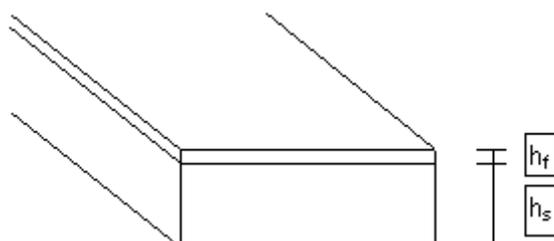


Figure 6: ZnO signal from a vibrating cantilever beam compared to theoretical vibrations.

## Measurement of Modulus of Elasticity for Thin Films

-Reason for Using Frequency Analysis

If a cantilever beam is sputter coated with a thin film, then the flexural rigidity will change. A change in stiffness will directly affect the frequency of the beam's vibrations. Thus, the elastic modulus of the film can be determined from this frequency shift. See Figure 7 for a schematic drawing of a sputtered beam; the subscript 's' refers to the substrate and 'f' refers to the film. Note, the height of the film is greatly exaggerated, since  $h_s \approx 100h_f$ .



### Figure 7: Cantilever Beam Sputter Coated with Thin Film

Theoretically, the stiffness could be determined from a stationary beam under a load,  $P$ , see equations (2a,b). Suppose the flexural rigidity was increased by adding a thin film to the beam. Then, assuming the applied load was constant, the deflection at the free end will decrease. For a typical beam described above, this displacement change is extremely small. For example, an aluminum beam with initial displacement (at the free end) of 0.55 mm will have an initial displacement of 0.49 mm with a relatively thick (8  $\mu$ m) ZnO film. The 0.06 mm change is too small to be measured with any conventional tools, and a thinner ZnO film (1  $\mu$ m) will give an immeasurable difference in displacement. Thus, stationary measurements of the film's modulus of elasticity are difficult, if not impossible.

However, the same beam coated with ZnO will have its fundamental frequency,  $\hat{\omega}_1$ , shift from 329.7 Hz to 342.0 Hz. Any oscilloscope can be used to measure the beam's vibrations, see Figure 6, above. If enough data points are taken, a Fourier transform of the signal can measure frequency shifts even as small as 0.2 Hz. Therefore, the frequency shift can then be known with high accuracy, and the film's elastic modulus can be calculated.

#### -Determination of Equations

All of the equations derived in the previous sections (1-13) can still be applied to a two component beam if several of the constants are changed:  $EI$  is replaced with  $E_s I_s + E_f I_f$  and  $\lambda_m$  is corrected as shown below. The assumptions stated previously must still be fulfilled: small deflections, linearly elastic, and uniform cross-section. The latter assumption implies that the film thickness cannot vary along the beam (difficult to do when sputter coating long beams).

The linear mass density of the coated beam is now

$$\lambda_{mf} = \rho_s A_s + \rho_f A_f$$

The center of mass as well as the inertia of the coated beam will shift

$$y_{cm} = \frac{E_s h_s^2 + E_f (2h_s h_f + h_f^2)}{2E_s h_s + 2E_f h_f}$$

$$I_s = \frac{w h_s^3}{12} + w h_s \left( y_{cm} - \frac{h_s}{2} \right)^2$$

$$I_f = \frac{w h_f^3}{12} + w h_f \left( \frac{h_f}{2} + h_s - y_{cm} \right)^2$$

Equation (6c) is used to measure the modulus of elasticity of the film from the frequency shift.

$$\Delta \hat{\omega}_n = k_n^2 \left[ \sqrt{\frac{E_s I_s + E_f I_f}{\lambda_{mf}}} - \sqrt{\frac{EI}{\lambda_m}} \right] \quad (14)$$

$$E_f = \frac{\lambda_{mf}}{I_f} \left( \frac{\Delta \hat{\omega}_n}{k_n^2} + \sqrt{\frac{EI}{\lambda_m}} \right)^2 - \frac{E_s I_s}{I_f} \quad (15)$$

Since the amplitudes of higher vibration modes are so small, see Figure 4, equation (15) is solved using the fundamental frequency,  $n=1$ .

## Conclusion

Due to the small size of thin films, conventional methods of measuring their properties often do not work. These thin film properties may differ from the bulk material properties. Therefore, alternative measurement methods must be developed. Vibration of cantilever beams is one of these methods. It is not limited to just determining the modulus of elasticity; other useful information, such as piezoelectric constants, can be determined from cantilever beams.

## Works Cited

Atkins, P. W. *Physical Chemistry*. 5<sup>th</sup> ed. New York: W. H. Freeman and Company, 1994.

Gere, J. M., Timoshenko, S. P. *Mechanics of Materials*. 4<sup>th</sup> ed. Boston: PWS Publishing Company, 1997.

Voltera, E., Zachmanoglou, E. C. *Dynamics of Vibrations*. Columbus, Charles E. Merrill Books, Inc., 1965.