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What is light?

Third version: On Thursday December 1, 2011, I gave an undergraduate colloquium talk on the subject of the title, and I wrote a 21 pages long text, including 150 footnotes, despite a warning by Knuth,[1] that footnotes tend to be distracting, although he added that *The History of the Decline and Fall of the Roman Empire* by Gibbon would not have been the same without footnotes,[2] and since I have not yet read it I cannot say if my excessive use of footnotes resembles that of Gibbon, but I also wonder if the organized attacks on the western academic systems have taken into account his analysis of the decline and then the collapse of the mighty Roman empire.

Although I articulated my first text along a somewhat chronological order, I was told that it is difficult to read, so that I wrote a second version by reorganizing the information contained in the first text, with small additions, and I put at the end of each chapter additional footnotes on people mentioned in the footnotes but not in the main text of the chapter, arranged in alphabetical order.

In this third version, I tried to improve the wording of my remarks about how good or bad or silly some models are, and about the efficient ways which engineers or physicists have for dealing with the necessary approximate models which one creates along the way for understanding how nature functions, and a mathematician's approach is different since he/she needs to separate what was proved from what is guessed. A part of my goal is to put all the known information together in order to assess which mathematical tools were useful, what can be explained more clearly now, and what type of mathematical tools are still missing for improving the understanding from a mathematician's point of view.

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[1] Donald Ervin Knuth, American mathematician, born in 1938. He worked at Caltech (California Institute of Technology), Pasadena, CA, and at Stanford University, Stanford, CA. He developed TEX, which I use for preparing all my texts, even those which do not include any mathematical sign.

[2] Edward Gibbon, English historian, 1817–1877.

Chapter I: reflection of light.

Ia: *flat mirrors*

Mirrors are based on reflection of light. The earliest mirrors were pieces of polished stone such as obsidian, a naturally occurring volcanic glass, and some obsidian mirrors found in Anatolia (now part of Turkey) have been dated to around 6 000 BCE;[3,4] polished stone mirrors from Central and South America appear around 2 000 BCE. Mirrors of polished copper appear in Mesopotamia around 4 000 BCE, in Egypt around 3 000 BCE, and bronze mirrors appear in China around 2 000 BCE.[5] Mirrors made of other metal mixtures like speculum metal may have also been produced in China and India,[6] but mirrors of speculum metal or any precious metal were hard to produce.

Metal-coated glass mirrors are said to have been invented in Sidon (now in Lebanon) in the first century, and glass mirrors backed with gold leaf are mentioned by Pliny in his *Natural History*,[7] written around 77. The Romans made crude mirrors by coating blown glass with molten lead. The Chinese began making mirrors using silver-mercury amalgams around 500, and it was not until the Renaissance that one found in Europe a better method of coating glass with a tin-mercury amalgam, and it was used in Venezia (Venice) in the 16th century.[8] The invention of the silvered-glass mirror is due to Liebig,[9] but nowadays mirrors are usually produced by the vacuum deposition of aluminum onto the glass.

In a thin flat glass coated on one side with a much thinner metallic substance, it is the metallic coating which is the mirror, which is then held by the transparent glass in front of it, and I wonder why the necessary presence of metal in a mirror had not given a hint earlier that *light must have something to do with electricity*, since metals are known for their high conductivity (of heat or electricity).

Ib: *curved mirrors*

Parabolic mirrors were described and studied by Diocles,[10] but his work *On Burning Mirrors* was only known through quotes by ancient authors, until a translation from Greek to Arabic was found (in the Shrine Library in Mashhad, Iran) and then translated into English in 1976 by Toomer.[11] Although there is a "well known" story that Archimedes used burning mirrors against the Romans,[12] Rashed argues that the story seems invented,[13] and that what Diocles wrote only concerned the work of theoreticians knowledgeable about conic sections, and that he did not say that any of them went to the practical step of constructing such mirrors. Three centuries later, Ptolemy worked with curved polished iron mirrors, and

[3] BCE = before common era.

[4] Unlike geologists, archaeologists do not trust carbon 14 dates for inanimate objects, so that the mirrors must have been found in strata dated by other considerations, or by carbon 14 dates for previously living samples.

[5] Bronze is an alloy of copper and tin, and modern bronze is typically 88% copper and 12% tin. Brass is an alloy of copper and zinc, and what one calls commercial bronze (90% copper and 10% zinc) and architectural bronze (57% copper, 3% lead, 40% zinc) are actually brass alloys.

[6] Speculum metal is a mixture of around two-thirds copper and one-third tin making a white brittle alloy, which can be polished to make a highly reflective surface.

[7] Gaius Plinius Secundus (known as Pliny the Elder), Roman author, naturalist, and naval and army commander, 23–79.

[8] Because of the secrecy concerning the mercury process, Venetian mirrors faced no competition from France or England until the 17th century, when the information became known (by some kind of industrial espionage).

[9] Justus von Liebig, German chemist, 1803–1873. He worked in Giessen and in München (Munich), Germany.

[10] Diocles of Carystus, Greek mathematician, 240 BCE–180 BCE.

[11] Gerald James Toomer, English-born historian of sciences, born in 1934. He worked at Brown University, Providence, RI.

[12] Archimedes, Greek mathematician, 287 BCE–212 BCE. He worked in Siracusa (Syracuse), then a Greek colony, now in Italy.

[13] Roshdi Hifni Rashed, Egyptian-born historian of science, born in 1936. He worked at CNRS (Centre National de la Recherche Scientifique) in Paris, France.

discussed plane, convex spherical, and concave spherical mirrors in his work *Optics*,[14] and more than eight centuries after, parabolic mirrors were also described by IBN SAHL,[15] while IBN AL-HAYTHAM (also known as ALHAZEN) wrote about concave and convex mirrors in cylindrical or spherical geometries a little after.[16] That a parabolic mirror focuses all the light coming parallel to its axis through its focus seems attributed to KEPLER.[17]

Ic: *is the moon a mirror?*

Around 450 BCE in Athens, ANAXAGORAS was sent to prison,[18] because he claimed that the sun is not a god (but a kind of red-hot stone) and that the moon reflects the sun's light.[19]

In February 1982, I visited Scuola Normale Superiore in Pisa, at the invitation of Ennio DE GIORGI,[20] and while in Pisa I was told by some newly met friends that I should go see an interesting exhibit in Firenze (Florence), of a manuscript by Leonardo (DA VINCI),[21] known then as Codex Hammer,[22] after the man who purchased it (in 1980 from the Leicester estate);[23] it was renamed Codex Leicester after W.H. GATES bought it at an auction in 1994 (for 30.8 million dollars).[24] One of the questions which Leonardo discussed in these "pages" was about the light from the moon being the reflected light from the sun.[25] As many other discoveries from the past, it had been forgotten and then rediscovered, but not everyone was convinced and someone had made the following observation: if one applies the law of reflection of light, there is only one point of the moon which can reflect the light from the sun directly into one's eye (at a given point of earth), so that one should only see one bright spot on the moon and not the totality of the illuminated part. Leonardo's suggestion was that there are seas on the moon and that, because of the waves, there is always the possibility to receive light from every illuminated point on the moon; there were skeptics arguing that there are no waves because there is no wind or, as one would say now, there are no seas on the moon.

Although reality is not according to Leonardo's hypothesis,[26] it contains the seed for an important improvement: Leonardo could have noticed that the size of the waves does not matter and that only the

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[14] Claudius PTOLEMY, Greek/Egyptian astronomer, 85–165. He worked in Alexandria, Egypt.

[15] Abu Sa'd al-'Ala IBN SAHL, Persian mathematician, c. 940–1000. He worked in Baghdad, now capital of Iraq. It was only found in 1990 (by RASHED) that IBN SAHL had in his 984 work *On Burning Mirrors and Lenses* a formula equivalent to the law of refraction.

[16] ALHAZEN (Abu 'Ali al-Hasan ibn al-Hasan IBN AL-HAYTHAM), Persian mathematician, 965–1040.

[17] Johannes KEPLER, German-born mathematician, 1571–1630. He worked in Graz, Austria, in Prague, now capital of the Czech republic, and in Linz, Austria, where the Johannes Kepler University is now named after him.

[18] ANAXAGORAS of Clazomenae, Ionian-born Greek philosopher and mathematician, 499 BCE–428 BCE. He worked in Athens, Greece, and then in Lampsacus, Ionia (now in Turkey).

[19] While in prison, ANAXAGORAS considered the problem of squaring the circle with straightedge and compass, and it is the oldest mention of that question. He was saved from prison by PERICLES, who was his friend, but he then had to leave Athens, so that all the story looked like a way the opponents of PERICLES had found to send the philosopher back to Ionia, where he came from.

[20] Ennio DE GIORGI, Italian mathematician, 1928–1996. He received the Wolf Prize in 1990, for his innovating ideas and fundamental achievements in partial differential equations and calculus of variations, jointly with Ilya PIATETSKI-SHAPIRO. He worked at Scuola Normale Superiore, Pisa, Italy.

[21] Leonardo DA VINCI, Italian artist, engineer and scientist, 1452–1519. He worked in Milano (Milan) and in Firenze (Florence), Italy.

[22] Armand HAMMER, American businessman, 1898–1990.

[23] The manuscript had been bought in 1717 by Thomas COKE, who was created Earl of Leicester in 1744.

[24] William Henry GATES III, American businessman and philanthropist, born in 1955.

[25] The Leicester codex is made of separate large sheets of paper, equivalent to two pages, where Leonardo put down some of his ideas which he planned to use later. His drawings are easy to understand, but his words are difficult to read, not only because he wrote in Italian, but because his handwriting only makes sense when seen in a mirror.

[26] There may be a few wrong guesses which are believed to be the truth in physics, chemistry, biology, and which are taught to generations of students, possibly even after some discrepancy has been discovered, making it more difficult for students to participate in the necessary detective work for discovering the truth.

angles made by the waves are important, so that he could have observed that the same result is expected for infinitesimal waves, and deduced that a rough surface made of a reflecting material may reflect light in many directions;[27] he could also have noticed that on earth there are no rocks which behave as mirrors, so that there is no reason to expect rocks on the surface of the moon to be made from a perfectly reflecting material, hence the question should have been dismissed until one obtained more information about the surface of the moon.

It is easy for us to argue in this way nowadays, but there was no one to teach Leonardo all these things which were not yet "known", and he was one of the brightest man of his time, so that I find useful to think about NEWTON's sentence,[28] *If I have seen a little further it is by standing on the shoulders of Giants.*[29] Indeed, our knowledge comes from what we learn from our teachers, and what we read in books (or on the Internet), so that the authors of these books become our teachers, hence we rely on the work of those before us who have analyzed the knowledge transmitted to them by the preceding generations, and we rely not only on those who created some new knowledge in the past, but also on those who after them simplified a few arguments and reorganized some previous work in order to make it simpler, hence easier to teach to a new generation of students, which is then given more chances to go further on the path of discovery.

Additional footnotes: BROWN N.,[30] COKE,[31] DESCARTES,[32] GRESHAM,[33] HOOKE,[34] LUCAS,[35] PERICLES,[36] PIATETSKI-SHAPIRO,[37] WOLF.[38]

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[27] One would say now that a rough surface scatters light in many directions.

[28] Sir Isaac NEWTON, English mathematician, 1643–1727. He worked in Cambridge, England, holding the Lucasian chair (1669–1701). There is an Isaac Newton Institute for Mathematical Sciences in Cambridge, England. The unit of force is named after him: a Newton is the force necessary to accelerate the unit of mass (a kilogram) to the unit of acceleration (a meter per square second).

[29] In a letter to HOOKE (dated February 5, 1676) NEWTON had written: *What Descartes did was a good step. You have added much several ways, and especially in taking the colours of thin plates into philosophical consideration. If I have seen a little further it is by standing on the shoulders of Giants.*

[30] Nicholas BROWN Jr., American merchant, 1769–1841. Brown University, Providence, RI, is named after him.

[31] Thomas COKE, 1st Earl of Leicester, wealthy English land-owner, 1697–1759. He was created Earl of Leicester in 1744.

[32] René DESCARTES, French mathematician and philosopher, 1596–1650. Université de Paris 5 is named after him. The terms Cartesian coordinates and Cartesian products are derived from his name (written in Latin as CARTESIUS, possibly DES CARTES in French).

[33] Sir Thomas GRESHAM, English merchant and financier, 1519–1579. He left the money used for founding Gresham College in London, England, in 1597.

[34] Robert HOOKE, English mathematician, 1635–1703. He was professor of geometry at Gresham College in London, England. He had ideas about light and about gravitation earlier than NEWTON (who may have read about them before developing his own ideas), but he lacked the technical ability for further mathematical developments. Hooke's law in elasticity is named after him.

[35] Reverend Henry LUCAS, English clergyman and philanthropist, 1610–1663. The Lucasian chair in Cambridge, England, is named after him.

[36] PERICLES, Greek (Athenian) statesman and general, 495 BCE–429 BCE.

[37] Ilya PIATETSKI-SHAPIRO, Russian-born mathematician, born in 1929. He received the Wolf Prize in 1990, for his fundamental contributions in the fields of homogeneous complex domains, discrete groups, representation theory and automorphic forms, jointly with Ennio DE GIORGI. He worked at Tel-Aviv University, Tel Aviv, Israel.

[38] Ricardo WOLF, German-born inventor, diplomat and philanthropist, 1887–1981. He emigrated to Cuba before World War I; from 1961 to 1973 he was Cuban Ambassador to Israel, where he stayed afterwards. The Wolf Foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900–1981, "to promote science and art for the benefit of mankind".

Chapter II: refraction of light.

IIa: *from Ptolemy to ibn Sahl*

Ptolemy had studied refraction around 140, but his formula is only accurate for small incidence angles. In 1990, Rashed published a translation of two fragments (found in two different libraries, one in Damascus, Syria, and one in Tehran, Iran) of a manuscript by ibn Sahl, *On Burning Mirrors and Lenses*, where the explanations about a drawing clearly show that ibn Sahl knew an equivalent form of the law of refraction, and he used it in order to find the shapes of anaclastic lenses, i.e. those which focus light with no geometric aberration. It is unlikely that any European had heard about the work of ibn Sahl on refraction before Rashed discovered it, since al Haytham who wrote just after him did not mention it, although he knew about Ptolemy's work.

IIb: *the invention of the telescope*

What the surface of the moon looks like was discovered by Galileo (Galilei) around 1610,[1] when he pointed his telescope (which magnified 20 times) towards the moon, and he observed its craters, but it now seems that Hariot had built a telescope and pointed it towards the moon before Galileo,[2] so that Hariot might be the inventor of the telescope, before one the three possible inventors of the telescope in The Netherlands, Janssen,[3] Lippershey,[4] both from Middelburg (in the province of Zeeland), and Metius,[5] from Alkmaar (in the province of Noord Holland); Galileo only heard about the invention (as a spy glass), and it was enough for him to understand how to make a telescope, on the basis of what he knew about refraction of light, and he learned to polish lenses for the purpose, because manufacturers did not produce what he needed.

IIc: *Hariot, Snell, Mydorge, and Descartes*

Galileo must have understood enough about refraction for building an efficient telescope, but it does not seem that he stated a precise law for refraction, while Hariot is said to have discovered the law of refraction in 1601 or 1602, about twenty years before Snell,[6] but none of them published anything about it. Descartes did not publish the sine-law immediately after discovering it, but he published it in 1637 in *La Dioptrique*, following *Le Discours de la Méthode*. Neither Hariot nor Snell expressed the law of refraction in the usual sine-law, and actually ibn Sahl had not expressed it in this way either, but Mydorge explained in 1626 a way to compute the angle of refraction which implies the sine-law,[7] and since he was a good friend of Descartes he may have expressed something which he learned from him;[8] however, the first time Descartes expressed himself the law in the way he published it in 1637 is in a 1632 letter which he wrote to Golius,[9] who may have mentioned to him some of the observations on conic sections from the manuscripts in Arabic which he collected.

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[1] Galileo Galilei, Italian mathematician, 1564–1642. He worked in Siena, in Pisa, in Padova (Padua), Italy, and again in Pisa. Galilean invariance is named after him.

[2] Thomas Hariot, English scientist, 1560–1621. He built a telescope and pointed it towards the moon before Galileo, and he discovered the law of refraction before Snell or Descartes, but he did not publish any of his 7 000 manuscript pages.

[3] Zacharias Janssen, Dutch spectacle-maker, 1580–1638. A possible inventor of the telescope, with Hans Lippershey and Jacobus Metius.

[4] Hans Lippershey, German-born lensmaker, 1570–1619. A possible inventor of the telescope, with Zacharias Janssen and Jacobus Metius.

[5] Jacobus Metius, Dutch instrument-maker, 1571–1624 (or 1631). A possible inventor of the telescope, with Zacharias Janssen and Hans Lippershey.

[6] Willebrord Snellius (Willebrord Snel van Royen), Dutch mathematician and astronomer, 1580–1626. He worked in Leiden, The Netherlands. His name is attached to the law of refraction of light, although it was discovered before him by Hariot in 1602, long after ibn Sahl, who had written about it in 984.

[7] Claude Mydorge, French mathematician, 1585–1647.

[8] Since I had previously read (and then repeated) that Descartes had published his law in 1626, it could be that someone considered the publication of Mydorge as an idea of Descartes.

[9] Jacobus Golius (Jacob van Gool), Dutch orientalist and mathematician, 1596–1667.

IId: *the controversy between Fermat and Descartes*

Descartes then spent almost four years in writing letters for answering a few objections made by mathematicians, among them Fermat,[10] who apparently did not focus on any of the physical or metaphysical considerations of Descartes, but on mathematical logic, although I read an argument of Fermat against Descartes which is about physics: Descartes must have mentioned a similarity with the propagation of sound in solids, so that Fermat pointed out that it is the opposite, that sound goes faster in denser solids,[11] while light goes slower in denser solids.[12]

One must remark that the speed of light was only measured in the last part of the 17th century, after both had died, and that the speed of sound in solids was only estimated in the first part of the 19th century, but in some way both of them were partially right: the similarity will be seen in the 1690 argument of Huygens concerning propagation of waves,[13] but the difference is that propagation of sound in solids is a question of *elasticity*, which only made sense in the 19th century after the work of Cauchy,[14] and of Lamé,[15] while propagation of light is a question of *electromagnetism*, which only made sense in the 19th century after the work of Maxwell,[16] who unified electricity and magnetism, and after the simplifications by Heaviside.[17]

After a few years, Descartes and Fermat stopped talking to each other, i.e. they stopped exchanging letters, since Descartes lived in the Dutch province of Holland after 1628 (with short visits to France in 1644, 1647, and 1648), while Fermat lived in the south of France, in Toulouse.

Fermat restarted the controversy in 1657 with the partisans of Descartes (who had died seven years earlier), and he had a new idea, a principle of least time (now named after him), but it was only in 1662 that he clearly showed that it implies the law of sines.

It is possible that Fermat criticized a type of argument which I call pseudo-logic: a game A implies B, and one observes something looking like B, but one mistakenly "deduces" that nature plays game A. Actually, the argument of Fermat concerning the principle of least time is logically flawed in this way: it is just a game whose consequence is the sine-law.

However, one must observe that such a discussion between Descartes and Fermat occurred before the invention of *differential calculus*, so that there could not have been a discussion yet about what form

[10] Pierre de Fermat, French mathematician, 1601–1665. He worked (as a lawyer and government official) in Toulouse, France. There are a few "theorems" attributed to him, but since he rarely explained his proofs in his letters, one should probably call them conjectures, and then some famous mathematicians (like Euler) proved or improved most of what he had said in letters.

[11] The first estimates of the speed of sound in solids seem to be due to Chladni, and to Biot, in the first years of the 19th century, so that Fermat was obviously guessing, but he was wrong, since the speed of sound in lead is about a third of the speed of sound in copper.

[12] Fermat may have been aware that the index of refraction of glass is about 1.5, and that addition of lead increases the index of refraction, up to 1.7.

[13] Christiaan Huygens, Dutch mathematician, astronomer and physicist, 1629–1695. He worked in Paris, France, and The Hague, The Netherlands. The Huygens–Fresnel principle is partly named after him.

[14] Augustin Louis Cauchy, French mathematician, 1789–1857. He was made baron by Charles x. He worked in Paris, France, went in exile after the 1830 revolution and worked in Torino (Turin), Italy, returned from exile after the 1848 revolution, and worked in Paris again. The Cauchy stress tensor in elasticity is named after him. Cauchy sequences are named after him, but were introduced before by Bolzano. The Cauchy–Schwarz inequality is partly named after him, but was proved before by Bunyakovsky.

[15] Gabriel Lamé, French mathematician, 1795–1870. He worked in St. Petersburg, Russia and in Paris, France. Lamé's system in linearized elasticity is named after him.

[16] James Clerk Maxwell, Scottish-born physicist, 1831–1879. He worked in Aberdeen, Scotland, in London, and in Cambridge, England, where he held the first Cavendish professorship of physics (1871–1879). Maxwell equation, which I call Maxwell–Heaviside equation, is named after him.

[17] Oliver Heaviside, English engineer, 1850–1925. He worked as a telegrapher, in Denmark, in Newcastle upon Tyne, England, and then did research on his own, living in the south of England. He transformed equations written by Maxwell into the system which one uses now under the name Maxwell equation, which I call Maxwell–Heaviside equation.

the laws of nature could take (and in those days physics was called *natural philosophy*): after differential calculus was invented, it became possible to talk about *ordinary differential equations*, which is the language for *classical mechanics*, the 18th century point of view about mechanics, which deals with rigid bodies, i.e. systems having a finite number of degrees of freedom; it then became possible to talk about *partial differential equations*, which is the language for *continuum mechanics*, the 19th century point of view about mechanics, which deals with deformable bodies (solids or fluids), all having an infinite number of degrees of freedom, but with no small scale effects which put us in the realm of the 20th century point of view about mechanics and physics, for which the mathematical tools have not been completely developed yet. Of course, it does not mean that all the mathematical questions concerning classical mechanics were solved in the 18th century, and all the mathematical questions concerning continuum mechanics were solved in the 19th century, but my point is that it was difficult for mathematicians and philosophers to discuss logically in the middle of the 17th century about some issues about light, since *light is not about geometry* but about solutions of particular systems of partial differential equations.

Additional footnotes: BIOT,[18] BOLZANO,[19] BUNYAKOVSKY,[20] CASORATI,[21] CAVENDISH,[22] Charles X,[23] CHLADNI,[24] EULER,[25] FRESNEL,[26] FULLER,[27] George II,[28] SCHWARZ,[29] WEIERSTRASS.[30]

[18] Jean-Baptiste BIOT, French mathematician and physicist, 1774–1862. He worked in Beauvais, and he held a chair at Collège de France (physique mathématique, 1801–1862) in Paris, France.

[19] Bernhard Placidus Johann Nepomuk BOLZANO, Czech mathematician and philosopher, 1781–1848. He worked in Prague (then in Austria, now capital of the Czech Republic). He introduced “Cauchy sequences” a few years before CAUCHY did. The Bolzano–Weierstrass theorem is partly named after him.

[20] Viktor Yakovlevich BUNYAKOVSKY, Ukrainian-born mathematician, 1804–1889. He worked in St Petersburg, Russia. He studied with CAUCHY in Paris (1825), and he proved the “Cauchy–Schwarz” inequality in 1859, 25 years before SCHWARZ.

[21] Felice CASORATI, Italian mathematician, 1835–1890. He worked in Pavia and in Milano (Milan), Italy. The Casorati–Weierstrass theorem (that in any neighbourhood of an essential singularity of a function of one complex variable it comes arbitrarily close to any given value) is partly named after him, but he included it in his 1868 treatise on complex numbers, while WEIERSTRASS only proved it in an article in 1876.

[22] Henry CAVENDISH, English physicist and chemist (born in Nice, not yet in France then), 1731–1810. He lived in London, England. He founded the Cavendish professorship of physics at Cambridge, England.

[23] Charles-Philippe de France, 1757–1836, comte d'Artois, duc d'Angoulême, pair de France, was king of France from 1824 to 1830 under the name Charles X.

[24] Ernst Florens Friedrich CHLADNI, German physicist and musician, 1756–1827.

[25] Leonhard Paul EULER, Swiss-born mathematician, 1707–1783. He worked in St Petersburg, Russia, in Berlin, Germany, and then again in St Petersburg. A few of the subjects to which his name is attached are the Euler equation for inviscid fluids, the Euler φ function, the Euler Γ function, and the Euler constant.

[26] Augustin-Jean FRESNEL, French engineer, 1788–1827. He worked in Paris, France. He invented the Fresnel lens for lighthouses, with many applications today.

[27] John FULLER, English politician and philanthropist, 1757–1834. He instituted Fullerian professorship in chemistry and in physiology at the Royal Institution of Great Britain, London, England.

[28] Georg Augustus, 1683–1760. Duke of Brunswick-Lüneburg (Hanover), he became king of Great Britain and Ireland in 1727, under the name of George II. Georg-August-Universität in Göttingen, Germany, is named after him.

[29] Hermann Amandus SCHWARZ, German mathematician, 1843–1921. He worked at ETH (Eidgenössische Technische Hochschule), Zürich, Switzerland, and in Berlin, Germany. The Cauchy–Schwarz inequality is partly named after him, but was proved before by BUNYAKOVSKY.

[30] Karl Theodor Wilhelm WEIERSTRASS, German mathematician, 1815–1897. He first taught in high schools in Münster, in Braunsberg, Germany, and then he worked in Berlin, Germany. The Bolzano–Weierstrass theorem is partly named after him. The Weierstrass theorem of approximation by polynomials is named after him. The Casorati–Weierstrass theorem (that in any neighbourhood of an essential singularity of a function of one complex variable it comes arbitrarily close to any given value) is partly named after him, but he published it in 1876, and CASORATI had included it in his 1868 treatise on complex numbers.

Chapter III: scalar wave equations.

IIIa: *the speed of light c*

Galileo had also pointed his telescope towards Jupiter and observed four of its moons.[1] Io,[2] the moon nearest to Jupiter, orbits in 42 hours 27 minutes 21 seconds, and some discrepancy was found along the months, since the period did not change but Io was late when Jupiter was further away from the earth. CASSINI,[3] probably before he became director of the Paris observatory, may have been the first to attribute this discrepancy to the fact that it took a different time to light for traveling from Io (or Jupiter) to the earth,[4] but then he looked for a more conventional explanation. It is RØMER,[5] who completed his observations made in Denmark with those which he made in Paris (as an assistant to CASSINI!), who is then credited now for the first measurement of the speed of light c, in 1676.

IIIb: *the argument of* HUYGENS

In 1690, HUYGENS introduced another idea, that light has a wave nature: his argument was to consider a plane wave-front propagating with velocity c in the medium of index 1 with incidence angle $i \neq 0$, and imagining that once the wave touches a boundary point on the interface with the medium of index n (and such a point "moves" with speed $\frac{c}{\sin i}$),[6] this point generates a centered wave in the medium of index n which propagates at speed $\frac{c}{n}$, and the envelope of the corresponding spherical fronts at time t is a plane wave-front in the medium of index n with refraction angle r, and one must have $\frac{c/n}{\sin r} = \frac{c}{\sin i}$, i.e. $\sin i = n \sin r$.

Said otherwise, it might be what the argument of DESCARTES was about: working in $\mathbb{R}^2$ with the x_1 axis as interface,[7] the medium of index 1 occupying the region $x_2 > 0$ and the medium of index n occupying the region $x_2 < 0$,[8] a plane wave propagating (downward) in the direction of the unit vector $a_- = \begin{pmatrix} \sin i \\ -\cos i \end{pmatrix}$ corresponds to a function $f\big((x, a_-) - c\,t\big)$ in the medium of index 1, and it creates a refracted wave propagating (downward) in the direction of the unit vector $b_- = \begin{pmatrix} \sin r \\ -\cos r \end{pmatrix}$, which corresponds to a function $h\big((x, b_-) - \frac{c}{n}\,t\big)$ in the medium of index n, and one then wants these two functions to coincide on the interface, so that one must have $f\big(x_1 \sin i - c\,t\big) = h\big(x_1 \sin r - \frac{c}{n}\,t\big)$ for all x_1, t: unless f and h are the same constant, one deduces that $f(z) = h\big(\frac{z}{n}\big)$ for all z and $\sin i = n \sin r$.

Unfortunately, like for the least time principle of FERMAT, the argument of HUYGENS (and possibly DESCARTES) is just an example of pseudo-logic: inventing a game which makes the observed relation $\sin i = n \sin r$ appear does not explain what is going on.[9] Actually, it is contradicted by the next step in the mathematical understanding: HUYGENS could have made a better guess, by realizing that the points on the

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[1] Jupiter, chief god in the Roman mythology, corresponding to Zeus in the Greek mythology.

[2] Io, nymph in the Greek mythology, seduced by Zeus, who transformed her into a heifer.

[3] Jean-Dominique (Giovanni Domenico) CASSINI, Italian-born astronomer, 1625–1712. He worked in Bologna, Italy, and in Paris, France, where he was the first director of the observatory.

[4] On the earth's orbit around the sun, the distance between the point nearest to Jupiter to the point farthest to Jupiter is about the diameter of the earth's orbit (around the center of mass of the solar system, so that it looks like it orbits around the sun), which is around 300 million kilometers, and it takes light around 1 000 seconds (16 minutes 40 seconds) to travel this distance.

[5] Ole Christensen RØMER, Danish astronomer, 1644–1710. He worked in Copenhagen, Denmark, and in Paris, France.

[6] That this "apparent velocity" $\frac{c}{\sin i}$ is greater than the speed of light does not contradict the hyperbolic character of the equation, whose maximum speed is c, since the velocity of plane waves is a *phase velocity*, different from a *group velocity*, which is the physical one.

[7] Working in $\mathbb{R}^3$ would also show that if $i \neq 0$, then the refracted ray is in the plane containing the incidence ray and orthogonal to the interface.

[8] This is the traditional picture, corresponding to air above water, but IBN SAHL drew a picture with a vertical interface, with the medium of index 1 on the left and the medium of index n on the right.

[9] Nevertheless, such games are useful for engineers or physicists, who go forward and will later discard a part of the games which are contradicted by other measurements. Actually, what I criticize is not the use of various games, but the fact that one often does not mention the already known defects of previous models.

interface should also generate centered waves (propagating at speed c) in the medium of index 1, and that the envelope of the corresponding spherical fronts at time t is a reflected plane wave-front in the medium of index 1, but if one takes into account both the reflected wave and the refracted wave, *the argument is incomplete*, since it gives no way for comparing the corresponding strengths of the various waves, incident, reflected, and refracted.

In the case of mirrors, there is an intuitive notion of *reflectivity*, since a dark metal reflects less than a clear metal, and similarly, in a bright sunny day, a wall painted in white obviously reflects more than a wall painted in black, so that besides the question of understanding what colours are about, there is a question of reflectivity of a surface to understand.

Maybe one could have asked a question: if for a bad mirror, only a small part of the light is reflected, what happens to the rest of the light? Of course, the idea that various quantities are conserved was not yet in the air, and if one thinks about *linear momentum* and *angular momentum*, it was difficult to talk about them before the invention of differential calculus. When he studied the pendulum, Galileo could have guessed that there is a notion of *potential energy* which can be transformed into *kinetic energy*, and the *total energy* (sum of the two) is constant; actually, he could also have observed that some damping mechanisms are at work, but it was much too early to think that the lost energy is transformed into something, which one calls "heat",[10] although it still is a badly understood notion.

IIIc: *differential calculus and the calculus of variations*

With the invention of differential calculus by NEWTON, and the improvements by LEIBNIZ,[11] it became possible to obtain information on the path of a "ray of light" according to Fermat's principle of least time,[12] and a branch of mathematics called *calculus of variations* was developed:[13] if the index of refraction varies smoothly one finds that rays of light are smooth curves, and in the limit of a piecewise constant index of refraction with smooth interfaces one finds that rays of light are continuous broken lines satisfying the law of sines at each crossing of an interface, but already one could have criticized the idea that "rays of light take the *shortest path in time* for going from A to B", and observe that it is obviously inexact, since the physical problem (in this approximation of *geometrical optics* which was the only one at the time) is that a ray of light starts at A in some direction, and then a differential equation permits to tell where it goes. Actually, there could be more than one direction at A which will make the corresponding rays of light go through B, with possibly different times taken between A and B, and if there are obstacles (where the index of refraction may be considered to be $+\infty$) there could be no direction, so that if there is only one light source, at A, then B is in the *shadow*.

IIId: *the 1-dimensional wave equation*

D'ALEMBERT must have noticed that in one dimension a wave propagating to the right with velocity c means considering a function of position and time of the form $f(x - c\,t)$,[14] while a wave propagating to the left with velocity $-c$ means considering a function of the form $g(x + c\,t)$; he must then have observed that a wave equation (at least a linear one where waves moving in opposite directions do not interact) should have solutions of the form $f(x - c\,t) + g(x + c\,t)$, and he looked for an equation having this property. Of

[10] Galileo was probably the first to think about a thermometer, although his own invention was just a thermoscope.

[11] Gottfried Wilhelm VON LEIBNIZ, German mathematician, 1646–1716. He worked in Frankfurt, in Mainz, Germany, in Paris, France, and in Hanover, Germany, but never in an academic position.

[12] In reflection by a mirror, HERON observed around 60 that light travels along the path of least length.

[13] One should be aware that nowadays this term of calculus of variations is often used wrongly: it should only be applied to questions of optimality when the unknowns are geometrical objects, so that it is not as simple as computing the derivative of a functional defined on an open set of a functional space (which was well developed in the beginning of the 20th century and goes with the name of functional analysis). There is an unfortunate tendency nowadays to create new names in mathematics, corresponding to very narrow specialties, and to distort the meaning of classical names, probably because the mathematical culture has been slowly shrinking for too long a time.

[14] Jean LE ROND, known as D'ALEMBERT, French mathematician, 1717–1783. He worked in Paris, France. He was co-editor of the Encyclopédie with DIDEROT.

course, this is another game of pseudo-logic, since it will not prove that nature plays with the equation which one invents. D'ALEMBERT observed that $a(x,t) = f(x - c\,t)$ satisfies $A_+ a = 0$, where A_+ is the operator $\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial x}$, and that $b(x,t) = g(x + c\,t)$ satisfies $A_- b = 0$, where A_- is the operator $\frac{\partial}{\partial t} - c\,\frac{\partial}{\partial x}$, and since A_+ and A_- commute, one has $A_+ A_- a = A_+ A_- b = 0$, so that $a + b$ satisfies $A_+ A_-(a + b) = 0$, and $A_+ A_-$ is the (one-dimensional) wave operator $\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$; of course, D'ALEMBERT also made the observation that every solution U of $\frac{\partial^2 U}{\partial t^2} - c^2\frac{\partial^2 U}{\partial x^2} = 0$ can be decomposed as $U(x,t) = F(x - c\,t) + G(x + c\,t)$ for some functions F, G, defined up to the addition of a constant (since for a constant C one may replace F by $F + C$ and G by $G - C$). Of course, one needs some smoothness hypotheses for doing these computations.

D. BERNOULLI almost discovered the one-dimensional wave equation in a much more physical way than what I just described,[15] which may not be the way D'ALEMBERT thought about it: D. BERNOULLI tried to understand the sound made by a violin chord, i.e. relate the frequency of vibration to the tension in the chord and its length.[16] He modeled the chord and the tension by little masses connected by springs, using Newton's law (that force is mass times acceleration) and Hooke's law (that the change in length of an elastic rod is proportional to the force with which one pulls on the rod) which he only *used in a linearized way*,[17] and it gives a system of differential equations, for which one easily computes the eigen-values and the eigen-functions, which is enough for what D. BERNOULLI was interested in; in this way the one-dimensional wave equation is hidden, since it only appears after a limiting process,[18] increasing the number of masses while scaling down each mass, and adapting the strength of the springs, and it is a little tricky: it was the subject of one lecture in a graduate course whose lecture notes became my third book *From Hyperbolic Systems to Kinetic Theory – A Personalized Quest*, namely chapter 25 (D. Bernoulli: from Masslets and Springs to the 1-D Wave Equation) and the next step, describing how CAUCHY went further by inventing the theory of (linearized) elasticity, is chapter 26 (Cauchy: from Masslets and Springs to 2-D Linearized Elasticity).

What is important to observe, is that there are many different physical situations which may produce similar mathematical equations to solve, eventually with different *boundary conditions*, so that a mathematician may see a similarity because the equations look the same, while a physicist may say that some phenomena are extremely different.

IIIe: *the 3-dimensional wave equations*

For comparing various equations, one needs to go to a more realistic three-dimensional situation, beginning by the (scalar) wave equation in an *isotropic medium*,

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta\, U = 0, \text{ with } \Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}, \tag{1}$$

which was studied by POISSON,[19] the Laplacian operator Δ having been used before for stationary problems,[20] in gravitation, electricity, or magnetism.

Then, one considers the equation of (linearized) elasticity, where the unknown is a displacement u, with components u_1, u_2, u_3 satisfying

$$\frac{\partial}{\partial t}\Big(\rho\,\frac{\partial u_i}{\partial t}\Big) - \sum_{j=1}^{3} \frac{\partial \sigma_{i,j}}{\partial x_j} = 0 \text{ for } i = 1, 2, 3, \tag{2}$$

[15] Daniel BERNOULLI, Swiss mathematician, 1700–1782. He worked in St Petersburg, Russia, and in Basel, Switzerland.

[16] Just before the beginning of a concert, there is a strange noise made by the musicians tuning their instruments, and for the violinists it consists in varying the tension of the chords.

[17] Hooke's law is still true if one pushes slightly on the rod, but nonlinear effects appear easily if one pushes too much, and the rod buckles.

[18] Indeed, one starts from a problem in classical mechanics, with a finite number of degrees of freedom, and one wants to describe a problem in continuum mechanics, with infinitely many degrees of freedom.

[19] Siméon Denis POISSON, French mathematician, 1781–1840. He worked in Paris, France.

[20] Pierre-Simon LAPLACE, French mathematician, 1749–1827. He was made comte in 1806 by Napoléon I and marquis in 1817 by Louis XVIII. He worked in Paris, France.

where ρ is the (possibly variable) density, and σ is the (symmetric) Cauchy *stress tensor*, which satisfies the (linearized) constitutive relation

$$\sigma_{i,j} = \sum_{k,\ell=1}^{3} C_{i,j,k,\ell}\varepsilon_{k,\ell} \text{ for } i,j = 1,2,3, \tag{3}$$

where the (possibly variable) elasticity coefficients $C_{i,j,k,\ell}$ satisfy some symmetry requirements, as well as some kind of *ellipticity condition*, and the symmetric (linearized) *strain tensor* ε is given by

$$\varepsilon_{k,\ell} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\Big), \text{ for } k,\ell = 1,2,3. \tag{4}$$

In this linearized setting, an isotropic medium corresponds to the special case of (3) given by

$$\sigma_{i,j} = 2\mu\,\varepsilon_{i,j} + \lambda\,\delta_{i,j}\sum_{k=1}^{3}\varepsilon_{k,k} \text{ for } i,j = 1,2,3, \tag{5}$$

where $\mu > 0$ is the shear modulus, λ is the Lamé parameter (satisfying some inequalities), and δ is the Kronecker symbol;[21] one should notice that practitioners put the accent on stress more than on displacements,[22] so that instead of the parameters μ, λ, they prefer to use the Young modulus E,[23] and the Poisson ratio ν. For linearized elasticity in an isotropic medium with ρ, μ, λ constant, the system (2) takes the form of the Lamé system

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} - (\mu+\lambda)\,\Delta\,u_i - \lambda\,\frac{\partial\big(div(u)\big)}{\partial x_i} = 0, \text{ for } i = 1,2,3, \text{ with } div(u) = \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}, \tag{6}$$

so that $div(u)$ satisfies a (scalar and isotropic) wave equation

$$\frac{\partial^2\big(div(u)\big)}{\partial t^2} - c_P^2\Delta\,\big(div(u)\big) = 0, \text{ with } c_P^2 = \frac{\mu+2\lambda}{\rho}, \tag{7}$$

which describes P-waves (or pressure waves, or primary waves), which are *longitudinal waves* where the particles move in the direction of propagation of the wave, and each component of $\omega = curl(u)$ satisfies a (scalar and isotropic) wave equation[24]

$$\frac{\partial^2\omega_i}{\partial t^2} - c_S^2\Delta\,\omega_i = 0, \text{ for } i = 1,2,3, \text{ with } c_S^2 = \frac{\mu+\lambda}{\rho}, \tag{8}$$

which describes S-waves (or shear waves, or secondary waves), which are *transverse waves* where the particles move in a direction orthogonal to the direction of propagation of the wave. The name of the waves is related to seismic waves, which are important in earthquakes, but *propagation of sound* in an isotropic solid is also

[21] Leopold KRONECKER, German mathematician, 1823–1891. He worked in Berlin, Germany.

[22] The elastic range of real materials is not infinite, and past some threshold, which depends upon stress, they enter a plasticity domain, or they just break.

[23] Thomas YOUNG, English scientist, 1773–1829. He worked at the Royal Institution in London, England. He then practiced as a physician, and he did some deciphering from the Rosetta stone, not as decisive as he thought, as the final deciphering of Egyptian hieroglyphs by CHAMPOLLION showed. The Young modulus in elasticity is named after him.

[24] One has $\omega_i = \sum_{j,k=1}^{3}\varepsilon_{i,j,k}\frac{\partial u_k}{\partial x_j}$ for $i = 1,2,3$, where ε is the completely skew-symmetric tensor, so that $\varepsilon_{i,j,k} = 0$ if two of the indices i,j,k coincide, and it is the signature of the permutation $(1,2,3)\mapsto(i,j,k)$ if the three indices are different.

about using (7),[25] which then is quite different from the one-dimensional wave equation, and has not much to do either with propagation of light.

One may have discontinuities of the coefficients, so that the partial differential equations should be understood *in the sense of distributions*, following the work of Laurent SCHWARTZ,[26] which came after some pioneering work by Sergei SOBOLEV,[27] and by Jean LERAY:[28] in the case of piece-wise smooth coefficients showing a discontinuity along a smooth interface, it is equivalent to writing the partial differential equations in a classical way on each side, and adding adapted transmissions conditions at the interface.

IIIf: *reflection for a 3-dimensional scalar wave equation*

With this understanding about three-dimensional wave equations, we can now look at something which resembles the reflection of light, but which is not satisfactory, because it has not much to do with light. For a scalar wave equation in a half space $x_3 > 0$, there are two types of boundary conditions which show a reflection effect, a Dirichlet condition,[29]

$$\frac{\partial^2 U_D}{\partial t^2} - c^2 \Delta\, U_D = 0, \text{ in } x_3 > 0, \text{ and } U_D = 0 \text{ at } x_3 = 0, \tag{9}$$

or a Neumann condition,[30]

$$\frac{\partial^2 U_N}{\partial t^2} - c^2 \Delta\, U_N = 0, \text{ in } x_3 > 0, \text{ and } \frac{\partial U_N}{\partial x_3} = 0 \text{ at } x_3 = 0. \tag{10}$$

Considering a downward incident unit vector $a_- = (\,\alpha, \beta, -\gamma\,)$, and the corresponding upward reflected unit vector $a_+ = (\,\alpha, \beta, +\gamma\,)$, with $\alpha^2 + \beta^2 + \gamma^2 = 1$ and $\gamma > 0$, there is a large family of explicit solutions of (9)

[25] As mentioned before, FERMAT seemed to have the wrong idea that the speed of sound in a solid is an increasing function of ρ, but (as pointed out by my collaborator Amit ACHARYA) it is the ratio of the stiffness to the density which appears in the formula (7). In the beginning of the 19th century, CHLADNI estimated the speed of sound by the note emitted by a rod when one hits it: since the wave rushes back and forth from one end of the rod to the other, the frequency of vibration is then the ratio of the speed of sound to the length; hitting a rod of lead and a rod of copper (of the same length) shows then easily that sounds propagates faster in copper than in lead.

[26] Laurent SCHWARTZ, French mathematician, 1915–2002. He received the Fields Medal in 1950 for his work in functional analysis. He worked in Nancy, in Paris, France, at École Polytechnique, which was first in Paris (when he was my teacher in 1965–1966), and then in Palaiseau, and at Université Paris 7 (Denis Diderot), Paris.

[27] Sergei L'vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Leningrad, in Moscow, and in Novosibirsk, Russia. I first met him when I was a student, in Paris in 1969, then at the International Congress of Mathematicians in Nice in 1970, and conversed with him in French, which he spoke perfectly (all educated Europeans did learn French in the beginning of the 20th century). I only met him once more, when I traveled with a French group from INRIA (Institut National de la Recherche en Informatique et Automatique) in 1976 to Akademgorodok near Novosibirsk, Russia, where he worked. There is now a Sobolev Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia.

[28] Jean LERAY, French mathematician, 1906–1998. He received the Wolf Prize in 1979, for pioneering work on the development and application of topological methods to the study of differential equations, jointly with André WEIL. He worked in Nancy, France, in a prisoner of war camp in Austria (1940–1945), at IAS (Institute for Advanced Study), Princeton, NJ, and he held a chair at Collège de France (théorie des équations différentielles et fonctionnelles, 1947–1978) in Paris, France.

[29] Johann Peter Gustav LEJEUNE DIRICHLET, German mathematician, 1805–1859. He worked in Breslau (then in Germany, now Wrocław, Poland), in Berlin, and at Georg-August-Universität, Göttingen, Germany. Dirichlet series, and the Dirichlet conditions are named after him. The Dirichlet principle was named after DIRICHLET by RIEMANN, who was probably unaware that GAUSS and GREEN had used the same idea before him.

[30] Franz Ernst NEUMANN, German mathematician, 1798–1895. He worked in Königsberg, then in Germany, now Kaliningrad, Russia.

or (10), corresponding to an incident plane wave described by an arbitrary (smooth) shape function f, and its corresponding reflected plane wave:

$$\begin{aligned} U_D(x,t) &= f\big((x,a_-)-c\,t\big)-f\big((x,a_+)-c\,t\big) \text{ in } x_3\geq 0,\\ U_N(x,t) &= f\big((x,a_-)-c\,t\big)+f\big((x,a_+)-c\,t\big) \text{ in } x_3\geq 0. \end{aligned} \tag{11}$$

In the one-dimensional wave equation, the situation considered by D. BERNOULLI comes with a natural Dirichlet condition, because the vibrating chord is fixed at its ends, and the function u is the displacement perpendicular to the chord, since the vibration of a violin string is a transverse wave.[31] For the three-dimensional elasticity model, one natural condition is to have $u=0$, i.e. $u_1=u_2=u_3=0$ (like if an elastic body is glued to a rigid body showing a flat surface), or a traction free condition, $\sigma_{i,3}=0$ for $i=1,2,3$; for the P-waves, the condition that $div(u)$ is 0 at the boundary does not seem natural, but for the S-waves, the component $\omega_3=\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}$ is 0 at the boundary if u is 0.

So there are wave equations for which an effect of reflection can be studied, but it will not help in understanding the similar questions about light if one does not identify why one needs a metal for making a mirror, and with the intuition that it might be related to conduction of electricity, the mechanical model of elasticity invoked above might be of little help. However, it helps for learning some general mathematical properties about wave propagation phenomena, and this might have been considered good enough for DESCARTES, who may have missed the points which were bothering FERMAT, but in the end FERMAT did not either understand the physics behind light, and he only came up with another type of mathematical insight about waves, which is not really satisfactory since it does not answer the basic physical question of quantifying the part of a wave which is reflected and the part which is refracted.

IIIg: *refraction for a 3-dimensional scalar wave equation*

Let us see now what the preceding models say about refraction, and for this question we do not have the choice of which boundary conditions to impose at the interface, since they are imposed to us by the way we write the equation (in order that they are compatible with using discontinuous coefficients). In order to understand the case of a non-homogeneous (isotropic) material, where the speed is a function $v(x)$ (the same in all directions), I now consider the equation[32]

$$w_{tt}-div\big(v^2grad(w)\big)=0 \text{ in the sense of distributions,} \tag{12}$$

where I now write ψ_t for $\frac{\partial\psi}{\partial t}$, and I use an orthonormal basis for writing $grad(w)$ for the vector whose ith component is $\frac{\partial w}{\partial x_i}$, and the divergence of a vector field D is $div(D)=\sum_i\frac{\partial D_i}{\partial x_i}$, but I use only two space variables x_1,x_2 for simplification. If $v(x)=c$ for $x_2>0$ and $v(x)=\frac{c}{n}$ for $x_2<0$ (with $n>1$), we shall see that the construction of HUYGENS does not correspond to a solution of the wave equation (12), except for a special incidence angle i_* for which there is no reflected wave.[33]

If one writes $a_\pm=\begin{pmatrix}\sin i\\ \pm\cos i\end{pmatrix}$ and $b_\pm=\begin{pmatrix}\sin r\\ \pm\cos r\end{pmatrix}$, one should consider functions of the form

$$w(x,t)=\begin{cases} f\big((x,a_-)-c\,t\big)+g\big((x,a_+)-c\,t\big) \text{ for } x_2>0\\ h\big((x,b_-)-\frac{c}{n}\,t\big) \text{ for } x_2<0\end{cases}, \tag{13}$$

[31] A violinist uses her/his bow by moving it perpendicularly to the string in order to create a transverse movement, and the bow must be in contact with the chord since the creation of the motion is due to friction. The same one-dimensional wave equation may also model a longitudinal wave, for example by considering a metallic rod and hitting it at one end with a hammer, in the direction of the rod, and this will send a wave along the rod, where the molecules of the metal mostly move in the direction of the rod, which is the direction of propagation of the wave.

[32] The wave equation in an anisotropic medium is $\rho\,\frac{\partial^2 w}{\partial t^2}-\sum_{i,j}\frac{\partial}{\partial x_i}\big(A_{i,j}\frac{\partial w}{\partial x_j}\big)=0$, hence the isotropic case is $\rho\,w_{tt}-div\big(\rho\,v^2grad(w)\big)=0$, and a discontinuity of ρ requires an adaptation of the computations shown.

[33] The special incidence angle i_* and refracted angle r_* correspond to $n\cos i_*=\cos r_*$, and it gives $i_*+r_*=\frac{\pi}{2}$, $\sin i_*=\cos r_*=\frac{n}{\sqrt{n^2+1}}$, $\cos i_*=\sin r_*=\frac{1}{\sqrt{n^2+1}}$, $\tan i_*=n$, $\tan r_*=\frac{1}{n}$.

i.e. an incident wave (corresponding to f) and a reflected wave (corresponding to g) in $x_2 > 0$, and a refracted wave (corresponding to h) in $x_2 < 0$; there are conditions of continuity to impose at the interface: w must be continuous (since $grad(w)$ contains the derivative $\frac{\partial w}{\partial x_2}$), and $v^2 \frac{\partial w}{\partial x_2}$ must be continuous (since its partial derivative in x_2 appears in the wave equation), so that one must add to (13) the continuity conditions

$$\begin{aligned} &f(x_1 \sin i - c\,t) + g(x_1 \sin i - c\,t) = h\Big(x_1 \sin r - \frac{c}{n}\,t\Big), x, t \in \mathbb{R} \\ &-\cos i\, f'(x_1 \sin i - c\,t) + \cos i\, g'(x_1 \sin i - c\,t) = \frac{-\cos r}{n^2}\, h'\Big(x_1 \sin r - \frac{c}{n}\,t\Big), x, t \in \mathbb{R}. \end{aligned} \tag{14}$$

The first equation gives $f(z) + g(z) = h\big(\frac{z}{n}\big)$ and the law of sines $\sin i = n \sin r$, but we should not stop here and be happy. Since $f'(z) + g'(z) = \frac{1}{n}\, h'\big(\frac{z}{n}\big)$, the second equation becomes $n \cos i\,\big(g'(z) - f'(z)\big) = -\cos r\,\big(f'(z) + g'(z)\big)$, so that one finds

$$\begin{aligned} g'(z) &= \frac{n\,\cos i - \cos r}{n\,\cos i + \cos r} f'(z) = \frac{\sin 2i - \sin 2r}{\sin 2i + \sin 2r} f'(z), \\ h'(z) &= \frac{2n\,\cos i}{n\,\cos i + \cos r} f'(n\,z) = \frac{2\sin 2i}{\sin 2i + \sin 2r} f'(n\,z), \end{aligned} \tag{15}$$

and (if $n > 1$) one cannot have $g = 0$ as in the construction of HUYGENS (except for the special incidence angle i_* with $\tan i_* = n$), which may be interpreted as using the function w defined by

$$\begin{aligned} &w(x_1, x_2, t) = 1 \text{ if } x_2 > 0 \text{ and } x_1 \sin i + x_2 \cos i < c\,t, \text{ or if } x_2 < 0 \text{ and } x_1 \sin i + x_2 n\, \cos r < c\,t, \\ &w(x_1, x_2, t) = 0 \text{ if } x_2 > 0 \text{ and } x_1 \sin i + x_2 \cos i > c\,t, \text{ or if } x_2 < 0 \text{ and } x_1 \sin i + x_2 n\, \cos r > c\,t, \end{aligned} \tag{16}$$

which is a particular case of

$$w(x_1, x_2, t) = \begin{cases} F(x_1 \sin i + x_2 \cos i - c\,t), \text{ for } x_2 > 0 \\ F(x_1 \sin i + x_2\, n\, \cos r - c\,t), \text{ for } x_2 < 0 \end{cases}, \tag{17}$$

which one uses for F smooth, and then one lets it tend to a piecewise constant function with one discontinuity; if $v = c$ for $x_2 > 0$, and $v = \frac{c}{n}$ for $x_2 < 0$, it is the jump of $v^2 \frac{\partial w}{\partial x_2}$ through the interface $x_2 = 0$ which is responsible for a nonzero distribution T (actually a Radon measure) in the wave equation:[34]

$$\begin{aligned} &\frac{\partial^2 w}{\partial t^2} - \sum_i \frac{\partial}{\partial x_i}\Big(v^2 \frac{\partial w}{\partial x_i}\Big) = T, \text{ with the distribution } T \in \mathcal{D}'(\mathbb{R}^3) \text{ given by} \\ &\langle T, \varphi \rangle = c^2 \frac{n\,\cos i - \cos r}{n} \int_{\mathbb{R}^2} F'(x_1\, \sin i - c\,t)\, \varphi(x_1, 0, t)\, dx_1 dt \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^3), \end{aligned} \tag{18}$$

so that the case (16), corresponding to the computation of HUYGENS, gives

$$\langle T, \varphi \rangle = -c^2 \frac{n\,\cos i - \cos r}{n\,\sin i} \int_{\mathbb{R}} \varphi\Big(\frac{c\,t}{\sin i}, 0, t\Big)\, dt \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^3), \tag{19}$$

and one has $T \neq 0$, except for the special incidence angle i_*. Of course, the same computations are valid in a N-dimensional setting for $N \geq 2$.

The coefficient of proportionality between g and f is then $\frac{n\,\cos i - \cos r}{n\,\cos i + \cos r}$ by (15), and the proportion between energies is the square of that, hence the proportion of reflected energy at normal incidence is $\frac{(n-1)^2}{(n+1)^2}$, it is 0 at the critical incidence i_*, and it goes to 1 when i tends to $\frac{\pi}{2}$.

[34] Johann RADON, Czech-born mathematician, 1887–1956. He worked in Hamburg, in Greifswald, in Erlangen, Germany, in Breslau (then in Germany, now Wrocław, Poland) before World War II, and in Vienna, Austria after 1947.

The classical arguments about refraction give a wrong impression about reversing the direction of light. The wave equation is invariant by changing t into $-t$, but reversing time in the solution described in (13)–(15) does not tell us what happens if one sends an incident wave along the direction $-b_-$ (except in the case of the special refraction angle r_*, which gives $g = 0$), because it tells us what happens if one sends two related incident waves, one along $-b_-$ and one along $-a_+$. The computation (13)–(15) could have been presented with media of index n_1 and n_2, and the lack of symmetry comes from the fact that there is a critical refraction angle r_c corresponding to the incidence angle $i = \frac{\pi}{2}$, i.e. $\sin r_c = \frac{1}{n}$, so that one cannot define the angle i if $r_c < r < \frac{\pi}{2}$.

For $r_c < r < \frac{\pi}{2}$, I was taught that the light is reflected, but if one checks what the wave equation tells us, one finds a different story, since *evanescent waves* (which decay exponentially fast away from the interface) appear in the upper part: changing the orientation of the incident wave, one looks for a solution of the form

$$w(x,t) = \begin{cases} U\big(x_1 \sin r - \frac{c}{n}\,t, x_2\big) \text{ with } \frac{\partial^2 U}{\partial t^2} - c^2 \Delta\, U = 0 \text{ for } x_2 > 0, \\ f\big((x, b_+) - \frac{c}{n}\,t\big) + g\big((x, b_-) - \frac{c}{n}\,t\big) \text{ for } x_2 < 0 \end{cases}, \tag{20}$$

where f is the incident wave in the direction b_+, g is the reflected wave in the direction b_-, U is the transmitted wave, and the continuity conditions are

$$\begin{aligned} &U(z, 0) = f(z) + g(z) \text{ for } z \in \mathbb{R}, \\ &n^2 \frac{\partial U}{\partial x_2}\Big|_{x_2=0} = \cos r\, f'(z) - \cos r\, g'(z) \text{ for } z \in \mathbb{R}. \end{aligned} \tag{21}$$

At this point, it is natural to decompose U (as well as f and g) in plane waves, i.e. use their *Fourier transform*,[35] for which I use the notation of Laurent SCHWARTZ. Since $n \sin r > 1$, or $v = \frac{c}{n \sin r} < c$,

$$u = e^{2i\,\pi\,\xi\,(x_1 \sin r - c\,t/n)} e^{-\alpha\, x_2} \text{ solves } \frac{\partial^2 u}{\partial t^2} - c^2 \Delta\, u = 0 \text{ for } x_2 > 0 \text{ if } \alpha = 2\pi\,\beta\,|\xi|, \beta = \sqrt{\sin^2 r - \frac{1}{n^2}}, \tag{22}$$

and the first compatibility condition in (21) gives

$$\begin{aligned} &\text{if } f(z) = \int_{\mathbb{R}} e^{2i\,\pi\,z\,\xi}\,\widehat{f}(\xi)\,d\xi \text{ and } g(z) = \int_{\mathbb{R}} e^{2i\,\pi\,z\,\xi}\,\widehat{g}(\xi)\,d\xi, \text{ then one has} \\ &U(z, x_2) = \int_{\mathbb{R}} \big(\widehat{f}(\xi) + \widehat{g}(\xi)\big)\, e^{2i\,\pi\,z\,\xi} e^{-2\pi\,\beta\,|\xi|\,x_2}\,d\xi, \end{aligned} \tag{23}$$

and the second compatibility condition in (21) becomes

$$-n^2 \beta\,|\xi|\,\big(\widehat{f}(\xi) + \widehat{g}(\xi)\big) = i\,\xi\,\cos r\,\big(\widehat{f}(\xi) - \widehat{g}(\xi)\big) \text{ for } \xi \in \mathbb{R}, \tag{24}$$

so that

$$\text{if } \gamma = \frac{i\,\cos r + n^2\beta}{i\,\cos r - n^2\beta}, \text{ then } \widehat{g}(\xi) = \begin{cases} \gamma\,\widehat{f}(\xi) \text{ for } \xi > 0 \\ \overline{\gamma}\,\widehat{f}(\xi) = \frac{1}{\gamma}\,\widehat{f}(\xi) \text{ for } \xi < 0 \end{cases} = \Re\gamma\,\widehat{f}(\xi) + \Im\gamma\,i\,sign(\xi)\,\widehat{f}(\xi), \tag{25}$$

which can be written using the *Hilbert transform* H,[36] i.e. $H\,f(x) = \frac{1}{\pi}\,\lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy$ for smooth functions with compact support:[37]

$$g = \Re\gamma\,f + \Im\gamma\,H\,f. \tag{26}$$

[35] Jean-Baptiste Joseph FOURIER, French mathematician, 1768–1830. He worked in Auxerre, in Paris, France, accompanied BONAPARTE in Egypt, was prefect in Grenoble, France, until the fall of Napoléon I, and worked in Paris again. The first of three universities in Grenoble, France, Université de Grenoble 1, is named after him, and the Institut Fourier is its department of mathematics.

[36] David HILBERT, German mathematician, 1862–1943. He worked in Königsberg (then in Germany, now Kaliningrad, Russia) and at Georg-August-Universität, Göttingen, Germany. The term Hilbert space was coined by his former student VON NEUMANN.

[37] The Hilbert transform is the convolution with the distribution $\frac{1}{\pi}\,pv\,\frac{1}{x}$, defined by Laurent SCHWARTZ as an extension of the classical notion of *principal value* introduced by CAUCHY; it is an isometry on $L^2(\mathbb{R})$ with $H^2 = -I$, since $\mathcal{F}(H\,f)(\xi) = i\,sign(\xi)\,\mathcal{F}f(\xi)$, and M. RIESZ showed that it maps $L^p(\mathbb{R})$ into itself for $1 < p < \infty$, and his proof set the framework for a theory of interpolation of Banach spaces.

so that the relation between g and f is not local. For r tending to r_c, β tends to 0, and γ tends to $+1$, while for r tending to $\frac{\pi}{2}$, $\cos r$ tends to 0 and γ tends to -1, and γ varies from $+1$ to -1 by keeping its modulus equal to 1, with its imaginary part < 0 (since $\gamma = \frac{-(i\,\cos r + n^2\beta)^2}{\cos^2 r + n^4\beta^2}$).

IIIh *on some defects of scalar wave equations*

HUYGENS could have thought about a reflected wave, and a problem of sharing something like energy between the reflected wave and the refracted wave. Since sunlight is decomposed into the colours of the spectrum by a prism made of crystal (usually of a type containing lead and having a higher index of refraction than glass), he knew that the index of refraction of a transparent material depends upon the colour of light, whatever colour is. Since FERMAT had guessed that the denser the transparent material is, the slower the light goes, it had to be that colour is related to an intrinsic length scale, like if a ray of light could be skinny and wiggle quickly amidst the atoms of the materials, or fat and be much slowed down while trying to squeeze itself inside the material, but people were not thinking about matter being made from atoms in those days.[38]

It was difficult (but not impossible) for the first people who played with the wave equation to understand the correct conditions to impose at an interface between two different materials (perhaps by thinking about some kind of analogy with mechanical devices). Fourier transform is natural, and is related to separating waves according to their frequency, or their *wave-length* λ, and some of the necessary mathematics was only developed during the 20th century, but imagining evanescent waves requires a more mathematical mind, I believe, and I think that it was Rayleigh who introduced them for questions of surface waves in elasticity.[39]

The above analysis still does not tell us what light is, and we still have to understand why one needs a metal for creating a mirror, i.e. change the coefficient of reflection, but there were other hints in the 17th century and the 18th century about other defects of a scalar wave equation for explaining what light is.

Additional footnotes: Amit ACHARYA,[40] BANACH,[41] BONAPARTE,[42] CHAMPOLLION,[43] DIDEROT,[44] .../...

[38] Actually, since physicists tend to believe that nature has no choice but playing the games which they invent, is matter really made in the way they tell us?

[39] John William STRUTT, third baron Rayleigh (known as Lord Rayleigh), English physicist, 1842–1919. He received the Nobel Prize in Physics in 1904, for his investigations of the densities of the most important gases and for his discovery of argon in connection with these studies. He worked in Cambridge, England, holding the Cavendish professorship (1879–1884), after MAXWELL.

[40] Amit ACHARYA, Indian-born engineer, born in 1965. He works at CMU (Carnegie Mellon University), Pittsburgh, PA.

[41] Stefan BANACH, Polish mathematician, 1892–1945. He worked in Lwów (then in Poland, now Lvov, Ukraine). There is a Stefan Banach International Mathematical Center in Warsaw, Poland. The term Banach space was introduced by FRÉCHET.

[42] Napoléon BONAPARTE (Napoleone BUONAPARTE), French general, 1769–1821. He became Premier Consul after his coup d'état in 1799, was elected Consul à vie in 1802, and he proclaimed himself emperor in 1804, under the name Napoléon I (1804–1814, and 100 days in 1815).

[43] Jean-François CHAMPOLLION, French classical scholar, philologist and orientalist, 1790–1832. He worked in Grenoble, and he held a chair at Collège de France (archéologie, 1831–1832), in Paris, France. He deciphered the Egyptian hieroglyphs with the help of the work of others, like Thomas YOUNG.

[44] Denis DIDEROT, French philosopher and writer, 1713–1784. He worked in Paris, France, and he was co-editor of the Encyclopédie with D'ALEMBERT. Université Paris 7, Paris, France, is named after him.

Fields,[45] Fréchet,[46] Green,[47] Hardinge,[48] Hardy,[49] Heron,[50] Lebesgue,[51] Louis xviii,[52] Napoléon i (see Bonaparte), Nobel,[53] Perse,[54] Riemann,[55] Riesz F.,[56] Riesz M.,[57] Sadleir,[58] Savile,[59] Von Neumann,[60] Weil,[61] Laurence Young,[62] Young W.H..[63]

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[45] John Charles Fields, Canadian mathematician, 1863–1932. He worked in Meadville, PA, and in Toronto, Ontario. The Fields Medal is named after him.

[46] Maurice René Fréchet, French mathematician, 1878–1973. He worked in Poitiers, in Strasbourg and in Paris, France. Fréchet spaces are named after him.

[47] George Green, English mathematician, 1793–1841. He was a miller, and he wrote interesting articles before starting studying at Cambridge, at age 40; he received a Perse fellowship at Cambridge, England, but he did not live long afterward.

[48] Sir Charles Hardinge, 1st baron Hardinge of Penshurst, English diplomat, 1858–1944. He was Viceroy and Governor-General of India (1910–1916).

[49] Godfrey Harold Hardy, English mathematician, 1877–1947. He worked in Cambridge, in Oxford, England, holding the Savilian chair of geometry (1920–1931), and in Cambridge again, holding the Sadleirian chair of pure mathematics (1931–1942).

[50] Heron of Alexandria, "Egyptian" mathematician, about 10–75. He worked in Alexandria, Egypt.

[51] Henri Léon Lebesgue, French mathematician, 1875–1941. He worked in Rennes, in Poitiers, and he held a chair at Collège de France (mathématiques, 1921–1941) in Paris, France. The spaces L^p were named Lebesgue spaces in his honour by F. Riesz, and the Lebesgue integration theory named after him was discovered two years before him by W.H. Young.

[52] Louis Stanislas Xavier de France, 1755–1824, comte de Provence, duc d'Anjou, was king of France from 1814 to 1824, under the name of Louis xviii.

[53] Alfred Bernhard Nobel, Swedish industrialist and philanthropist, 1833–1896. He created a fund to be used as awards for people whose work most benefited humanity.

[54] Stephen Perse, English philanthropist, 1548–1615.

[55] Georg Friedrich Bernhard Riemann, German mathematician, 1826–1866. He worked at Georg-August-Universität, Göttingen, Germany. The Riemann ζ function is named after him, although Euler studied it before. Riemannian manifolds and Riemannian geometry are named after him, as well as Riemann surfaces for functions of a complex variable, and Riemann invariants for conservation laws in continuum mechanics.

[56] Frigyes (Frederic) Riesz, Hungarian mathematician, 1880–1956. He worked in Kolozsvár (then in Hungary, now Cluj-Napoca, Romania), in Szeged and in Budapest, Hungary. He introduced the spaces L^p in honor of Lebesgue and the spaces $\mathcal{H}^p$ in honor of Hardy, but no spaces are named after him; the Riesz operators have been introduced by his younger brother Marcel Riesz.

[57] Marcel Riesz (younger brother of Frigyes (Frederic) Riesz), Hungarian-born mathematician, 1886–1969. He worked in Stockholm and in Lund, Sweden. The Riesz operators are named after him.

[58] Lady Mary Sadleir (born Lorymer), –1706. In 1701, she funded lectures in algebra at Cambridge, England, which started in 1710; it transformed into a professorship in 1860.

[59] Sir Henry Savile, English mathematician, 1549–1622. In 1619, he founded professorships of geometry and astronomy at Oxford, England.

[60] János (John) von Neumann, Hungarian-born mathematician, 1903–1957. He worked in Berlin, in Hamburg, Germany, and at IAS (Institute for Advanced Study), Princeton, NJ.

[61] André Weil, French-born mathematician, 1906–1998. He received the Wolf Prize in 1979, for his inspired introduction of algebro-geometry methods to the theory of numbers, jointly with Jean Leray. He worked in Aligarh, India, in Haverford, PA, in Swarthmore, PA, in São Paulo, Brazil, in Chicago, IL, and at IAS (Institute for Advanced Study), Princeton, NJ.

[62] Laurence Chisholm Young, English-born mathematician, 1905–2000. He worked in Cape Town, South Africa, and at University of Wisconsin, Madison, WI, where I first met him during my first trip to United States, in the spring of 1971. Young measures are named after him, and he introduced them in the Calculus of Variations. I pioneered their use in partial differential equations (from continuum mechanics) in the late 1970s, not knowing at the time that he introduced them, as I heard about them as parametrized measures in seminars on control theory.

[63] William Henry Young, English mathematician, 1863–1942. He worked in Liverpool, England, in Cal-

Chapter IV: other aspects of light and wave equations.

IVa *birefringence*

In his 1690 article, HUYGENS also presented measurements made on a crystal from Iceland (Iceland spar, which is calcite, i.e. crystallized $CaCO_3$) which for one incident ray of light shows two refracted rays, an effect discovered in 1669 by BARTOLIN,[1] and called *double refraction*, or birefringence. Although the wave equation was unknown to HUYGENS, since the one-dimensional case is attributed to D'ALEMBERT who worked a century after him, it is important to notice that birefringence effects cannot be explained by a scalar wave equation in an anisotropic medium (like a crystal which does not show a cubic symmetry), in which plane waves move with a speed depending upon the direction, but such an effect is compatible with the Maxwell–Heaviside equation in an anisotropic medium.

IVb *polarized light*

In 1809, MALUS observed polarized light,[2] and the year after he published a theory for what had puzzled HUYGENS, the double refraction of light in some crystals, but finding a practical formula which works does not mean that one understands the phenomenon which one studies: polarized light is not a property of scalar wave equations, but it occurs for Maxwell–Heaviside equation in some anisotropic materials.[3]

IVc *diffraction*

In 1802, T. YOUNG studied questions of *diffraction*, reviving the work of HUYGENS on the wave nature of light, and opposing then the ideas of NEWTON on the particle nature of light, and since a cult of personality towards NEWTON had developed in England, it might have not been so easy for him. The main contributors to diffraction in 19th century were FRESNEL, FRAUNHOFER,[4] and AIRY who must have observed that it is not possible that a non-zero solution of the wave equation vanish identically in the shadow region predicted by geometrical optics,[5] since the Airy function appears in connection with this fact, but I have heard that POISSON may have thought about it before him. These effects rely crucially on the wavelength of the wave: for a wave of speed v, one may observe functions of the form $f(x - v\,t)$, but introducing frequency is about considering periodic solutions, i.e. $e^{i\,\omega\,t}\psi(x)$, which forces $\psi(x)$ to be periodic too, and the period in time is $\tau = \frac{2\pi}{\omega}$ and in space it is the wavelength $\lambda = v\,\tau$, so that the particular function considered is $e^{i\,(\omega\,t - \frac{2\pi\,x}{\lambda})}$.

Fourier transform is a natural tool which has not much to do with what light is, and it serves for all kind of partial differential equations with constant coefficients, or where periodic functions appear, in which case one uses *Fourier series*. For example, PTOLEMY had a model of solar system with the sun, the moon and the planets turning around the earth, and they were on circles rolling over other circles,[6] COPERNICUS used

cutta, India, holding the first Hardinge professorship (1913–1917), in Aberystwyth, Wales, and in Lausanne, Switzerland. He is said to have discovered Lebesgue integration two years before LEBESGUE. There are many results attributed to him which may be joint work with his wife, Grace CHISHOLM-YOUNG, English mathematician, 1868–1944, as they collaborated extensively; their son Laurence is known for his own results.

[1] Rasmus BARTHOLIN (Erasmus BARTHOLINUS), Danish mathematician, 1625–1698. He worked in Copenhagen, Denmark.

[2] Étienne Louis MALUS, French mathematician, 1775–1812. He worked at École Polytechnique in Paris, accompanied BONAPARTE in Egypt, and then he worked in Antwerp, Belgium, in Strasbourg, and in Paris, France.

[3] The refraction of linearized elastic waves in anisotropic materials also shows some kind of polarization.

[4] Joseph VON FRAUNHOFER, German optician, 1787–1826. He worked in Benediktbeuern, Germany. He invented the spectroscope in 1814, and discovered 574 dark lines appearing in the solar spectrum, named Fraunhofer lines after him, although he was not the first to observe them.

[5] George Biddell AIRY, English mathematician and astronomer, 1801–1892. He was Lucasian professor of mathematics (1826–1828), and then Plumian professor of astronomy (1828–1835) in Cambridge, England, before becoming the seventh Astronomer Royal (1835–1881). The Airy stress function in elasticity is named after him. The Airy function, named after him, is defined by $Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\,(x\,t+t^3/3)}\,dt$; it solves $Ai'' + x\,Ai = 0$ on $\mathbb{R}$, and decays exponentially fast as $x \to +\infty$.

[6] This is natural for those who observe the sky and the position of a planet among the stars, or more precisely among the zodiacal constellations which were invented around the milky way (which Galileo found

the same model but with the sun at the center,[7] Brahe used the same model with the sun and the moon moving around the earth and the planets turning around the sun,[8] and Galileo may have been convinced about the models of Copernicus or Brahe after observing in his telescope that Venus has phases,[9] i.e. it appears with variable crescent form, hence it is lit differently by the sun in its rotation around the sun.[10] Since a Fourier series is precisely about decomposing a periodic motion into a sum of circular motions, the circles rolling over circles are both inexistent from a physical point of view, and real from a mathematical point of view, and this explains why Ptolemy's model and Copernicus's model give good predictive results.[11]

IVd *Maxwell's unification of electricity and magnetism*

Maxwell unified electricity and magnetism by making a synthesis of the laws found by Ampère,[12] Faraday,[13] Gauss,[14] and Weber,[15] although the Gauss–Weber law ($div(B) = 0$) is also called Pèlerin's law,[16] and, because he believed in a somewhat "solid aether", he used mechanical devices for transmitting the electric field and the magnetic fields, so that he arrived at a quite large system of equations, which Heaviside later simplified by using vector calculus (somewhat new at the time, when people thought hat the good framework was the quaternions of Hamilton),[17] and devised the smaller system of equations which one uses now, so that I find it unfair to only refer to Maxwell for it, and I call it the Maxwell–Heaviside equation. However, in 1862, Maxwell had computed the speed of propagation of electro-magnetic waves in his complicated model, and found that they travel at the speed of light, so that he wrote *We can scarcely avoid the conclusion that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena*, but it obviously looked new and unclear to him, although he and others should have understood much before that something in the reflection of light indicates that it is coupled with at least electricity.

In the simplified form due to Heaviside, there is an *electric field* E, a *magnetic field* H, an electric *polarization* field D, and a magnetic *induction* field B, but contrary to what the names may imply, E and B play together (they are the coefficients of an exact 2-differential form in space-time) and H and D play together (they are the coefficients of another 2-differential form in space-time, which is not exact, but is

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to be made by a multitude of stars), which corresponds more or less to the plane of our galaxy: a planet may advance, then go backward a little, then advance again, so that it is like if the planet is attached to a circle rolling on the "sphere of the stars".

[7] Nicolaus Copernicus (Mikołaj Kopernik), Polish mathematician, 1473–1543. He worked in Frombork, Poland.

[8] Tyge Brage (Tycho Brahe), Danish-born astronomer, 1546–1601. He worked in Prague, now the capital of Czech republic.

[9] Venus, goddess of love in the Roman mythology, corresponding to Aphrodite in the Greek mythology.

[10] Neither Venus nor the earth rotates around the sun: they move around the center of mass of the solar system, in exactly the same way as the sun does, but since the distance from the center of mass of the sun to the center of mass of the solar system is smaller than the radius of the sun, it is a good first approximation to use the center of mass of the sun as origin for studying the solar system.

[11] Among the models with which physicists play, which one are based on similar inexistent things?

[12] André Marie Ampère, French mathematician, 1775–1836. He worked in Bourg, in Lyon, and he held a chair at Collège de France (Physique générale et expérimentale, 1824–1836) in Paris, France.

[13] Michael Faraday, English chemist and physicist, 1791–1867. He worked in London, England, as Fullerian professor of chemistry at the Royal Institution of Great Britain.

[14] Johann Carl Friedrich Gauss, German mathematician, 1777–1855. He worked at Georg-August-Universität, Göttingen, Germany. Gaussian functions, and many theorems are named after him.

[15] Wilhelm Eduard Weber, German physicist, 1804–1891. He worked at Georg-August-Universität, Göttingen, and in Leipzig, Germany.

[16] Petrus Peregrinus de Maricourt (Pierre de Maricourt, known as "Le Pèlerin"), French scientist, 13th century. He is mostly known from a letter from 1269 (in Latin) about magnets, with abbreviated title *Epistola de Magnete*.

[17] Sir William Rowan Hamilton, Irish mathematician, 1805–1865. He worked in Dublin, Ireland.

related to conservation of *electric charge*): they satisfy the general system

$$\begin{aligned}&B_t + curl(E) = 0, div(B) = 0\\&D_t - curl(H) = j, div(D) = \rho, \text{ so that } \rho_t + div(j) = 0 \text{ (conservation of electric charge)},\end{aligned} \tag{27}$$

where ρ is the density of charge and j the density of current, and in the *vacuum*, the constitutive relation is

$$D = \varepsilon_0 E, \text{ and } B = \mu_0 H, \tag{28}$$

with ε_0 the *dielectric permittivity* of the vacuum, μ_0 the *magnetic susceptibility* of the vacuum, such that

$$\varepsilon_0 \mu_0 c^2 = 1. \tag{29}$$

MAXWELL must have looked at plane waves moving at speed v in the direction of a unit vector $e \in \mathbb{R}^3$, in his more complicated model, and here it consists in assuming that E and H are functions of $(x, e) - v\,t$, then one deduces what B and D are by (28), one puts all this in (27), and out comes $\varepsilon_0 \mu_0 v^2 = 1$, so that MAXWELL must then have looked at the measured values of ε_0 and μ_0 to find that v just looks like the speed of light c, and he was somewhat surprised. MAXWELL would have been more surprised if he had understood that electromagnetism describes X-rays, only discovered in 1895 by RÖNTGEN,[18] which correspond to wavelength from 0.01 to 10 nanometers (and a tenth of a nanometers is an Angström,[19] which is of the order of distances between atoms in a crystal), (invisible) ultraviolet light whose wavelength is from 10 nanometers to 400 nanometers, visible light whose wavelength is from 380 nanometers (0.38 micron, violet) to 740 nanometers (0.74 micron, red), (invisible) infrared light whose wavelength is from 740 nanometers (0.74 micron) to 300 microns, and radio waves, discovered by HERTZ in 1887,[20] whose wavelength is from 1 millimeter (frequency of 300 GHz) to 100 kilometers (frequency of 3 kHz).

The good boundary condition for total reflection is $E = 0$, corresponding to a *perfect conductor*, so that at least one has to use a good conductor if one wants a good reflection, and it is then not surprising after all that *metal is needed in mirrors*, since *light is electromagnetism*!

IVe *completing the Maxwell–Heaviside equation*

One should not stop here, and a natural question is about matter, like ρ and j which appear in the Maxwell–Heaviside equation. One needs other equations for studying the *interaction of light and matter*, like the so-called *Lorentz force*,[21] which says that an electrical charge q moving at velocity v feels a force

$$F = q\,(E + v\, \times B), \tag{30}$$

but such a force already appears in an 1862 article by MAXWELL, maybe in connection with Ohm's law,[22] while LORENTZ introduced the "Lorentz" force in 1892. Around 1900, POINCARÉ wondered how there could

[18] Wilhelm Conrad RÖNTGEN, German physicist, 1845–1923. He received the Nobel Prize in Physics in 1901 "in recognition of the extraordinary services he has rendered by the discovery of the remarkable rays subsequently named after him". He worked in Hohenheim, Germany, in Strasbourg, France (but occupied by Germany at the time), in Giessen,Würzburg, and in München (Munich), Germany.

[19] Anders Jonas ÅNGSTRÖM, Swedish physicist, 1814–1874. He worked in Uppsala, Sweden.

[20] Heinrich Rudolf HERTZ, German physicist, 1857–1894. He worked in Karlsruhe, and in Bonn, Germany. The unit of frequency, the hertz (inverse of a second) is named after him.

[21] Hendrik Antoon LORENTZ, Dutch physicist, 1853–1928. He received the Nobel Prize in Physics in 1902, jointly with Pieter ZEEMAN, in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena. He worked in Leiden, The Netherlands. The Institute for Theoretical Physics in Leiden, The Netherlands, is name after him, the Lorentz Institute.

[22] Georg Simon OHM, German mathematician, 1789–1854. He taught in various high-schools, before working in München (Munich), Germany. Ohm's law in electricity is named after him.

be an action (the Lorentz force) without a reaction,[23] and he concluded that the density of energy of the electromagnetic field, sum of the density of electric energy and the density of magnetic energy,

$$e = \frac{(E, D)}{2} + \frac{(B, H)}{2}, \tag{31}$$

is equivalent to a density of mass according to the formula

$$e = m\,c^2. \tag{32}$$

In (31), one admits that in a material which is not the vacuum, the constitutive relation takes the form

$$D = \varepsilon\,E, B = \mu\,H, \text{ for symmetric positive definite tensors } \varepsilon, \mu, \tag{33}$$

so that studying the propagation of waves is less obvious than in the vacuum, but the next approximation is to consider a crystalline material and observe that, even in the case of an isotropic material (which occurs for crystals having cubic symmetry), ε and μ are positive scalars which depend upon frequency: here comes the index of refraction depending upon the colour, since different colours are a sign of different frequency, or different wavelength (from 0.38 micron for violet to 0.74 micron for red).

Maxwell imagined a mechanistic model for unifying electricity and magnetism because he believed in a somewhat solid material called aether, since he thought that electromagnetic waves are transverse waves which could only be transmitted by a solid. This is quite strange from a mathematical point of view, since the notion of longitudinal or transverse waves pertains to elasticity which describes displacements, and it is not clear why one considers that electromagnetic waves are transverse: for example, a particle at rest feels only the part $q\,E$ of the force and in a plane wave in the vacuum E is perpendicular to the direction of propagation of the wave, but the particle starts moving and acquires a velocity, so that the part of the force in $q\,v \times B$ starts having an effect, and there is a differential equation to study.[24]

It is a bad idea to couple a 19th century point of view for light (the Maxwell–Heaviside equation, which is a partial differential equation) with an 18th century point of view for matter (the Lorentz force, which creates an ordinary differential equation), but physicists were going to invent later an even more bizarre $18\frac{1}{2}$th century point of view, an object which is sometimes a particle and sometimes a wave!

Matter should also be described by a partial differential equation, which should preclude it to move faster than light, and such a coupled system was proposed around 1930 by Dirac,[25] although with some limitations, and his equation has the advantage of giving Planck's constant its physical role of coupling light and matter,[26] instead of the mysterious role that it plays in the rules of quantum mechanics (which I find silly), and in Schrödinger equation,[27] but since letting c tend to ∞ in Dirac's equation gives (at least

[23] Jules Henri Poincaré, French mathematician, 1854–1912. He worked in Paris, France. There is now an Institut Henri Poincaré (IHP), dedicated to mathematics and theoretical physics, part of UPMC (Université Pierre et Marie Curie), Paris.

[24] The result of the study (which I did a few years ago, but could have been done by Maxwell after his 1862 article) is that if the plane wave is strong enough it sweeps the particle, which ends up surfing the wave at a limiting angle of $\frac{\pi}{4}$ and acquiring a speed near $c\sqrt{2}$, well over the speed of light, so that this effect can hardy be attributed to a transverse wave!

[25] Paul Adrien Maurice Dirac, English physicist, 1902–1984. He received the Nobel Prize in Physics in 1933, jointly with Erwin Schrödinger, for the discovery of new productive forms of atomic theory. He worked in Cambridge, England, holding the Lucasian chair (1932–1969).

[26] Max Karl Ernst Ludwig Planck, German physicist, 1858–1947. He received the Nobel Prize in Physics in 1918, in recognition of the services he rendered to the advancement of Physics by his discovery of energy quanta. He worked in Kiel and in Berlin, Germany. There is a Max Planck Society for the Advancement of the Sciences, which promotes research in many institutes, mostly in Germany (I spent my sabbatical year 1997–1998 at the Max Planck Institute for Mathematics in the Sciences in Leipzig, Germany).

[27] Erwin Rudolf Josef Alexander Schrödinger, Austrian-born physicist, 1887–1961. He received the Nobel Prize in Physics in 1933, jointly with Paul Adrien Maurice Dirac, for the discovery of new productive forms of atomic theory. He worked in Vienna, Austria, in Jena and in Stuttgart, Germany, in Breslau (then in Germany, now Wrocław, Poland), in Zürich, Switzerland, in Berlin, Germany, in Oxford, England, in Graz, Austria, and in Dublin, Ireland.

formally) Schrödinger's equation, it would be silly to use Schrödinger's equation (which then is a model with $c = +\infty$) in order to compare results with the real speed of light c, isn't it?

MAXWELL died before the theory of aether was tested by a series of measurements of the speed of light c carried out in 1887 at the Case School of Applied Science (now Case Western Reserve University),[28] in Cleveland, OH, by MICHELSON and MORLEY,[29,30] who arrived at a quite intriguing result: if one moved in a frame with fixed velocity v with respect to a fixed frame, one expected that light traveling with velocity c in either direction in the moving frame would result in its traveling with speed $c+v$ or $c-v$ in the fixed frame; with the idea that a fixed frame called aether existed, they made a quite precise experiment,[31] with light moving in perpendicular arms, and creating interferences, and the pattern of the interferences should have changed by rotating the experiment, but nothing significant was detected, so that the theory of aether was shattered, and it was admitted that light travels with speed c in the vacuum, in every direction, and in every frame moving at constant velocity!

FITZGERALD then made a bold suggestion in 1892,[32] that the results of the Michelson–Morley experiment could be explained by the contraction of a body along its direction of motion, and this was then also proposed by LORENTZ, who went further, by introducing a local change of time too, and there is a clear piece of mathematics (which I think was first done by POINCARÉ) which explains how the Lorentz group of transformations occurs, and it was then that POINCARÉ invented the theory of (special) relativity.

There is a long way between a mirror in obsidian from 6 000 BCE to Maxwell–Heaviside equation for understanding what light is, but it is not so clear what the answer is for the next question: *what is matter*?

Additional footnotes: BECQUEREL,[33] CURIE P. & M.,[34] PLUME,[35] ZEEMAN.[36]

[28] Leonard CASE Jr., American philanthropist, 1820–1880. He endowed the Case School of Applied Science in Cleveland, OH, founded in 1881, which became the Case Institute of Technology in 1948, and merged in 1967 with Western Reserve University to form Case Western Reserve University.

[29] Albert Abraham MICHELSON, Polish-born physicist, 1852–1931. He received the Nobel Prize in Physics in 1907, for his optical precision instruments and the spectroscopic and metrological investigations carried out with their aid. He worked in Chicago, IL.

[30] Edward Williams MORLEY, American chemist, 1838–1923. He worked in Cleveland, OH.

[31] To avoid vibrations, the experiment was set up in the basement of a stone building, on top of a huge block of marble floating in a pool of mercury, so that the whole experiment could be rotated easily.

[32] George Francis FITZGERALD, Irish physicist, 1851–1901. He worked in Dublin, Ireland.

[33] Antoine Henri BECQUEREL, French physicist, 1852–1908. He received the Nobel Prize in Physics in 1903, in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity, jointly with Pierre CURIE and Marie SKLODOWSKA-CURIE. He worked in Paris, France.

[34] Pierre CURIE, French physicist, 1859–1906, and his wife Marya Salomea SKŁODOWSKA-CURIE, Polish-born physicist, 1867–1934, received the Nobel Prize in Physics in 1903, in recognition of the extraordinary services they have rendered by their joint researches on the radiation phenomena discovered by Professor Henri BECQUEREL, jointly with Henri BECQUEREL; Marie SKŁODOWSKA-CURIE also received the Nobel Prize in Chemistry in 1911, in recognition of her services to the advancement of chemistry by the discovery of the elements radium and polonium, by the isolation of radium and the study of the nature and compounds of this remarkable element. They worked in Paris, France. Université Paris VI, Paris, is named after them, UPMC (Université Pierre et Marie Curie).

[35] Thomas PLUME, English churchman and philanthropist, 1630–1704. He founded the chair of Plumian professor of astronomy and experimental philosophy in 1704 in Cambridge, England.

[36] Pieter ZEEMAN, Dutch physicist, 1865–1943. He received the Nobel Prize in Physics in 1902, jointly with Hendrik LORENTZ, in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena. He worked in Leiden, and in Amsterdam, The Netherlands.