On the three-dimensional Filon construct for dislocations

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Dedicated to Professor V.L.Berdichevskii on his sixty-fifth birthday.

Abstract

The relationship between dislocation theory and the difference of linear elastic solutions for two different sets of elastic moduli, derived by Filon in two-dimensions, is generalised to three-dimensions. Essential features are developed and illustrated by the examples of the edge and screw dislocations. The inhomogeneity problem is discussed within the same context, and related to Somigliana dislocations, and in the limit to the interstitial atom.

Key words: Dislocations, linear elasticity, variation of moduli.

1 Introduction

This paper generalises to three-dimensions the relationship, or construct, established by Filon [13] for two-dimensional isotropic linear elasticity, between dislocation theory and the difference of linear elastic solutions to the same boundary value problem but for two different sets of elastic moduli.The connexion between dislocations and linear isotropic thermoelasticity has been demonstrated by Muskhelishvili [32] for the plane theory, and in [2] for three-dimensions. The relation between three-dimensional thermoelasticity and a variation of Poisson's ratio is discussed in [17].

Here, the principal aim is to explicitly describe, apparently for the first time, the structure common to these three-dimensional theories facilititing the transposition of properties from one theory to another. Of course,the unifying concept is that of residual or initial stress which creates the opportunity for other two- and three- dimensional physical theories to be similarly interrelated. In this respect, the heuristic operations of cutting and welding customarily employed to explain the action and consequences of initial stress in principle are not restricted to linear theories, so that Filon's construct may be applicable to nonlinear theories. The topic has been explored, for example, by Kondo [22]. To maintain reasonable length, however, we prefer to confine attention to describing the generalisation to three-dimensiosn of Filon's original construct. The extension, not entirely straightforward, is illustrated by simple well-known examples from both linear elasticity and dislocation theory to best convey the approach's main structural elements. Accordingly, while the account is purposely introductory, and is not intended to be either comprehensive or to solve any new problems, its aim includes the provision of sufficient description to indicate prospects for future development. Consequently, we omit discussion of such theories as plasticity, magnetostriction, functionally graded materials, and linearised elasticity, along with a study of Riemannian structure. Furthermore, we do not consider, for instance, how arrays of discrete dislocations, dislocation dipoles, dislocation loops, self-energies, interactive energies, and the Peach-Koehler formula may be generated from known solutions to corresponding elastic problems. These also are topics for possible later consideration.

As with most dislocation studies, we develop our investigation within the context of linear elasticity, but for the nonhomogeneous anisotropic theory. Specific examples, however, are chosen from known isotropic boundary value problems. These applications illustrate both the advantages and limitations of the Filon construct, at least as it applies to linear elasticity, and emphasise that its effectiveness in homogeneous isotropic elasticity depends upon the presence both of some kind of singularity, and of multiply-connected regions. Nevertheless, it must be stressed that Filon's construct is not restricted to isotropic elasticity, but is equally valid for the nonhomogeneous anisotropic theory and probably under less restrictive conditions. Furthermore, another anticipated chief benefit is the mutual enrichment of the constituent theories since known features of one theory can be transferred to unknown properties in the other. These possibilities are partially illustrated by the edge and screw dislocations in isotropic elasticity. Filon's extended construct is used to derive their well-known discontinuous solutions on a multiply-connected region from the elastic displacement due respectively to a uniform linear distribution of point-forces, and to anti-plane shear. The latter connexion appears to be new.

Section 2 presents relevant parts of the linear theory of anisotropic elasticity, and defines the boundary value problems to be considered. The incompatibility tensor is introduced and some properties discussed. Section 3 derives equations governing the difference between the displacement, strain, and stress for the same boundary value problem but for two different sets of elastic moduli. Section 4.1 identifies that part of the difference strain that produces no extra stress when the moduli are varied, and provides a heuristic interpretation in terms of cut-and-weld operations. The difference stress is similarly interpreted . Section 4.2 treats the inhomogeneity problem by means of a variation in the elastic moduli, and, in particular, obtains a complete solution, regardless of the inhomogeneity's shape, in isotropic elasticity when only Poisson's ratio varies. Section 5 introduces pertinent elements of dislocation theory, including the Saint-Venant-Cesaro integral (cf, for example, [27]) and expressions for the Burgers vector and incompatibility tensor. Filon's construct, established in Section 6, is achieved by simple comparison of the respective formulations, and relates the total dislocation displacement, the plastic strain, and elastic stress and strain to appropriate components belonging to the difference between solutions obtained by varying the elastic moduli in the same nonhomogeneous anisotropic elastic problem. Only the symmetric part of the dislocation density can be similarly related. Expressions for the Burgers vector and incompatibility tensor are established for isotropic elasticity in terms respectively of the strain and dilatation belonging to the elastic problem for a definite set of moduli. Implications of Carlson's conclusions [3, 4] with respect to homogeneous isotropic elastic solutions independent of elastic moduli are briefly examined in Section 7, while in Section 8 we generate, as already mentioned, the solution for both an edge and screw dislocation from elastic solutions for a linear uniform distribution of point-forces, and for anti-plane shear respectively. The construct is employed in Section 9 to derive expressions for an array of dislocations continuously distributed over a bounded region from the elastic inhomogeneity problem. When the inhomogeneity is spherical, a connexion is demonstrated to Somigliana dislocations distributed over the interface, and in the limit to the interstitial atom.

Both an indicial and direct notation are used as convenient, with the standard conventions adopted of summation over repeated subscripts and a subscript comma to denote partial differentiation. Latin subscripts range over $[1, 2, 3]$ while Greek indices assume the values $1, 2$. In the direct notation, the gradient, divergence, and rotation operators are denoted by ∇, Div, and $\nabla\times$, the trace operator by tr, the identity tensor by I, and the scalar and tensor products by their usual symbols. We assume the existence of a solution suitable to our needs.

We deal only with equilibrium problems, but obviously the Filon construct may be generalised to relate anisotropic linearised and linear elastodynamics to dislocations in motion.

2 General theory

2.1 Linear anisotropic elasticity

We consider a nonhomogeneous anisotropic compressible linear elastic body occupying a region $\Omega \subseteq \mathbb{R}^3$ and in equilibrium subject to given body-force, mass density, and specified (mixed) boundary conditions. The surface $\partial\Omega$ of Ω

is continuously differentiable, with unit outward vector normal n. We consider two different sets of suitably smooth elastic moduli whose components with respect to a given Cartesian orthogonal coordinate system possess the major and minor symmetries

$$c^{(\alpha)}_{ijkl}(x) = c^{(\alpha)}_{jikl}(x) = c^{(\alpha)}_{klij}(x), \qquad x \in \Omega, \quad \alpha = 1, 2. \tag{2.1}$$

The elastic *compliances* $C^{(\alpha)}_{ijkl}(x)$ are the inverse of the elastic moduli, have corresponding symmetries, and satisfy the relations

$$c^{(\alpha)}_{ijpq} C^{(\alpha)}_{pqkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.2}$$

where δ_{ij} denotes the Kronecker delta.

For compressible isotropic elasticity, we have

$$c^{(\alpha)}_{ijkl} = \lambda^{(\alpha)}\delta_{ij}\delta_{kl} + \mu^{(\alpha)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.3}$$

$$C^{(\alpha)}_{ijkl} = \frac{1}{4\mu^{(\alpha)}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda^{(\alpha)}}{2\mu^{(\alpha)}(3\lambda^{(\alpha)} + 2\mu^{(\alpha)})}\delta_{ij}\delta_{kl}, \tag{2.4}$$

where $\lambda^{(\alpha)}$ and $\mu^{(\alpha)}$ are the respective Lamé moduli related to Poisson's ratio $\nu^{(\alpha)}$ by

$$\lambda^{(\alpha)} = \frac{2\mu^{(\alpha)}\nu^{(\alpha)}}{(1 - 2\nu^{(\alpha)})}. \tag{2.5}$$

Substitution of (2.5) in (2.4) gives the alternate expression

$$C^{(\alpha)}_{ijkl} = \frac{1}{4\mu^{(\alpha)}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu^{(\alpha)}}{2\mu^{(\alpha)}(1 + \nu^{(\alpha)})}\delta_{ij}\delta_{kl}. \tag{2.6}$$

In plane strain linear isotropic elasticity, the corresponding expressions are

$$c^{(\alpha)}_{\alpha\beta\gamma\delta} = \lambda^{(\alpha)}\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu^{(\alpha)}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{2.7}$$

$$C^{(\alpha)}_{\alpha\beta\gamma\delta} = \frac{1}{4\mu^{(\alpha)}}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - \frac{\lambda^{(\alpha)}}{4\mu^{(\alpha)}(\lambda^{(\alpha)} + \mu^{(\alpha)})}\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2.8}$$

$$= \frac{1}{4\mu^{(\alpha)}}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - \frac{\nu^{(\alpha)}}{2\mu^{(\alpha)}}\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{2.9}$$

The Cartesian components of the symmetric linear strain tensor $e^{(\alpha)}$ are derived from the continuously differentible displacement vector field $u^{(\alpha)}$ according to

$$e^{(\alpha)}_{ij} = \frac{1}{2}\left(u^{(\alpha)}_{i,j} + u^{(\alpha)}_{j,i}\right), \tag{2.10}$$

and are compatible in the sense that for $u^{(\alpha)} \in C^3(\Omega, \mathbb{R}^3)$ there holds

$$\nabla \times \nabla \times e^{(\alpha)} = 0, \tag{2.11}$$

or equivalently

$$e_{irs}e_{jpk}e^{(\alpha)}_{ks,pr} = 0, \tag{2.12}$$

where e_{ijk} is the usual alternating tensor.The compatibility condition (2.11), necessary for the existence of a continuously differentiable displacement vector field, is also sufficient provided that the region Ω is simply-connected. (See, for example,[14, sect.14.2, p.40].)

Let $\sigma^{(\alpha)}$ be the stress tensor, which for each α is related to the strain $e^{(\alpha)}$ by the constitutive assumptions

$$\sigma^{(\alpha)} = c^{(\alpha)}e^{(\alpha)}, \qquad x \in \Omega. \tag{2.13}$$

The equilibrium equations and boundary conditions satisfied by the elastic fields are

$$\begin{aligned} Div\,\sigma^{(\alpha)} + \rho f &= 0, \qquad x \in \Omega, && (2.14)\\ u^{(\alpha)} &= g, \qquad x \in \partial\Omega_1, && (2.15)\\ n.\sigma^{(\alpha)} &= F, \qquad x \in \partial\Omega_2, && (2.16) \end{aligned}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and the mass density ρ, body force vector f per unit mass, surface traction F, and surface displacement vector g are prescribed and remain the same for both sets of moduli.

In what follows, we additionally assume in general that each set of elastic moduli $c^{(\alpha)}$ are positive-definite and therefore satisfy the well-known Kirchhoff uniqueness condition. This assumption may be relaxed to admit moduli $c^{(2)}$, say, that do not satisfy any definiteness conditions, but which are chosen to simplify the boundary value problem. The second set $c^{(1)}$, however, usually is selected to be within the range sufficient for uniqueness. Such ranges include the Kirchhoff range for the displacement and traction boundary value problems, although for the mixed boundary value problem the ranges coincide. (See [21] and [36].)

2.2 Incompatible strains

We wish to investigate the implications when the compatibility condition (2.11) is *not* satisfied and for this purpose we introduce both the antisymmetric linear rotation tensor W, specified by

$$W = \frac{1}{2}(\nabla u - (\nabla u)^T) = -W^T, \tag{2.17}$$

and the axial vector ω defined by

$$\omega = \frac{1}{2}\nabla \times u. \tag{2.18}$$

Assume that $u \in C^2(\Omega, \mathbb{R}^3)$ and let e be the corresponding linear strain derived according to (2.10). Then we have the identity ([14, Sect.14,p.39])

$$\nabla \times e = \nabla\omega, \tag{2.19}$$

which in suffix notation becomes

$$e_{ipk}e_{jk,p} = \omega_{i,j} = \frac{1}{2}e_{ipk}W_{kp,j}, \tag{2.20}$$

where W_{ij} are the Cartesian components of the tensor W.

Let $[f]_B^A$ denote the change in the function f along a given simple smooth curve connecting the points A and B in the simply-connected region Ω. Then we have

$$\begin{aligned} [\omega]_A^B &= \int_A^B \nabla\omega\,.dx \\ &= \int_A^B \nabla\times e\,.dx, \end{aligned} \tag{2.21}$$

and consequently the jump in ω around the closed curve $\partial\Sigma$ bounding the open smooth surface $\Sigma \subset \Omega$ is by Stokes theorem

$$[\omega]_{\partial\Sigma} = \int_\Sigma \nabla\times\nabla\times e.n\,dS \tag{2.22}$$

$$= -\int_\Sigma \eta.n\,dS \tag{2.23}$$

where the symmetric *incompatability* tensor η , defined by

$$\eta = -\nabla\times\nabla\times e, \tag{2.24}$$

is further discussed in Section 5

Note that when Ω is a region where $u \in C^3(\Omega)$, then $\eta = 0$ and the jump in ω around $\partial\Sigma$ is zero. Furthermore, when $u \in C^2(\Omega)$, but the strain vanishes identically, then by (2.19) the rotation vector ω is constant, and its jump again vanishes. When $u \in C(\Omega)$, or less, these arguments must be formulated in a suitably weak form. This aspect is not developed here.

3 Variation of elastic moduli

We derive equations for the differences between the quantities introduced in Section 2.1 when the moduli are varied, but when the body force, mass density, and boundary conditions remain unaltered. Accordingly, we define the *difference* fields to be

$$u = u^{(1)} - u^{(2)}, \qquad x \in \bar{\Omega}, \tag{3.1}$$

$$e = e^{(1)} - e^{(2)}, \qquad x \in \bar{\Omega}, \tag{3.2}$$

$$\sigma = \sigma^{(1)} - \sigma^{(2)}, \qquad x \in \bar{\Omega}, \tag{3.3}$$

where $\bar{\Omega}$ denotes the closure of Ω.

It follows by subtraction of the respective equilibrium equations and boundary conditions (2.14)-(2.16) that the difference fields satisfy

$$\begin{aligned} Div\,\sigma &= 0, \qquad x \in \Omega, && (3.4)\\ u &= 0, \qquad x \in \partial\Omega_1, && (3.5)\\ n.\sigma &= 0, \qquad x \in \partial\Omega_2, && (3.6) \end{aligned}$$

while subtraction of the constitutive relations (2.13) leads to

$$\sigma = c^{(1)} \left\{ e + (I - D) e^{(2)} \right\}, \tag{3.7}$$

where I is the identity tensor, the tensor D is given by

$$D = C^{(1)} c^{(2)}, \tag{3.8}$$

and $C^{(1)}$ is the elastic compliance tensor satisfying (2.2).

We remark that the "strain" $De^{(2)}$ is incompatible in the sense that in general

$$\nabla \times \nabla \times De^{(2)} \neq 0. \tag{3.9}$$

The constitutive relations (3.7) enable the boundary value problem (3.4)-(3.6) to be alternatively expressed as

$$\begin{aligned} Div\, c^{(1)} \left[e + (I - D)\, e^{(2)} \right] &= 0, \qquad x \in \Omega, && (3.10)\\ u &= 0, \qquad x \in \partial\Omega_1, && (3.11)\\ n.c^{(1)} e &= n.\,(D - I)\, e^{(2)}, \qquad x \in \partial\Omega_2, && (3.12) \end{aligned}$$

which corresponds to the standard (mixed) boundary value problem with non-zero body force and surface traction, but homogeneous surface displacement. Classical methods may be applied to solve this problem but generally offer no advantage compared to the same methods applied to the constituent problems. Certain simplications, however, might arise from judicious choice of one set of moduli, say, $c^{(2)}$, and, moreover, the above formulation is employed in Section 4.2 where elastic inhomogeneities are discussed.

We list for later reference, the indicial form of several of the expressions introduced above. In three-dimensions, (3.7) is alternatively given by

$$\sigma_{ij} = c^{(1)}_{ijkl} \left\{ e_{kl} + \left(\frac{1}{2} [\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}] - D_{klpq} \right) e^{(2)}_{pq} \right\}, \tag{3.13}$$

while, in particular, for isotropic compressible linear elasticity, we use (2.3) and (2.6) to express (3.8) as

$$D_{ijpq} = \frac{\mu^{(2)}(\nu^{(2)} - \nu^{(1)})}{\mu^{(1)}(1 + \nu^{(1)})(1 - 2\nu^{(2)})} \delta_{ij}\delta_{pq} + \frac{\mu^{(2)}}{2\mu^{(1)}} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right), \tag{3.14}$$

so that (3.13) yields

$$\begin{aligned}\sigma_{ij} &= \Big[\lambda^{(1)}\delta_{ij}\delta_{rs} + \mu^{(1)}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\Big]\Big[e_{rs} + \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}e^{(2)}_{rs} \\ &\qquad + \frac{(\lambda^{(1)}\mu^{(2)} - \lambda^{(2)}\mu^{(1)})}{\mu^{(1)}(3\lambda^{(1)} + 2\mu^{(1)})}e^{(2)}_{kk}\delta_{rs}\Big] \qquad (3.15) \\ &= \Big[\lambda^{(1)}\delta_{ij}\delta_{rs} + \mu^{(1)}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\Big]\Big[e_{rs} + \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}e^{(2)}_{rs} \\ &\qquad + \frac{\mu^{(2)}(\nu^{(1)} - \nu^{(2)})}{\mu^{(1)}(1 + \nu^{(1)})(1 - 2\nu^{(2)})}e^{(2)}_{kk}\delta_{rs}\Big]. \qquad (3.16)\end{aligned}$$

The corresponding formulae in two-dimensions become

$$\sigma_{\alpha\beta} = c^{(1)}_{\alpha\beta\gamma\delta}\left[e_{\gamma\delta} + \left\{\frac{1}{2}(\delta_{\gamma\mu}\delta_{\nu\delta} + \delta_{\gamma\nu}\delta_{\mu\delta}) - D_{\gamma\delta\nu\mu}\right\}e^{(2)}_{\nu\mu}\right], \qquad (3.17)$$

which for isotropic compressible linear elasticity reduces to

$$\begin{aligned}\sigma_{\alpha\beta} &= \Big[\lambda^{(1)}\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu^{(1)}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\Big] \\ &\qquad \times\Big[e_{\gamma\delta} + \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}e^{(2)}_{\gamma\delta} + \frac{(\lambda^{(1)}\mu^{(2)} - \lambda^{(2)}\mu^{(1)})}{2\mu^{(1)}(\lambda^{(1)} + \mu^{(1)})}e^{(2)}_{\kappa\kappa}\delta_{\gamma\delta}\Big] \qquad (3.18) \\ &= \Big[\lambda^{(1)}\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu^{(1)}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\Big] \\ &\qquad \times\Big[e_{\gamma\delta} + \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}e^{(2)}_{\gamma\delta} + \frac{\mu^{(2)}(\nu^{(1)} - \nu^{(2)})}{\mu^{(1)}(1 - 2\nu^{(2)})}e^{(2)}_{\kappa\kappa}\delta_{\gamma\delta}\Big]. \qquad (3.19)\end{aligned}$$

4 Selected properties of the difference field

We review properties relevant to the subsequent discussion.

4.1 Stress free strain and the difference stress

We partially analyse the structure of the difference elastic field, and demonstrate that certain terms in the difference constitutive relation (3.7) produce no stress. The following heuristic intrepretation differs from Eshelby's treatment [9, 12] of the inhomogeneity problem,which is more aligned to the discussion of Section 4.2.

Let us set

$$\sigma = c^{(1)}e^{*}, \qquad x \in \Omega, \qquad (4.1)$$

where

$$\begin{aligned}e^{*} &= e + (I - D)e^{(2)} \qquad (4.2) \\ &= e + e^{**}, \qquad e^{**} = (I - D)e^{(2)} \qquad (4.3)\end{aligned}$$

and let us reformulate the constitutive relation for the unperturbed stress $\sigma^{(2)}$ as

$$\begin{aligned} \sigma^{(2)} &= c^{(2)}e^{(2)} \\ &= c^{(1)}De^{(2)} \\ &= c^{(1)}(e^{(2)} - e^{**}). \end{aligned} \tag{4.4}$$

We conclude *that the strain* $-e^{**}$ *produces no extra stress when the moduli are varied from* $c^{(2)}$ *to* $c^{(1)}$*, and the strain* $e^{(2)}$ *is held fixed.*

The last assertion is explained as follows. Suppose the linear elastic material occupying the region Ω has elastic moduli $c^{(2)}$ and is in equilibrium subject to zero body force and given mixed boundary conditions which create in Ω the nonhomogeneous stress $\sigma^{(2)}(x)$ and strain $e^{(2)}(x)$, related by $\sigma^{(2)} = c^{(2)}e^{(2)}, x \in \Omega$. The traction on the surface of an arbitrary subregion Ω^{**}_m of Ω is given by $t^{(2)}_m = \sigma^{(2)}n = c^{(2)}e^{(2)}n$. Now let a set of the closures of non-intersecting subregions $\Omega^{**}_m, m = 1, 2, \dots$ cover Ω. Detach each subregion Ω^{**}_m from the others, alter its elastic moduli from $c^{(2)}$ to $c^{(1)}$, and apply the traction $t^{(2)}_m$ to its surface. Within the approximations assumed for the linear theory, the stress distribution $\sigma^{(2)}$, while maintaining Ω^{**}_m in equilibrium, causes it to experience a further deformation. Consequently, the corresponding strain is no longer $e^{(2)}$ but $e^{(2)} - e^{**}$, as shown by (4.4). In this sense, a variation of the moduli has created an additional strain field $-e^{**}$ but has produced no extra stress beyond the original stress distribution $\sigma^{(2)}$.

A physical interpretation also may be provided for the difference stress σ. Separate from the loaded region Ω the arbitrary regions Ω^{**}_m, as just defined, vary the elastic moduli in each from $c^{(2)}$ to $c^{(1)}$, and apply tractions $t^{(2)}_m$ to the respective surfaces. Equilibrium requires that the strain be altered from $e^{(2)}$ to $e^{(2)} - e^{**}$, and this alteration deforms Ω^{**}_m into a new shape Ω^{***}_m which does not exactly fit the space from which it was originally cut. A perfect fit is achieved by applying additional surface tractions $t^{(3)}_m$ to Ω^{***}_m in order to return it to its original shape Ω^{**}_m. This operation generates a further stress additional to $\sigma^{(2)}$. The resulting subregions Ω^{**}_m can now be fitted coherently together and cemented in place to recover the region Ω. There is continuity of the displacement across the interfaces $\partial\Omega^{**}_m$ over which now is distributed a surface layer of body force due to the additional traction $t^{(3)}_m$. Remove the surface traction $t^{(3)}_m$ while retaining the bonding at the interface $\partial\Omega^{**}$. This relaxation causes each Ω^{**}_m to undergo an additional strain e^* which produces the additional stress σ, so that finally we have

$$\sigma^{(2)} + \sigma = c^{(1)}\left(e^{(2)} - e^{**} + e^*\right), \tag{4.5}$$

from which we conclude that the additional stress σ is given by (4.1). The sequence of operations outlined above is schematically described in Figure 1.

When the displacement is specified on part Σ_m of the surface $\partial\Omega^{**}_m$ of the subregion Ω^{**}_m, we may repeat the previous cut-and-weld operations. The stress

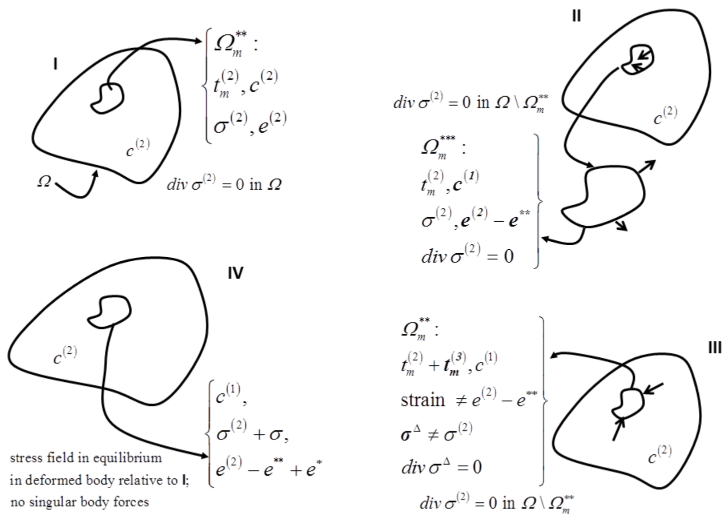

Figure 1: Cut and weld operations corresponding to variation of moduli argument.

$\sigma^{(2)}$ corresponds to the surface traction $t_m^{(2)}$ on all parts of $\partial\Omega_m^{**}$ which after the moduli are altered to $c^{(1)}$ requires additional strain to maintain the stress $\sigma^{(2)}$, which changes the shape of Ω_m^{**} to Ω_m^{***}. To recover Ω_m^{**}, the surface tractions $t_m^{(3)}$ that are applied to $\partial\Omega_m^{***}$ must ensure that the part deformed from Σ_m returns to the originally specified displacement.The argument now proceeds as before.

The strain $-e^{**}$ is analogous to the strain produced in a self-stressed body and provides a relation between various elastic theories in which initial or residual stress can be identified. Eshelby [11, 9] has generally treated systems containing this type of strain and presented examples in dislocation theory and inhomogeneities in stressed bodies. By means of a different approach based upon the variation of elastic moduli, we provide in Section 4.2 an alternative discussion of this interpretation with respect to the inhomogeneity problem, while in Section 6 we consider in detail the relationship with dislocations.

4.2 Elastic inhomogeneities

In this section, we derive certain solutions to simple elastic inhomogeneity problems that are used subsequently to discuss arrays of continuous dislocations. The treatment, which may easily be extended to less simple problems, is based on [18] and [19] and complements the analysis by Eshelby [9, 10, 12], and by Kröner [23], amongst others.

We consider a bounded or unbounded three-dimensional region Ω that contains a region Γ whose surface $\partial\Gamma$ is supposed closed and smooth. Linear elastic solids of different elastic moduli occupy the regions $\Omega\backslash\bar{\Gamma}$ and Γ, and are bonded together across the interfacial surface $\partial\Gamma$. That is, under prescribed loads and boundary displacement, the traction and displacement are continuous across $\partial\Gamma$(the problem in which either bonding or slippage of the displacement occurs on the interface is treated in [27], while general discontinuity relations are discussed in [16]). The elastic moduli, of course, are discontinuous across the interface, but otherwise are supposed continuously differentiable. Several such closed surfaces of discontinuity may be included, but for convenience attention is confined to a single surface. The region Γ is called the *inclusion*, while the complement $\Omega\backslash\bar{\Gamma}$ is called the *matrix*.

Let Ω be in equilibrium subject to specified body force f, mass density ρ, surface tractions F on $\partial\Omega_2$ and displacement g on the remainder, $\partial\Omega_1$. To facilitate the calculations of this Section, it is convenient to reverse the role of the moduli adopted in Section 4.1. Accordingly, we consider the *unperturbed* problem in which the moduli $c^{(1)}$ are continuously differentiable *everywhere in* Ω *and denote the corresponding stress, strain, and displacement by* $(\sigma^{(1)}, e^{(1)}, u^{(1)})$. For the *perturbed* problem, with elastic field $(\sigma^{(2)}, e^{(2)}, u^{(2)})$, under the same loads and surface displacement, we suppose that the moduli inside the inhomogeneity Γ are $c^{(2)}$, while those outside remain unaltered and are $c^{(1)}$.

We use the notation (3.1)-(3.3), and as before, subtract the respective equilibrium and boundary conditions to obtain

$$\begin{aligned}
\sigma &= c^{(1)}\left[e + (I-D)e^{(2)}\right], \qquad x \in \Gamma, && (4.6)\\
\sigma &= c^{(1)}e, \qquad x \in \Omega\backslash\bar{\Gamma}, && (4.7)\\
Div\,\sigma &= 0, \qquad x \in \Omega, && (4.8)\\
u &= 0, \qquad x \in \partial\Omega_1, && (4.9)\\
n\sigma &= 0, \qquad x \in \partial\Omega_2, && (4.10)\\
[n\sigma]_{\partial\Gamma} &= [u]_{\partial\Gamma} = 0. && (4.11)
\end{aligned}$$

In (4.11), square brackets denote the jump across the interface $\partial\Gamma$ in the sense, for example, given by

$$[u^{(\alpha)}]_{\partial\Gamma} = u^{(\alpha)}(inclusion) - u^{(\alpha)}(matrix), \tag{4.12}$$

where the unit normal n on $\partial\Gamma$ is taken outward from Γ.

The particular stress distribution, defined everywhere in Ω by

$$\bar{\sigma} = c^{(1)}e, \qquad x \in \Omega, \tag{4.13}$$

is maintained in equilibrium under zero (mixed) boundary conditions on $\partial\Omega$, unit mass density, body force given by

$$\begin{aligned}
\bar{X} &= Div\left\{(c^{(1)} - c^{(2)})e^{(2)}\right\}, \qquad x \in \Gamma, && (4.14)\\
&= 0, \qquad x \in \Omega\backslash\bar{\Gamma}, && (4.15)
\end{aligned}$$

and a layer of distributed surface force in the interface $\partial\Gamma$ of amount $-n(c^{(1)} - c^{(2)})e^{(2)}$ per unit surface area.

The solution to this problem may be expressed in terms of Green's function, which for linear homogeneous anisotropic elasticity is determined, for example, in [37]. For convenience of presentation, however, we confine attention to homogeneous isotropic elasticity, assume that Ω occupies the whole space $\mathbb{R}^3$, and for the next stage of the argument employ indicial notation. Accordingly, the Green's function $G_{ij}^{(1)}(x,y)$ is given by

$$G_{ij}^{(1)}(x,y) = \frac{1}{4\pi\mu^{(1)}}\frac{\delta_{ij}}{R(x,y)} - \frac{1}{16\pi\mu^{(1)}(1-\nu^{(1)})}\frac{\partial^2}{\partial x_i \partial x_j}R(x,y), \tag{4.16}$$

where

$$R^2 = (x_i - y_i)(x_i - y_i), \tag{4.17}$$

A standard procedure, that includes integration by parts, leads to the representation

$$\begin{aligned} u_i(x) \quad = \quad & \frac{1}{4\pi\mu^{(1)}}\frac{\partial}{\partial x_j}\int_\Gamma \frac{\rho_{ij}(y)}{R(x,y)}\,dy \\ & - \frac{1}{16\pi\mu^{(1)}(1-\nu^{(1)})}\frac{\partial^2}{\partial x_i \partial x_k}\int_\Gamma R(x,y)\rho_{kj}(y)\,dy, \end{aligned} \tag{4.18}$$

where

$$\rho_{ij}(y) = \rho_{ji}(y) = \left(\lambda^{(1)} - \lambda^{(2)}\right) e_{kk}^{(2)}(y)\delta_{ij} + 2\left(\mu^{(1)} - \mu^{(2)}\right) e_{ij}^{(2)}(y). \tag{4.19}$$

Expression (4.18), differently derived by Eshelby [9, 10, 12] (see also Kupradze[24]), leads easily to the corresponding difference strain components. We obtain

$$\begin{aligned} e_{ij}(x) \quad = \quad & \frac{1}{8\pi\mu^{(1)}}\left[\frac{\partial^2}{\partial x_k \partial x_j}\int_\Gamma \frac{\rho_{ik}(y)}{R(x,y)}\,dy + \right. \\ & \left. + \frac{\partial^2}{\partial x_k \partial x_i}\int_\Gamma \frac{\rho_{jk}(y)}{R(x,y)}\,dy\right] \\ & - \frac{1}{8\pi\mu^{(1)}(1-\nu^{(1)})}\frac{\partial^4}{\partial x_i \partial x_j \partial x_r \partial x_s}\int_\Gamma \rho_{rs}(y)R(x,y)\,dy, \end{aligned} \tag{4.20}$$

and the difference dilatation becomes

$$e_{ii}(x) = \frac{1}{4\pi(\lambda^{(1)} + 2\mu^{(1)})}\frac{\partial^2}{\partial x_i \partial x_i}\int_\Gamma \frac{\rho_{ij}(y)}{R(x,y)}\,dy. \tag{4.21}$$

These integrals are not readily evaluated in closed form for the general problem of an arbitrary inhomogeneity Γ, but a solution is possible for a single ellipsoidal inhomogeneity perturbing a stress field uniform at infinity.(See, for example, [9, 10, 12], and [19].)

An exact solution, however, is possible for homogeneous isotropic elasticity provided only that Poisson's ratio is varied ([18]). Then, by standard results in potential theory, we conclude that (4.21) reduces to

$$e_{ii} = -\left(\frac{\lambda^{(1)} - \lambda^{(2)}}{\lambda^{(1)} + 2\mu}\right) e_{ii}^{(2)}, \qquad x \in \Gamma, \tag{4.22}$$

in the inhomogeneity, while for the matrix we have

$$e_{ii} = 0, \qquad x \in \Omega\backslash\bar{\Gamma}. \tag{4.23}$$

Consequently, we have within the inhomogeneity

$$e_{ii}^{(2)} = \left(\frac{\lambda^{(1)} + 2\mu}{\lambda^{(2)} + 2\mu}\right) e_{ii}^{(1)}, \qquad x \in \Gamma, \tag{4.24}$$

while for the matrix

$$e_{ii}^{(2)} = e_{ii}^{(1)}, \qquad x \in \Omega\backslash\bar{\Gamma}, \tag{4.25}$$

where $e_{ii}^{(1)}$, possibly non-constant, is known from the unperturbed problem in which linear elastic material of constant moduli $\lambda^{(1)}, \mu$ occupies the whole space $\mathbb{R}^3$. In fact, the same result holds ([18]) for nonhomogeneous isotropic linear elasticity provided μ is constant and the Poisson's ratios $\nu^{(1)}$ and $\nu^{(2)}$ are continuously differentiable everywhere in $\mathbb{R}^3$ and Γ respectively.

The material therefore experiences no change in its dilatation in the matrix, while in the inhomogeneity the change is (4.22). Substitution of (4.24) in (4.19) and (4.18) enables the perturbed displacement to be derived in the form

$$u_i^{(2)}(x) = -\frac{(\lambda^{(1)} - \lambda^{(2)})}{4\pi(\lambda^{(2)} + 2\mu)} \frac{\partial}{\partial x_i} \int_\Gamma \frac{e_{ii}^{(1)}(y)}{R(x,y)}\, dy + u_i^{(1)}(x), \qquad x \in \mathbb{R}^3,\ y \in \Gamma. \tag{4.26}$$

The integration just outlined is essentially the Somigliana procedure. See, for instance, [26].

We observe from (4.26) that the difference displacement $u = u^{(1)} - u^{(2)}$ may be regarded as the gravitational attraction due to a potential distribution of density $e_{kk}^{(1)}$ over the inhomogeneity Γ. For loads that produce a uniform unperturbed dilatation, the appropriate gravitational attraction is known for several different regions Γ. In particular, when Γ is a hollow ellipsoid, the attraction is zero within the hollow. Moreover, by standard potential theory, the displacement u is harmonic in the matrix, and accordingly, each component achieves its maximum and minimum value on the interface $\partial\Gamma$. The solution to (4.26) is also known for several non-constant densities.

To illustrate the method, we consider the simple example of a spherical inhomogeneity of radius a in an infinite medium that perturbs a uniform hydrostatic pressure P. We let the origin of coordinates be located at the centre of the sphere. The unperturbed dilatation is

$$e_{kk}^{(1)} = -3A, \qquad A = \frac{P}{(3\lambda^{(1)} + 2\mu)}, \tag{4.27}$$

and from (4.26), the difference displacement becomes

$$u_i(x) = -3A \frac{(\lambda^{(1)} - \lambda^{(2)})}{4\pi(\lambda^{(2)} + 2\mu)} \frac{\partial}{\partial x_i} \int_\Gamma \frac{1}{R(x,y)}\, dy. \tag{4.28}$$

The integral in (4.28) is the potential of a homogeneous sphere of unit density. Consequently, we have within the inhomogeneity,

$$u_i(x) = \frac{(\lambda^{(1)} - \lambda^{(2)})}{(\lambda^{(2)} + 2\mu)} A x_i, \qquad x \in \Gamma, \tag{4.29}$$

and within the matrix

$$u_i(x) = a^3 \frac{(\lambda^{(1)} - \lambda^{(2)})}{(\lambda^{(2)} + 2\mu)} \frac{A x_i}{R^3(x,0)}, \qquad x \in \mathbb{R}^3 \backslash \bar{\Gamma}. \tag{4.30}$$

The corresponding difference stress from (3.15),(4.24), and (4.25) becomes for the inhomogeneity

$$\sigma_{ij}(x) = -\frac{4\mu(\lambda^{(1)} - \lambda^{(2)})}{(\lambda^{(2)} + 2\mu)} A \delta_{ij}, \qquad x \in \Gamma, \tag{4.31}$$

and for the matrix

$$\sigma_{ij}(x) = 2\mu A a^3 \frac{(\lambda^{(1)} - \lambda^{(2)})}{(\lambda^{(2)} + 2\mu)} \left(\frac{\delta_{ij}}{R^3(x,0)} - \frac{3 x_i x_j}{R^5(x,0)} \right), \quad x \in \mathbb{R}^3 \backslash \bar{\Gamma}. \tag{4.32}$$

We note that the difference stress does not vanish identically in the matrix.

On letting $a \to 0$ and P increase such that $a^3 P$ remains constant, we recover from (4.29)-(4.32) the displacement and stress for a centre of dilatation.

Clearly, when more than one inhomogeneity is present, the above treatment holds but with the region of integration extended over all inhomogeneities.

5 Dislocation theory

Let $\beta^{(E)} \in C^2(\Omega, \mathbb{R}^3 \times \mathbb{R}^3)$ denote an "elastic" distortion tensor that does not necessarily correspond to either a deformation, or displacement, gradient, and therefore whose symmetric part $E^{(E)}$ may be incompatible.

Let $\Sigma \subset \Omega$ denote any open smooth surface bounded by the closed smooth curve $\partial\Sigma$ described in a right-handed sense with respect to the unit outward normal on Σ. Define the Burgers vector for $\partial\Sigma$ by

$$b = -\oint_{\partial\Sigma} \beta^E . dx, \tag{5.1}$$

which by Stokes Theorem becomes equivalently

$$\begin{aligned} b &= -\int_\Sigma (\nabla \times \beta^{(E)})^T . n \, dS & (5.2)\\ &= \int_\Sigma \alpha^T . n \, dS, & (5.3) \end{aligned}$$

where the second order tensor α, termed the *dislocation (line) density*, is defined by

$$\alpha = -\nabla \times \beta^{(E)}, \tag{5.4}$$

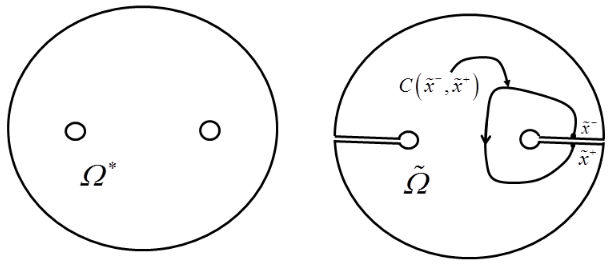

Figure 2: Saint-Venant/Cesaro set-up for Burgers vector calculation

and in consequence

$$Div\, \alpha = 0. \tag{5.5}$$

On taking the surface Σ to be infinitesimally small and threaded by a single dislocation line with continuously distributed core, we conclude that

$$\alpha = l \otimes b, \tag{5.6}$$

where l is the unit vector tangent to the dislocation line. The first subscript of the component α_{ij} in the representation (5.6) provides the direction of the dislocation line, while the second gives the direction of the Burgers vector.

Remark 5.1 ***Specific assumptions***
Let $u^{(E)}$ be a displacement vector defined on Ω whose singularities are distributed within subregions $\Omega_i \subset \Omega, i = 1, 2, \ldots$. For example, Ω_i can be the site of discrete dislocation lines, or a continuous distribution of dislocations. We suppose that the subregions Ω_i are such that the multiply-connected region $\Omega^* = \Omega \backslash (\cup_i \Omega_i)$ can be reduced to a simply-connected region $\tilde{\Omega}$ by the introduction of appropriate cuts. Motivated by the property that the potential for an irrotational vector field on a multiply-connected region is generally discontinuous, we assume that $u^{(E)}$ may be discontinuous across the cuts $\Omega^* \backslash \tilde{\Omega}$ and that $u^{(E)} \in C^2(\tilde{\Omega})$. Furthermore, we assume that $\beta^{(E)}$ is the gradient of $u^{(E)}$ in $\tilde{\Omega}$. Consider a simple closed curve C drawn in Ω^* to cross one and only one cut at the point $\tilde{x}$. Let $\tilde{x}^+$ and $\tilde{x}^-$ be points on either side of the cut immediately adjacent to $\tilde{x}$, and denote by $C(\tilde{x}^-, \tilde{x}^+)$ that part of the curve C that starts at the point $\tilde{x}^-$ and ends at $\tilde{x}^+$ without crossing the cut; that is, the curve $C(\tilde{x}^-, \tilde{x}^+)$ is traversed entirely in $\tilde{\Omega}$. See Figure 2.

Subject to these assumptions, we seek to obtain an alternative expression for the Burgers vector using the classic argument due to Saint-Venant and Cesaro (see, e.g., [17,21,22]).

Let $E^{(E)}, W^{(E)}$ be the symmetric and anti-symmetric parts of $\beta^{(E)}$ respectively, and set $y(x) = x - \tilde{x}^-$ so that $y(\tilde{x}^+) = y(\tilde{x}^-) = 0$. Then

$$\begin{aligned} b &= -\int_{C(\tilde{x}^-,\tilde{x}^+)} \nabla u^{(E)}.dx \\ &= -\int_{C(\tilde{x}^-,\tilde{x}^+)} E^{(E)}.dx - \int_{C(\tilde{x}^-,\tilde{x}^+)} W^{(E)}.dy \\ &= -\int_{C(\tilde{x}^-,\tilde{x}^+)} E^{(E)}.dx + \int_{C(\tilde{x}^-,\tilde{x}^+)} (dx.\nabla)W^{(E)}y - \Big[W^{(E)}.y\Big]_{\tilde{x}^-}^{\tilde{x}^+}, \end{aligned} \tag{5.7}$$

where we have integrated by parts. The last term on the right of (5.7) vanishes since $y = 0$ at both endpoints. Moreover, by virtue of (2.19), in an obvious notation, we have for $x \in \tilde{\Omega}$,

$$\begin{aligned} \nabla W^{(E)}y &= \nabla\omega^{(E)} \times y \\ &= -y \times (\nabla \times E^{(E)}), \end{aligned}$$

and in consequence (5.7) may be expressed as

$$b = -\oint_{\partial\Sigma} U.dx, \tag{5.8}$$

where the non-symmetric second order tensor U is given by

$$U = E^{(E)} + y \times (\nabla \times E^{(E)}). \tag{5.9}$$

This expression is used later to determine the Burgers vector for the edge and screw dislocations.

Remark 5.2 *The representations (5.1) and (5.8) deliver the same value of Burgers vector only under assumptions stipulated in Remark 5.1. For example, suppose $\beta^{(E)}$ is the gradient of a sufficiently smooth displacement vector whose symmetric part vanishes, but whose anti-symmetric part is non-zero: $E^{(E)} = 0, W^{(E)} \neq 0$ on $\tilde{\Omega}$. Then from (5.8) we have $b = 0$. This conclusion is consistent with (5.1) when it is recalled that (2.19) and (2.20) imply that $W^{(E)}$ is constant and hence that the integral (5.1) vanishes when the curves C and $\partial\Sigma$ coincide. Of course, when $\beta^{(E)}$ is not the gradient of a vector field, no comparison is possible between (5.1) and (5.8).*

Remark 5.3 *The last comment is further illustrated by examples discussed in [34] in which the elastic strain but not the elastic distorsion vanish and consequently the Burgers vector and dislocation density are non-zero. These examples explicitly demonstrate conditions under which (5.1) and (5.8) are not equivalent.*

Remark 5.4 *On appealing to the constitutive relation (5.17) for the elastic stress, we conclude that under the assumptions of Remark 5.1, when $\beta^{(E)} = \nabla u^{(E)}$, with $u^{(E)} \in C^2(\tilde{\Omega})$, the conditions $E^{(E)} = 0, W^{(E)} \neq 0$ imply not only*

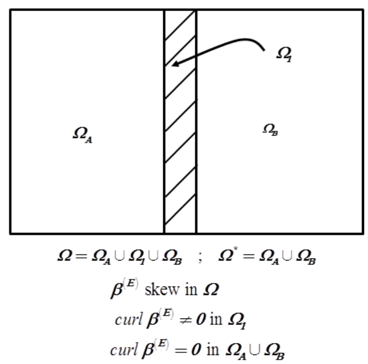

Figure 3: Example where Burgers vector cannot be computed from elastic strain field.

that $b = 0$ but also that the stress vanishes. Furthermore, as mentioned in Remark 5.2, $W^{(E)}$ is constant on $\tilde{\Omega}$ and consequently $u^{(E)}$ is there a rigid body displacement. This conclusion is not surprising since $\tilde{\Omega}$ is supposed simply-connected. Perhaps of greater significance is the implication that in order to have non-uniform $W^{(E)}$ or $b \neq 0$ for at least one closed curve in Ω^*, we must exclude those Ω_i of the type where Ω^* can be rendered simply-connected by the introduction of cuts. See Figure 3.

To continue this brief description of dislocation theory, we introduce the second order tensor $\beta^{(P)}$ such that the sum

$$\beta = \beta^{(E)} + \beta^{(P)} \tag{5.10}$$

corresponds to the gradient of a continuously differentiable displacement vector field u. We let the symmetric and anti-symmetric parts of β and $\beta^{(P)}$ be denoted respectively by E, W and $E^{(P)}, W^{(P)}$. We have $\nabla \times \beta = 0$, and consequently

$$\alpha = \nabla \times \beta^{(P)} = -\nabla \times \beta^{(E)}. \tag{5.11}$$

We may operate on (5.11) to obtain

$$\begin{aligned} (\nabla \times \alpha)^T &= -(\nabla \times \nabla \times \beta^{(E)})^T \\ &= -\nabla \times \nabla \times (\beta^{(E)})^T, \end{aligned}$$

which leads to

$$\begin{aligned} (\nabla \times \alpha) + (\nabla \times \alpha)^T &= -\nabla \times \nabla \times (\beta^{(E)} + (\beta^{(E)})^T) \\ &= -2\nabla \times \nabla \times E^{(E)}. \end{aligned} \tag{5.12}$$

The strain $E^{(E)}$ by hypothesis may be incompatible so that the incompatibility tensor, given by

$$\begin{aligned} \eta &= -\nabla \times \nabla \times E^{(E)} && (5.13) \\ &= \nabla \times \nabla \times E^{(P)}, && (5.14) \end{aligned}$$

is in general non-zero, and by (5.12) alternatively may be expressed in terms of the dislocation density as

$$\eta = \frac{1}{2}\left(\nabla \times \alpha + (\nabla \times \alpha)^T\right). \tag{5.15}$$

Although (5.13) may be solved for $E^{(E)}$ in terms of given η (see [11, p.92]), the solution can be used to determine the Burgers vector only by appeal to relation (5.8), subject to its validity, and not directly from (5.1).

We immediately infer from (5.15) that

$$\alpha = 0 \qquad \Rightarrow \eta = 0, \tag{5.16}$$

but the reverse implication is not necessarily true. The symmetric part of $\beta^{(P)}$ may be zero but its anti-symmetric part non-zero, and then by (5.14) and (5.11) we have $\eta = 0$, but $\alpha \neq 0$.

The (incompatible) elastic strain $E^{(E)}$ in a linear elastic body of elastic moduli c generates a stress σ according to the constitutive relation

$$\begin{aligned} \sigma &= c\, E^{(E)} && (5.17)\\ &= c\,(E - E^{(P)}). && (5.18) \end{aligned}$$

In the absence of body force and surface traction, and for homogeneous displacement on $\partial\Omega_1$, the stress σ in equilibrium satisfies

$$\begin{aligned} Div\, c(E - E^{(P)}) &= 0, \qquad x \in \Omega, && (5.19)\\ u &= 0, \qquad x \in \partial\Omega_1, && (5.20)\\ n.c(E - E^{(P)}) &= 0, \qquad x \in \partial\Omega \backslash \partial\Omega_1. && (5.21) \end{aligned}$$

Kröner [23] solves the boundary value problem (5.19)-(5.21) for the dislocation stress in terms of stress functions derived from the incompatibility tensor and the biharmonic Green's function. The method could be employed in the present treatment, but we prefer, however, the alternative procedure already adopted in Section 4.2 that involves the elastic Green's function. Willis [40] also employs the elastic Green's function to calculate the dislocation stress in terms of the dislocation density.

6 Relation between dislocations and variation of elastic moduli

The various analogies and relationships discussed in this Section are three-dimensional generalisations of results which in two dimensions were established by Filon [13] (see also [6]), [32], and the recent review [20]).

6.1 Basic derivation

Inspection of the respective boundary value problems formulated in Sections 3 and 5 indicates that they are analogous. To be precise, let us retain the same notation for the tensors σ and E that appear in both Sections, and set $c = c^{(1)}$ together with

$$E^{(P)} = -(I-D)\,e^{(2)}, \tag{6.1}$$
$$E^{(E)} = e + (I-D)e^{(2)}. \tag{6.2}$$

Such substitution renders the boundary value problem (3.4)-(3.7) (or (3.10)-(3.12)) identical to the dislocation boundary value problem (5.17)-(5.21). Consequently, we have extended Filon's construct from two- to three-dimensions.

Before developing other implications, we make two observations. First, let us note that the construct may be employed to generate anisotropic solutions for both discrete and continuous distributions of dislocations subject to appropriate conditions introduced later, especially for the isotropic theory. The second observation concerns the application of the construct to the conclusions of Section 4.1 for stress- free strains. In terms of the respective notations, upon setting

$$E^{(P)} = -e^{**}, \qquad E^{(E)} = e^{*}, \qquad E = e, \tag{6.3}$$

we infer from the arguments presented in Section 4.1 that the plastic strain does not produce stress irrespective of the boundary conditions imposed on the surface $\partial\Omega$. The equivalences defined in (6.3) enable the plastic, elastic, and total strains to be interpreted in the light of the cut-and-weld operations described in Section 4.1.

The plastic strain is not arbitrary but subject to a condition obtained by inversion of (6.1). For this purpose, we suppose that $(I-D)$ is invertible, and then it follows from the compatibility of the strain $e^{(2)}$ that

$$\nabla\times\nabla\times(I-D)^{-1}E^{(P)} = 0, \tag{6.4}$$

which restricts the plastic strains produced by the Filon construct.

Since only strains appear in the construct, the dislocation density defined in (5.11) can only partially be determined, and indeed from (5.12) we have

$$\begin{aligned}(\nabla\times\alpha)+(\nabla\times\alpha)^T &= 2\nabla\times\nabla\times E^{(P)} && (6.5)\\ &= 2\nabla\times\nabla\times De^{(2)}, && (6.6)\end{aligned}$$

by virtue of (6.1) and the compatibility of $e^{(2)}$. In consequence, the incompatibility tensor (5.15) becomes

$$\eta = \nabla\times\nabla\times De^{(2)}. \tag{6.7}$$

Furthermore, for a simple closed curve $\partial\Sigma$ drawn through a region in which $\beta^{(E)}$ is the gradient of a continuously differentiable displacement, the Burgers vector may be obtained from (5.8) in which the tensor U from (5.9) becomes

$$\begin{aligned} U &= E^{(E)} + y \times (\nabla \times E^{(E)}) && (6.8) \\ &= E - E^{(P)} - y \times (\nabla \times E^{(P)}). && (6.9) \end{aligned}$$

But by hypothesis, E is derived from a continuously differentiable displacement vector, u, and consequently may be omitted from the integral (5.8). The expression for the Burgers vector then simplifies to

$$\begin{aligned} b &= \oint_{\partial\Sigma} \left(E^{(P)} + y \times (\nabla \times E^{(P)}) \right) .dx && (6.10) \\ &= \oint_{\partial\Sigma} \left(De^{(2)} - y \times (\nabla \times [e^{(2)} - De^{(2)}]) \right) .dx, && (6.11) \end{aligned}$$

where we have substituted from (6.1).

In contrast, the dislocation density depends on the plastic distorsion and not only on its symmetric part, and therefore may be non-zero even for zero plastic strain. Its explicit representation is given by

$$\begin{aligned} \alpha &= \nabla \times E^{(P)} + \frac{1}{2}\nabla \times (\beta^{(P)} - (\beta^{(P)})^T) && (6.12) \\ &= -\nabla \times (I - D)e^{(2)} + \frac{1}{2}\nabla \times (\beta^{(P)} - (\beta^{(P)})^T). && (6.13) \end{aligned}$$

Mean values of the dislocation solution are easily calculated. In the displacement boundary value problem when $u = 0$ on $\partial\Omega$, we have from (5.10) that

$$\int_\Omega E^{(E)}\,dx = -\int_\Omega E^{(P)}\,dx. \qquad (6.14)$$

On the other hand, in the traction boundary value problem, when $n\sigma = 0$ on $\partial\Omega$, for zero body force and fixed number k, we have by integration of the identity

$$x_k Div\,\sigma = 0, \qquad (6.15)$$

that the mean value of the stress is

$$\int_\Omega \sigma\,dx = 0. \qquad (6.16)$$

The last result holds irrespective of the particular constitutive relation satisfied by the stress.

It is convenient to specialise several of the above general relationships to isotropic linear elasticity. On adopting an indicial notation, we conclude from (3.16) that

$$E^{(P)}_{ij} = -\left[\frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}} e^{(2)}_{ij} + \frac{\mu^{(2)}(\nu^{(1)} - \nu^{(2)})}{\mu^{(1)}(1+\nu^{(1)})(1-2\nu^{(2)})} e^{(2)}_{kk}\delta_{ij} \right], \qquad (6.17)$$

while in two-dimensions we have from (3.19)

$$E_{\alpha\beta}^{(P)} = -\left[\frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}} e_{\alpha\beta}^{(2)} + \frac{\mu^{(2)}(\nu^{(1)} - \nu^{(2)})}{\mu^{(1)}(1 - 2\nu^{(2)})} e_{\kappa\kappa}^{(2)} \delta_{\alpha\beta}\right]. \tag{6.18}$$

We use these expressions to calculate the corresponding incompatibility tensor for the plastic strain, for which purpose we further specialise to homogeneous isotropic linear elasticity. In three-dimensions, we substitute (6.17) in (5.14) and after recalling that $e_{ij}^{(2)}$ is compatible and that $e_{kk}^{(2)}$ is harmonic on the simply-connected region Ω or multiply-connected region Ω^* that excludes singularities present in the elastic fields, we obtain

$$\eta_{ij} = \gamma \frac{\mu^{(1)}}{\mu^{(2)}} e_{kk,ij}^{(2)}, \qquad x \in \Omega^*, \tag{6.19}$$

where γ is given by

$$\gamma = \frac{(\nu^{(1)} - \nu^{(2)})}{(1 + \nu^{(1)})(1 - 2\nu^{(2)})}. \tag{6.20}$$

For plane strain elastic problems in which $e_{\kappa\kappa}^{(\alpha)}$ depends on the variables x_1, x_2, the only possible non-zero component of the incompatibility tensor is η_{33}, which now reduces to

$$\eta_{33} = -\tau e_{\kappa\kappa,\delta\delta}^{(2)}, \qquad x \in \Omega^*, \tag{6.21}$$

where

$$\tau = (1 + \nu^{(1)})\gamma. \tag{6.22}$$

The plane strain dilatation $e_{\kappa\kappa}$ is harmonic on those regions where it is defined. In particular, it is defined on multiply-connected regions Ω^* that exclude singularities. Accordingly, on such regions the incompatibility tensor vanishes, and the plane elastic strain $E^{(E)}$ is derivable from a discontinuous displacement vector defined on Ω^*.

In the next section, we discuss for isotropic linear elasticity the solution generated from the construct when either the shear modulus or Poisson's ratio separately vary.

7 Moduli independent fields

We investigate the stress and strain associated with dislocations derived from linear elastic equilibrium problems with zero body force whose displacement or stress are independent of the moduli.(Necessary and sufficient conditions for the stress to be independent of Poisson's ratio are shown in [28] to follow from a Cosserat spectral decomposition, which is also employed to discuss the traction boundary value problem with divergences free body force. Other properties of the Cosserat spectrum are derived in [27].) For present purposes, however, it suffices to confine attention to the problems in homogeneous isotropic elasticity studied by Carlson [3, 4] who requires the displacement to be twice continuously differentiable on the simply -connected bounded region Ω. As expected, the corresponding incompatibility tensor and dislocation density tensor, determined respectively from (6.19) and (6.13), are zero. Thus, the plastic and elastic strains in the dislocation problem are compatible and derivable from continuous displacements, from which we conclude that the corresponding Burgers vector vanishes. Under the same conditions, however, the dislocation stress σ vanishes only for homogeneous tractions everywhere on the boundary, confirming conditions derived by Mura [31] that ensure "impotent stress" (or "zero-stress everywhere") for continuous dislocation distributions; see also [15, p.597]. We examine conditions for the displacement, mixed, and traction boundary value problems. These implications lose their validity once the region Ω becomes multiply-connected, or the fields $(u^{(\alpha)}, \sigma^{(\alpha)})$ contain singularities. Further examples are considered when inhomogeneities are discussed.

7.1 Variation of the shear modulus alone

In homogeneous isotropic linear elasticity, when the shear modulus alone is varied and Poisson's ratio is fixed, (6.20) shows that the constant γ vanishes, and consequently by (6.19) that the incompatibility tensor is zero. The plastic strain, which by (6.17) reduces to

$$E^{(P)} = \frac{(\mu^{(2)} - \mu^{(1)})}{\mu^{(1)}} e^{(2)}, \tag{7.1}$$

is compatible and corresponds to a continuous displacement $u^{(P)}$ given by

$$u^{(P)} = \frac{(\mu^{(2)} - \mu^{(1)})}{\mu^{(1)}} u^{(2)} + f + x \times d, \tag{7.2}$$

for constant vectors f, d. We may then determine the plastic distorsion $\beta^{(P)}$ as the gradient of $u^{(P)}$, and conclude from (5.11) that the dislocation density tensor α vanishes. The vanishing of η also implies that the elastic strain $E^{(E)}$ is compatible and derivable from a continuous displacement $u^{(E)}$. But since Ω is supposed simply-connected, (5.1) and (5.8) are consistent and independently lead to a zero Burgers vector.

The stress in the dislocation problem from either (3.16) or (5.18) and (6.17) becomes

$$\sigma = \frac{(\mu^{(1)} - \mu^{2})}{\mu^{(2)}} \sigma^{(2)} + \lambda^{(1)} I \, tr \, e + 2\mu^{(1)} e, \tag{7.3}$$

and vanishes only subject to additional conditions.

We now apply Carlson's results [4] and examine specific conditions under which either the displacement or stress are independent of the shear modulus. We let the body-force vanish, and keep fixed Poisson's ratio and the boundary conditions.

In the *displacement boundary value problem*, the displacement does not alter [4], and consequently we have

$$
\begin{aligned}
u &= 0, && (7.4)\\
e &= E = 0, && (7.5)\\
E^{(E)} &= -E^{(P)} &&\\
&= \frac{\left(\mu^{(1)}-\mu^{(2)}\right)}{\mu^{(1)}}e^{(2)}, && (7.6)\\
\sigma &= \frac{\left(\mu^{(1)}-\mu^{(2)}\right)}{\mu^{(2)}}\sigma^{(2)}. && (7.7)
\end{aligned}
$$

In the *traction boundary value problem*, the stress does not depend upon the shear modulus, so that for fixed body force and Poisson's ratio we have $\sigma = 0$, $x \in \bar{\Omega}$. This is the unique solution to the homogeneous traction boundary problem provided $-1 < \nu < 1, \mu^{(1)} \neq 0$ ([36]). Consequently, by (2.6) the stress-elastic strain relations may be inverted to give $E^{(E)} = 0$, and therefore $E = E^{(P)}$, where the compatible plastic strain $E^{(P)}$ remains given by (7.6).

In the *mixed boundary value problem*, with zero body-force and homogeneous boundary traction specified on $\partial\Omega_2$, the displacement is independent of the shear modulus so that $u = e = E = 0$. The compatible plastic strain is again given by (7.6), while the dislocation stress σ remains given by (7.7). It does not vanish provided $\sigma^{(2)}$ is the non-trivial solution to the given mixed boundary problem for $\mu^{(2)}$.

We now consider the problem in which the shear modulus is fixed and Poisson's ratio alone varies. It is immediately apparent from (6.19) that the incompatibility tensor in general does vanish.

7.2 Variation of Poisson's ratio alone

Carlson [3] considers dependence upon Poisson's ratio for fixed body force and shear modulus. It is supposed that the elastic moduli $\lambda^{(1)}, \mu$ lie in the range sufficient for uniqueness, and that both sets of moduli are constant.

For the *displacement boundary value problem*, a uniformly constant dilatation $\bar{\theta}$ for one value of Poisson's ratio $\nu^{(2)}$ implies that the displacement is independent of Poisson's ratio. (For example, when $\nu^{(2)} = 1$, then $u^{(2)} = (x-d) \times \nabla\phi + \nabla\psi$, where d is a constant vector, $\phi(x)$ is harmonic, and $\psi(x)$ is an arbitrary function.) Thus, we have that $u = e = E = 0$, $E^{(E)} = -E^{(P)}$, and from (6.17) that

$$E^{(P)} = -\gamma\bar{\theta}I, \tag{7.8}$$

where γ is defined by (6.20). Accordingly, $\eta = 0$, and the elastic and plastic strains are compatible, with the plastic displacement given by

$$u^{(P)} = -\frac{\gamma\bar{\theta}}{3}x + f + x \times d, \tag{7.9}$$

from which we deduce that $\alpha = 0$ and $b = 0$. The dislocation stress, however, is not zero but

$$\sigma = \gamma\bar{\theta}(3\lambda^{(1)} + 2\mu)I. \tag{7.10}$$

In the *traction boundary value problem*, when for one value of Poisson's ratio, say $\nu^{(2)}$, the dilatation is linear so that

$$tr\, e^{(2)} = a.(x - c) + q, \tag{7.11}$$

where $a, c \in \mathbb{R}^3, q \in \mathbb{R}$ are constant, the stress is independent of Poisson's ratio. In consequence, $\sigma = 0$, and provided Poisson's ratio $\nu^{(1)}$ lies in the uniqueness range $-1 < \nu^{(1)} < 1, \mu \neq 0$, then $E^{(E)} = 0$, and $E = E^{(P)}$. The plastic strain is compatible and by (6.17) becomes

$$E^{(P)} = -\gamma\, (a.(x - c) + q)\, I. \tag{7.12}$$

The continuous displacement $u^{(P)}$ is easily derived to be

$$u^{(P)} = -\gamma\, \{(a.(x - c) + q)(x - c) - (1/2)a(x - c).(x - c)\} + f + x \times d, \tag{7.13}$$

where $f, d \in \mathbb{R}^3$ are constants.

We deduce directly from (5.11) and (7.12) that $\alpha = 0$, so that (5.1) and (5.8) are consistent and give $b = 0$.

In the *mixed boundary value problem*, the vanishing of the dilatation for one value of Poisson's ratio implies that both the displacement and stress are independent of Poisson's ratio. Consequently, let us suppose that the Poissons's ratio $\nu^{(2)}$ can be found such that $tr\, e^{(2)} = 0$ (for example, set $\nu^{(2)} \neq 1$, $u^{(2)} = \nabla\phi$, where ϕ is harmonic). We conclude that $u = e = E = \sigma = 0$, $E^{(E)} = -E^{(P)}$ and by (6.17) that $E^{(P)} = 0$, so that $\eta = 0$. The plastic strain is compatible and corresponds to a rigid body displacement, which implies that $\alpha = 0$. Therefore, subject to the stated assumptions, the generalised Filon construct for mixed boundary value problems on simply-connected regions fails to generate dislocations in the sense that the Burgers vector derived from (5.1) or (5.8) vanishes.

7.3 Comments

The examples discussed in Sections 7.1 and 7.2 involve non-trivial but compatible plastic strains leading to zero dislocation density and Burgers vector. The dislocation stress does not always vanish in each case. Nevertheless, our discussion indicates that for the standard boundary value problems of linear homogeneous isotropic elasticity on simply-connected regions Ω and for twice continuously differentiable displacements, the Filon construct does not generate dislocations and indeed leads to the vacuous result that $b = \eta = \alpha = 0$. Furthermore, it is only in the traction boundary value problem that the dislocation stress vanishes. In the displacement and mixed boundary value problems, the difference, or dislocation, stress is tacitly non-zero on the part $\partial\Omega_1$ of the boundary , and the condition of homogeneous boundary tractions required for Mura's impotent stress is not satisfied. Consequently, displacement and mixed boundary conditions are consistent with non-zero dislocation stress.

Filon's construct consequently possesses limitations, certainly in regard to its application to linear homogeneous isotropic elasticity as considered in this paper. In order to generate meaningful results that admit incompatibilities, some or all of the assumptions of Sections 7.1 and 7.2 must be abandoned. For example, the next section applies Filon's construct to the fundamental examples of the edge and screw dislocations, obtained from elastic solutions possessing certain singularities. Both η and α become singular at the origin which therefore is excluded from Ω causing it to become multiply-connected. Equally important is the assumption of homogeneous isotropic linear elasticity. As remarked in Section 1, Filon's construct remains valid for nonhomogeneous anisotropic elastic solutions, and leads to solutions for continuous distributions of dislocations in simply-connected regions, generated even from regular elastic solutions.

In the final section, we consider the inhomogeneity problem. This is excluded from Carlson's analysis precisely because the displacement does not satisfy the differentiability assumptions on the interface of the inclusion.

8 Special examples

We derive solutions for an edge and a screw dislocation by means of Filon's construct, and show how a singular elastic solution yields a discontinuous displacement in the corresponding dislocation problem. Other elastic problems may be similarly employed to generate dislocation solutions (e.g., dislocation loops) that combine the basic edge and screw dislocations. An account is postponed to elswhere.

8.1 Edge dislocation and point forces

The stress and strain for the edge dislocation are obtained from the homogeneous isotropic linear elastic solution for a linear uniform distribution of point forces in the whole space. A complex variable treatment is presented in [20]. Suppose the edge dislocation is aligned along the infinite x_3- axis and consider the plane strain elastic field due a point force X_1 uniformly distributed along the x_3- axis in the whole space $\mathbb{R}^3$ and directed along the positive x_1- axis. The continuous displacement in the elastic problem (see, e.g., [25, p.209]) for shear modulus $\mu^{(\alpha)}$ and Poisson's ratio $\nu^{(\alpha)}$ is given by

$$\begin{aligned} u_1^{(\alpha)} &= -\frac{X_1}{8\pi\mu^{(\alpha)}}\frac{(3-4\nu^{(\alpha)})}{(1-\nu^{(\alpha)})}\ln r - \frac{X_1}{8\pi\mu^{(\alpha)}(1-\nu^{(\alpha)})}\frac{x_2^2}{r^2}, \\ u_2^{(\alpha)} &= \frac{X_1}{8\pi\mu^{(\alpha)}(1-\nu^{(\alpha)})}\frac{x_1x_2}{r^2}, \end{aligned}$$

where $r^2 = x_1^2 + x_2^2 \neq 0$ and $x \in \Omega^* = \mathbb{R}^3\backslash\{x : x_1 = x_2 = 0\}$.

We deduce that

$$e_{\alpha\alpha}^{(\alpha)} = -\frac{x_1 X_1}{4\pi\mu^{(\alpha)} r^2}\frac{(1-2\nu^{(\alpha)})}{(1-\nu^{(\alpha)})}, \tag{8.1}$$

$$e_{11}^{(\alpha)} = -\frac{x_1 X_1}{8\pi\mu^{(\alpha)}(1-\nu^{(\alpha)}) r^2}\left((3-4\nu^{(\alpha)}) - \frac{2x_2^2}{r^2}\right), \tag{8.2}$$

$$e_{22}^{(\alpha)} = \frac{x_1 X_1 (x_1^2 - x_2^2)}{8\pi\mu^{(\alpha)}(1-\nu) r^4}, \tag{8.3}$$

$$e_{12}^{(\alpha)} = -\frac{x_2 X_1}{8\pi\mu^{(\alpha)}(1-\nu^{(\alpha)}) r^2}\left((1-2\nu^{(\alpha)}) + \frac{2x_1^2}{r^2}\right). \tag{8.4}$$

Now let the shear modulus remain fixed at the value μ and consider the difference in the strains (8.2)-(8.4) for a variation in Poisson's ratio from $\nu^{(2)}$ to $\nu^{(1)}$.

The corresponding elastic strain $E^{(E)}$, obtained from (6.2), becomes

$$E_{\alpha\beta}^{(E)} = e_{\alpha\beta} + \frac{(\nu^{(1)} - \nu^{(2)})}{(1-2\nu^{(2)})} e_{\kappa\kappa}^{(2)} \delta_{\alpha\beta}, \tag{8.5}$$

which by (8.2)-(8.4) yields

$$E_{11}^{(E)} = B\left(\frac{x_1(x_1^2 + 3x_2^2)}{4\pi r^4 (1-\nu^{(1)})} - \frac{x_1}{2\pi r^2}\right), \tag{8.6}$$

$$E_{22}^{(E)} = B\left(\frac{x_1(x_1^2 - x_2^2)}{4\pi r^4 (1-\nu^{(1)})} - \frac{x_1}{2\pi r^2}\right), \tag{8.7}$$

$$E_{12}^{(E)} = -B\frac{x_2(x_2^2 - x_1^2)}{4\pi r^4 (1-\nu^{(1)})}, \tag{8.8}$$

where

$$B = \frac{(\nu^{(1)} - \nu^{(2)}) X_1}{2\mu(1-\nu^{(2)})}. \tag{8.9}$$

We either may show that the elastic strain (8.6)-(8.8) is compatible on Ω^* by appealing to either (5.13) or (6.7), or we may directly prove the elastic discontinuous displacement for $x \in \Omega^*$ ([33, p.57]) to be

$$u_1^{(E)} = \frac{B}{4\pi(1-\nu^{(1)})}\left(\frac{x_1^2}{r^2} - (1-2\nu^{(1)})\log\frac{r}{b}\right), \tag{8.10}$$

$$u_2^{(E)} = B\left(\frac{x_1 x_2}{4\pi r^2(1-\nu^{(1)})} - \frac{1}{2\pi}\tan^{-1}\frac{x_2}{x_1}\right). \tag{8.11}$$

But the existence of the elastic displacement $u^{(E)}$ means that either (5.1) or (5.8) can be employed to derive a consistent value of the Burgers vector, provided the relevant curve C is drawn in the cut region $\tilde{\Omega}$. It is simpler, however, to employ the expression (6.10) and the plastic strain which from (6.18) beomes

$$\begin{aligned} E^{(P)} &= -\frac{(\nu^{(1)} - \nu^{(2)})}{(1 - 2\nu^{(2)})} I \, tr \, e^{(2)} \\ &= \frac{(\nu^{(1)} - \nu^{(2)}) x_1 X_1}{4\pi\mu(1 - \nu^{(2)}) r^2} I \\ &= B \frac{x_1}{2\pi r^2} I. \end{aligned} \tag{8.12}$$

Insertion of (8.12) into (6.10) followed by integration around a plane circle $\partial\Sigma$ of fixed radius a centre the origin leads to the expression for the corresponding Burgers vector. On taking C to be the plane circle $\partial\Sigma$ of fixed radius a and centre at the origin, we obtain from (5.8) the expressions

$$\begin{aligned} 2B^{-1} b_\alpha &= \oint_{\partial\Sigma} \left(\frac{x_1}{a^2} - y_\gamma \left(\frac{x_1}{r^2} \right)_{,\gamma} \right) dx_\alpha \\ &\quad + \oint_{\partial\Sigma} \left(\frac{x_1}{r^2} \right)_{,\alpha} y_\beta \, dx_\beta \\ &= \oint_{\partial\Sigma} \left(\frac{2x_1}{a^2} + \frac{\bar{x}_1}{a^2} - \frac{2x_1 x_\gamma \bar{x}_\gamma}{a^4} \right) dx_\alpha \\ &\quad - \oint_{\partial\Sigma} \left(\frac{\delta_{1\alpha}}{a^2} - \frac{2x_1 x_\alpha}{a^4} \right) \bar{x}_\gamma \, dx_\gamma, \end{aligned}$$

since on the circle $\partial\Sigma$ we have

$$y_\gamma \, dx_\gamma = -\bar{x}_\gamma \, dx_\gamma, \qquad y_\gamma = x_\gamma - \bar{x}_\gamma.$$

Evaluation of the respective integrals leads to

$$b_1 = 0, \tag{8.13}$$

$$b_2 = B = \frac{(\nu^{(1)} - \nu^{(2)}) X_1}{2\mu(1 - \nu^{(2)})}. \tag{8.14}$$

On substituting the value of B from (8.14) in (8.6)-(8.8), we recover the well-known expressions for the elastic strain belonging to an edge dislocation along the x_3-axis (see [33]).

Let us also remark that the continuous displacement $u = u^{(1)} - u^{(2)}$ corresponds to the total strain $E = e^{(1)} - e^{(2)}$, which is compatible on Ω^*.

Expressions for the general edge dislocation may be derived by introduction of the point force (X_1, X_2) uniformly distributed along the x_3-axis. The derivation follows the same pattern as just described.

8.2 Screw Dislocation and anti-plane shear

We recover the displacement and strain for a single screw dislocation from the problem in homogeneous isotropic linear elasticity of anti-plane shear with singularity at the origin. Let $\theta = \tan^{-1}(x_2/x_1), x_1 \neq 0$. Then the displacement and strain components are

$$
\begin{aligned}
u_1^{(\alpha)}(x) &= u_2^{(\alpha)}(x) = 0, \qquad x \in \mathbb{R}^3, &(8.15)\\
u_3^{(\alpha)} &= u_3^{(\alpha)}(x_1, x_2) = \theta, \qquad x \in \Omega^*, &(8.16)\\
e_{13}^{(\alpha)} &= (1/2)u_{3,1}^{(\alpha)} = -\frac{x_2}{2r^2}. \qquad x \in \Omega^*, &(8.17)\\
e_{23}^{(\alpha)} &= (1/2)u_{3,2}^{(\alpha)} = \frac{x_1}{2r^2}, \qquad x \in \Omega^*, &(8.18)
\end{aligned}
$$

with the remaining strain components all identically zero. The non-zero stress components are given by

$$
\begin{aligned}
\sigma_{13}^{(\alpha)} &= 2\mu^{(\alpha)} e_{13}^{(\alpha)} = -\mu^{(\alpha)} \frac{x_2}{r^2}, &(8.19)\\
\sigma_{23}^{(\alpha)} &= 2\mu^{(\alpha)} e_{23}^{(\alpha)} = \mu^{(\alpha)} \frac{x_1}{r^2}, &(8.20)
\end{aligned}
$$

where, as before, $r^2 = x_\alpha x_\alpha \neq 0$, and $\Omega^* = \mathbb{R}^3 \backslash \{x : x_1 = x_2 = 0\}$.

We now determine the dislocation generated from the difference in the anti-plane shear fields (8.15)-(8.20) for two distinct values of the shear modulus. The difference displacement is $u = u^{(1)} - u^{(2)} \equiv 0$, as the constituent displacements are independent of the shear modulus. Consequently, the total elastic strain, E, is identically zero, while from (8.17),(8.18), and (6.1) we have

$$
\begin{aligned}
E_{\alpha 3}^{(P)} &= -\frac{(\mu^{(1)} - \mu^{(2)})}{2\mu^{(1)}} \theta_{,\alpha} &(8.21)\\
E_{\alpha\beta}^{(P)} &= E_{33}^{(P)} = 0. &(8.22)
\end{aligned}
$$

We take $\partial\Sigma$ to be a circle of radius a centre the origin in the $x_1 x_2$-plane. The Burgers vector from (6.10) or (6.11) then has components $b_1 = b_2 = 0$, and

$$
\begin{aligned}
b_3 &= \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}} \oint_{\partial\Sigma} \theta_{,\alpha}\, dx_\alpha\\
&= 2\pi \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}, &(8.23)
\end{aligned}
$$

so that the corresponding non-zero "plastic" strains are

$$
E_{\alpha 3}^{(P)} = -\frac{b_3}{4\pi} \theta_{,\alpha}. \tag{8.24}
$$

The non-zero components of elastic strain and stress appropriate for a screw dislocation in an infinite medium: ([33, p.57]) are then $E_{\alpha 3}^{(E)} = -E_{\alpha 3}^{(P)}$ and

$$
\begin{aligned}
\sigma_{13} &= -\frac{\mu^{(1)} b_3}{2\pi} \frac{x_2}{r^2}, &(8.25)\\
\sigma_{23} &= \frac{\mu^{(1)} b_3}{2\pi} \frac{x_1}{r^2}, &(8.26)
\end{aligned}
$$

where as before $r^2 = x_\alpha x_\alpha \neq 0$.

We note that the circuit $\partial\Sigma$ is drawn to enclose the coordinate origin and therefore does not intercept but encircles the single screw dislocation located at the origin.

As a further illustration, we may apply Filon's construct to derive the solution when a screw dislocation is located along the axis of a circular cylinder of radius R with lateral free boundary. The anti-plane distribution now is augumented by that for torsion. The analysis is straightforward and leads to $E = 0$, and to non-zero components of the elastic dislocation strain and stress given by

$$\begin{aligned} E_{13}^{(E)} &= \frac{b_3}{4\pi}\left(\frac{2x_2}{R^2} - \frac{x_2}{r^2}\right), && (8.27)\\ E_{23}^{(E)} &= \frac{b_3}{4\pi}\left(-\frac{2x_1}{R^2} + \frac{x_1}{r^2}\right), && (8.28)\\ \sigma_{13} &= \mu^{(1)}\frac{b_3}{2\pi}\left(\frac{2x_2}{R^2} - \frac{x_2}{r^2}\right), && (8.29)\\ \sigma_{23} &= \mu^{(1)}\frac{b_3}{2\pi}\left(-\frac{2x_1}{R^2} + \frac{x_1}{r^2}\right), && (8.30) \end{aligned}$$

where $R^2 = r^2 + x_3^2$. The corresponding elastic dislocation displacement becomes

$$u_1^{(E)} = \frac{b_3 x_2}{\pi R^2}, \quad u_2^{(E)} = -\frac{b_3 x_1}{\pi R^2}, \quad u_3^{(E)} = \frac{b_3\theta}{2\pi}.$$

The torsion of the cylinder represented by the distributions (8.27)-(8.30) is observed in long thin whiskers containing a screw dislocation ([11]).

9 Inhomogeneities and (Somigliana) dislocations

It is clear from the relationships identified in Section 6 that the inhomogeneity problem introduced in Section 4.2 is analogous to an elastic body containing an array of dislocations continuously distributed over the region Γ of the inhomogeneity. The inhomogeneity problem is also related to the Somigliana dislocation in which the displacement exhibits a discontinuity across a surface.

A comprehensive examination of the relation between Volterra and Somigliana dislocations and the inhomogeneity problem as defined in Section 4.2 is conducted in [12, §5] for a bonded interface. Expressions for the displacement, derived in the latter article mainly from heuristic cut-and-weld operations, are recovered here using the relationship with the variation of moduli. The problem when interfacial slipping is allowed is analysed in [27].

Throughout this section, we adopt the notation of Section 4.2, for which superscripts (1) and (2) refer respectively to unperturbed and perturbed fields. We compare the constitutive relations (5.18) and (3.7) to obtain

$$E^{(P)} = (1/2)\left(\beta^{(P)} + (\beta^{(P)})^T\right),$$

and consequently

$$E^{(P)} = \begin{cases} -(I-D)e^{(2)}, & x \in \Gamma, \\ 0, & x \in \Omega \backslash \bar{\Gamma}. \end{cases} \tag{9.1}$$

The elastic moduli in the undislocated region,or matrix, $\Omega\backslash\bar{\Gamma}$, are $c = c^{(1)}$, and correspond to those in the unperturbed problem. Note also that $E = e$.

We obtain the total dislocation displacement u and strain E once we know either the plastic distorsion tensor $\beta^{(P)}$ or the solution to the inhomogeneity problem. Let us suppose the latter, so that the plastic strain $E^{(P)}$ is given by (9.1).By analogy with (4.18), the total dislocation displacement is

$$u_i(x) = \frac{\partial}{\partial x_j} \int_\Gamma c^{(1)}_{kjpq} E^{(P)}_{pq} G^{(1)}_{ik}(x,y)\, dy. \tag{9.2}$$

When the plastic distorsion tensor is known, the previous integral gives the difference displacement in the inhomogeneity problem which, when added to the displacement in the unperturbed problem, leads to the perturbed displacement everywhere in Ω. The perturbed strain in the inhomogeneity from (9.1) is

$$e^{(2)} = -(I-D)^{-1} E^{(P)}, \qquad x \in \Gamma.$$

We examine the various relationships for homogeneous isotropic linear elasticity. We have shown that the corresponding incompatibility tensor is independent of a variation in the shear modulus and accordingly we consider only variations in Poisson's ratio. We denote the common shear modulus by μ.

The expression (9.1) simplifies and the plastic strain becomes

$$E^{(P)} = \begin{cases} -\gamma tr\, e^{(2)} I, & x \in \Gamma, \\ 0, & x \in \Omega \backslash \bar{\Gamma}, \end{cases} \tag{9.3}$$

where γ is given by (6.20). The plastic strain tensor is therefore polar.

On recalling (4.24) and (4.25), we may alternatively write these expressions as

$$E^{(P)} = \begin{cases} -\gamma_1 tr\, e^{(1)} I, & x \in \Gamma, \\ 0, & x \in \Omega \backslash \bar{\Gamma}, \end{cases} \tag{9.4}$$

where

$$\gamma_1 = \gamma \frac{(\lambda^{(1)} + 2\mu)}{(\lambda^{(2)} + 2\mu)} = \frac{(\nu^{(1)} - \nu^{(2)})(1-\nu^{(1)})}{(1+\nu^{(1)})(1-\nu^{(2)})(1-2\nu^{(1)})}. \tag{9.5}$$

It follows from (6.19) that the incompatibility tensor reduces to

$$\eta_{ij} = \gamma_1 e^{(1)}_{kk,ij}, \tag{9.6}$$

and therefore in general does not vanish, so that both $E^{(E)}$ and $E^{(P)}$ are incompatible.

The integral (9.2) determines the total dislocation (or difference) displacement and hence the total strain E. We have

$$u_i(x) = -\gamma_1 \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_\Gamma \frac{e^{(1)}_{kk}(y)}{R(x,y)}\, dy, \qquad x \in \Omega, \tag{9.7}$$

$$E_{ij}(x) = -\gamma_1 \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_\Gamma \frac{e^{(1)}_{kk}(y)}{R(x,y)}\, dy, \qquad x \in \Omega, \tag{9.8}$$

which enables the elastic dislocation strain $E^{(E)}$ to be calculated from

$$E^{(E)} = E - E^{(P)} \tag{9.9}$$

$$= E + \gamma_1 tr\, e^{(1)} I, \qquad x \in \Omega. \tag{9.10}$$

Thus, in principle, we may obtain the total dislocation displacement and corresponding strain for an array of continuous dislocations derived from the unperturbed dilatation in the whole space subject to various loadings. The practical determination of the dislocation quantities depends upon the exact evaluation of the attraction in (9.7). As already stated, this is known for several non-uniform densities, and also for uniform densities contained in an ellipsoid. In particular, for a dislocation array uniformly distributed in an ellipsoidal shell, the total dislocation displacement vanishes inside the shell, and achieves its maximum and minimum value at points on the shell's outer surface.

As a simple illustration of the approach, let us return to the example of the spherical inhomogeneity Γ of radius a in an infinite medium subject to uniform hydrostatic pressure P at infinity. From the solution derived in Section 4.2, we have that the plastic strain is uniform and is given by

$$E^{(P)}(x) = \begin{cases} QI, & x \in \Gamma, \\ 0, & x \in \mathbb{R}^3 \backslash \bar{\Gamma}, \end{cases} \tag{9.11}$$

where

$$Q = 3\gamma_1 A, \tag{9.12}$$

$$\gamma_1 = \gamma \frac{(\lambda^{(1)} + 2\mu)}{(\lambda^{(2)} + 2\mu)}, \tag{9.13}$$

$$A = \frac{P}{(3\lambda^{(1)} + 2\mu)}, \tag{9.14}$$

on recalling previously introduced notation.

On further appealing to the results established in Section 4.2, we conclude that the total displacement in terms of Q within the dislocated region Γ is expressed as

$$u(x) = \frac{Q}{3} \frac{(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + 2\mu)} x, \qquad x \in \Gamma, \tag{9.15}$$

and in the undislocated region as

$$u(x) = \frac{a^3 Q}{3} \frac{(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + 2\mu)} \frac{x}{R^3(x,0)}, \qquad x \in \mathbb{R}^3 \backslash \bar{\Gamma}. \tag{9.16}$$

We proceed slightly differently to the treatment in Section 4.2 of the corresponding inhomogeneity problem to obtain the elastic stress. The total strain from (9.15) in the dislocated region becomes

$$E(x) = \frac{Q}{3}\frac{(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + 2\mu)} I, \qquad x \in \Gamma, \tag{9.17}$$

while from (9.16) in the undislocated region, the total strain is

$$E(x) = \frac{a^3 Q}{3}\left(\frac{3\lambda^{(1)} + 2\mu}{\lambda^{(1)} + 2\mu}\right)\left[\frac{I}{R^3(x,0)} - \frac{3x \otimes x}{R^5(x,0)}\right], \; x \in \mathbb{R}^3 \backslash \bar{\Gamma}, \tag{9.18}$$

which by (9.9) gives the elastic strain in the respective regions as

$$E^{(E)}(x) = -\frac{4\mu Q}{3(\lambda^{(1)} + 2\mu)} I, \qquad x \in \Gamma, \tag{9.19}$$

and

$$E^{(E)}(x) = \frac{a^3 Q}{3}\left(\frac{3\lambda^{(1)} + 2\mu}{\lambda^{(1)} + 2\mu}\right)\left[\frac{I}{R^3(x,0)} - \frac{3x \otimes x}{R^5(x,0)}\right], \; x \in \mathbb{R}^3 \backslash \bar{\Gamma}, \tag{9.20}$$

while the elastic stress is

$$\sigma(x) = -\frac{4\mu Q}{3}\left(\frac{3\lambda^{(1)} + 2\mu}{\lambda^{(1)} + 2\mu}\right) I, \qquad x \in \Gamma, \tag{9.21}$$

and

$$\sigma(x) = \frac{2\mu a^3 Q}{3}\left(\frac{3\lambda^{(1)} + 2\mu}{\lambda^{(1)} + 2\mu}\right)\left[\frac{I}{R^3(x,0)} - \frac{3x \otimes x}{R^5(x,0)}\right], \; x \in \mathbb{R}^3 \backslash \bar{\Gamma}, \tag{9.22}$$

which from (9.12) are the stresses (4.31) and (4.32) otherwise obtained.

The incompatibility tensor associated with either $E^{(P)}$ or $E^{(E)}$, and from (5.4), the dislocation density, vanish everywhere except on $\partial\Gamma$. Indeed, the elastic strain is derivable from the elastic displacement which in the respective regions is given by

$$u^{(E)}(x) = -\frac{4\mu Q}{3(\lambda^{(1)} + 2\mu)} x, \qquad x \in \Gamma,$$

and

$$u^{(E)}(x) = \frac{a^3 Q}{3}\frac{(3\lambda^{(1)} + 2\mu)}{(\lambda^{(1)} + 2\mu)}\frac{x}{R^3(x,0)}, \qquad x \in \mathbb{R}^3 \backslash \bar{\Gamma}.$$

By inspection $u^{(E)}(x)$ suffers the discontinuity across $\partial\Gamma$ given by

$$\left[u^{(E)}\right]_{\partial\Gamma} = -Qx, \tag{9.23}$$

equivalent to a non-uniform array of Somigliana dislocations distributed over $\partial\Gamma$.

Remark 9.1 *We note that had the incompatibility tensor vanished everywhere, then the present example would have contradicted [15, Thm.2], because the elastic stress (9.21) and (9.22) is non-zero in the regions* Γ *and* $\mathbb{R}^3\backslash\bar{\Gamma}$ *and vanishes at infinity. But singularities in the dislocation density and incompatibility tensor fail to satisfy the conditions of [15, Thm.2], and consequently no contradiction occurs.*

Finally, on letting $a \to 0$ and $Q \to \infty$ such that a^3Q remains constant, the displacement (9.16), and stress (9.22), are those for an interstitial atom at the origin. (See, for example, [11].)

References

[1] Biot, M.A.A general property of the two-dimensional thermal stress distribution. Phil.Mag., (1935), **19** 540-549.

[2] Boley, B.A. and Weiner, J.H. Theory of Thermal Stresses. John Wiley and Sons. New York (1960).

[3] Carlson, D.E. Dependence of linear elasticity solutions on the elastic constants. J. Elast., **1**(2) (1971), 145-151.

[4] Carlson, D.E, Dependence of linear elasticity solutions on the elastic constants. J. Elast., **2**(2)) (1972), 129-134.

[5] Cosserat, E.and F. Sur les fonctions potential de la theorie de l'élasticité. C.R.Acad.Sci. Paris (1898) **126**, 1129-1132.

[6] Coker, E. G. and Filon,L.N.G. A Treatise on Photo-Elasticity. (Revised by H.T.Jessop).University Press. Cambridge (1957)

[7] Dundurs, J. Load and self-stress in an elastic body. J. Elast.,(1972) **2**(3) , 211-213.

[8] Dundurs, J. and Markenscoff, X. Invariance of stresses under a change of elastic compliances. Proc. R. Soc. Lond. (1993), **443**,289-300.

[9] Eshelby, J D. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc.Roy.Soc.(1957) **A241**, 376-396.

[10] Eshelby, J.D. The elastic field outside an ellipsoidal inclusion. Proc.Roy.Soc. (1959) **252**, 561-569.

[11] Eshelby, J. D. The Continuum Theory of Lattice Defects. In: Progress in Solid State Physics (Eds F.Seitz and D. Turnbull) (1956), **3**, 79-144.New York Academic Press.

[12] Eshelby, J.D. Elastic Inclusions and Inhomogeneities. Progress in Solid Mechanics. (Ed. by I.N.Sneddon and R.Hill) **2**,89-140. Amsterdam: North-Holland Publ. Co. (1961).

[13] Filon, L.N.G. On stresses in multiply-connected plates. Report of the Eighty-Ninth Meeting of the British Association for the Advancement of Science (1921). London (1922).

[14] Gurtin, M.E. The Linear Theory of Elasticity. Handbuch der Physik (S,Flügge, ed.) **VIa/2**,1-295. Springer- Verlag. Berlin. (1972).

[15] Head, A. K., Howison, S.D., Ockendon, J.R. and Tighe, S.P. An equilibrium theory of dislocation continua. S.I.A.M Review (1993) **35**, 580-609.

[16] Hill,R. Discontinuity Relations in Mechanics of Solids.Progress in Solid Mechanics.(Ed. by I.N.Sneddon and R.Hill). Amsterdam: North-Holland Publ. Co. (1961), **2**,247-276.

[17] Knops, R.J. A method for solving linear thermoelastic problems. J. Mech. Phys. Solids, (1959) **7**, 182-192.

[18] Knops, R.J. On the use of Poisson's ratio in studying certain non-homogeneous elastic inclusions. Z.A.M.M. (1960) **40**, 542-550.

[19] Knops, R.J. Further considerations of the elastic inclusion problem. Proc. Edinburgh Math. Soc., (1964) **14**, 61-70.

[20] Knops, R.J. Filon's construct for dislocations and related topics. J. Eng.Maths.,(2003) **46**, 331-354.

[21] Knops, R.J. and Payne, L.E. Uniqueness Theorems in Linear Elasticity. Springer-Verlag. Berlin Heidelberg New York. (1971).

[22] Kondo, K. Geometry of elastic deformation and incompatibility. *RAAG* Memoirs of the Unifying Study of Basic Problems in Engineering and Physical Sciences by means of Geometry. (ed. by K. Kondo). **1**, 361-373.Megutoki Tokyo:Gakujutsu Bunken Fukyu-Kai (1955).

[23] Kröner, E. Kontinuumstheorie der Versetzungen und Eigenspannungen. Erg. Angew. Math,. **5** Springer-Verlag. Berlin Göttingen, Heidelberg (1958).

[24] Kupradze. V.D. Dynamical Problems in Elasticity. Progress in Solid Mechanics. **III** (1963). (Eds I. N. Sneddon and R. Hill). North-Holland Publishing Company. Amsterdam.

[25] Love, A.E.H. A Treatise on the Mathematical Theory of Elasticity,4th Edition. Cambridge University Press. Cambridge (1927)..

[26] Maiti, M. and Makan, G.R. Somigliana's method applied to elastic inclusion and dislocations. J. Elast.,(1973) **3**(1), 45-49.

[27] Markenscoff, X. A note on strain jump conditions and Cesé0 integrals for bonded and slipping inclusions. J. Elast., (1996). **83**, 45-61.

[28] Markenscoff, X. Stress independence of Poisson's ratio and divergence-free body forces.. J. Elast., (2006). **45**, 65-74.

[29] Markenscoff, X. and Paukshito, M. The Cosserat spectrum in the theory of elasticity and applications. Proc.R. Soc. Lond. A (1998), **454**,631-643.

[30] Micbell, J.H.. On the direct determination of stress in an elastic solid with applications to the theory of plates. Proc. London Math. Soc., (1900)**31**,100-124.

[31] Mura, T. Impotent dislocation walls. Math. Sci. and Engrg., (1989) **A113**, 149-152.

[32] Muskhelishvili, N.I. Some Basic Problems of the Mathematical Theory of Elasticity. (Transl. By J R M Radok) Noordhoff. Groningen (1963)

[33] Nabarro, F.R.N.Theory of Crystal Dislocations. Oxford University Press. Oxford (1967).

[34] Roy, A. and Acharya, A. Finite element approximation of field dislocation mechanics. J. Mechs. Phys.Solids, (2005), **53**, 143-170.

[35] Sokolnikoff,I.S. Mathematical Theory of Elasticity. 2nd Edition. McGraw-Hill. New York (1956).

[36] Shield, R.T. On uniqueness for the traction boundary-value problem of linear elastostatics. (S.Nemat-Nasser ed.) Mechanics Today, **5**,437-453. Pergamon Oxford (1980).

[37] Steeds, J.W. and Willis, J.R. Dislocations in anistropic media. In: Dislocations in Solids (F.R.N.Nabarro, ed.), 144-165. North-Holland (1979).

[38] Sternberg, E. and Muki,R. Note on the expansion in powers of Poisson's ratio of solutions in elastostatics. Arch. Rational Mech.Anal., (1959),**3** 229-234.

[39] Westergaard, H.M.Effects of a change in Poisson's ratio analyzed by twinned gradients. J.Appl. Mech. (1940), **7** 113-116.

[40] Willis, J. R. Second-order effects of dislocations in anistropic crystals. Int. J. Engng.Sci., (1967), **5**, 171-190.