CARTESIAN PRODUCT

Tuple is a collective name for double, triple, quadruple, etc. An *n*-tuple is an ordered list of *n* items. The Cartesian product of *n* sets $S_1,...,S_n$ is the collection of all tuples, each of which draws the first item from S_1,..., and draws the *n*th item from S_n. Tuples and Cartesian products are everywhere in our daily life.

Tuple

Tuple. A *tuple* is an ordered list of items. An *n*-tuple is an ordered list of *n* items. The integer *n* is called the *length* of the tuple. We write a tuple by listing the items between parentheses, $(\ )$.

Examples. The notation

$$(6,1,7,4,9,5,3,7,8,9)$$

denotes a ten-tuple, an ordered list of ten numbers. The ten-tuple happens to be a telephone number in the United States.

The meaning of a tuple is understood in context. For example, we often write the three-tuple $(3,0,9)$ as 309. The tuple can mean Room 9 on the third floor. The same tuple can also mean the number 309 in the base-10 numeral system. In this case, the tuple is shorthand for an expression:

$$309 = 3\times 10^2 + 0\times 10^1 + 9\times 10^0 .$$

The items listed in a tuple, of course, need not be numbers. For example, the notation

$$(\text{Eltz, Metternich, Reichsburg})$$

denote three castles along the Mosel River.

Tuple vs. set. Tuple and set are both collections of elements, but they differ in two ways. The order of elements is significant in a tuple, but not in a set. The repetition of an element is significant in a tuple, but not in a set.

In defining a tuple, the order of the items is significant. Thus, $(\text{Eltz, Metternich, Reichsburg})$ and $(\text{Reichsburg, Metternich, Eltz})$ are distinct tuples. A particular order is important, for example, when we plan to visit the three castles in that order.

By contrast, in defining a set, the order of elements is insignificant. Thus, $(\text{Eltz, Metternich, Reichsburg})$ and $(\text{Reichsburg, Metternich, Eltz})$ denote the same set.

In a tuple, the same item may appear multiple times. Thus, the three-tuple $(\text{Eltz, Metternich, Reichsburg})$ differs from the four-tuple $(\text{Eltz, Metternich, Eltz, Reichsburg})$. Each tuple, for example, represents a plan of visits. The three-tuple represents a plan to visit each castle once, in that order. The four-tuple represents a plan including two visits to Eltz.

As another example, $(6,1,7,4,9,5,3,7,8,9)$ is a ten-tuple, in which the number 7 appears twice, and the number 9 also appears twice. The phone number works only when we dial all the ten digits, in this order, with these repetitions.

By contrast, each element should appear only once in a set. Thus, if we regard $\{6,1,7,4,9,5,3,7,8,9\}$ as set, we should remove the repeated digits from the set, and write the eight distinct elements in any order, e.g., $\{1,3,4,5,6,7,8,9\}$.

Single, double, triple, and quadruple. We sometimes use special words for tuples.

A one-tuple is called a single, or a singleton.

An ordered list of two items is called a *couple*, a *double*, a *twin*, a *dual*, or an *ordered pair*. That is, a couple is a two-tuple. A husband and a wife form a couple. A pair of shoes is an ordered pair, which contains two different shoes, the left and the right. A pair of socks is also an ordered pair, even if the two socks are identical.

Similarly, an ordered list of three things is called a three-tuple, or a *triple*. An ordered list of four things is called a four-tuple, or a *quadruple*.

Next come quintuple, sextuple, septuple, octuple, nonuple, and decuple. Somewhere down the list come centuple and milluple. But these words are rarely used. Instead, we call them five-tuple, six-tuple, seven-tuple, eight-tuple, nine-tuple, ten-tuple, 100-tuple, and 1000-tuple.

Cartesian Product

Cartesian product of two sets. The *Cartesian product* of two sets A and B is the collection of all ordered pairs of the form (a,b), where $a \in A$ and $b \in B$. We call the two sets, A and B, the factors of the Cartesian product.

We write the Cartesian product as $A \times B$, and express its definition in the set-building notation:

$$A \times B = \{(a,b) | a \in A \text{ and } b \in B\}.$$

The Cartesian product of the two sets defines a new set, each element of which is an ordered pair, drawing upon the two sets A and B.

Examples. *Genders of animals*. Let A be a set of two genders:

$$A = \{\text{male, female}\}.$$

Let B be a set of three animals:

$$B = \{\text{dog, cat, pig}\}.$$

The Cartesian product of the two sets is a set of six elements:

$$A \times B = \left\{ \begin{matrix} (\text{male, dog}) & (\text{male, cat}) & (\text{male, pig}) \\ (\text{female, dog}) & (\text{female, cat}) & (\text{female, pig}) \end{matrix} \right\}$$

We write the Cartesian product of the two sets as a 2×3 table.

Personal belongings. Let A be a set of three people:

$$A = \{\text{mom, dad, son}\}.$$

Let B be a set of four things:

$$B = \{\text{car, hat, bag, phone}\}.$$

The Cartesian product of the two sets is a set of twelve elements:

$$A \times B = \left\{ \begin{matrix} (\text{mom, car}) & (\text{mom, hat}) & (\text{mom, bag}) & (\text{mom, phone}) \\ (\text{dad, car}) & (\text{dad, hat}) & (\text{dad, bag}) & (\text{dad, phone}) \\ (\text{son, car}) & (\text{son, hat}) & (\text{son, bag}) & (\text{son, phone}) \end{matrix} \right\}$$

Again we write the Cartesian product of the two sets as a 3×4 table.

Playing cards. Let set A be the four suits of playing cards:

$$A = \{\text{spade, heart, diamond, club}\}.$$

Let set B be the set of thirteen face values:

$$B = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J, Q, K, A}\}.$$

The Cartesian product $A \times B$ describes the whole set of 4×13 cards.

Specify Cartesian product by a table. Let m be the number of elements in A, and n be the number of elements in B. The elements in the two sets form a total of mn ordered pairs. Thus, the Cartesian product $A \times B$ has mn members, which can be listed in an $m \times n$ table.

Let A be a set of two elements:

$$A = \{a_1, a_2\}.$$

Let B be a set of three elements:

$$B = \{b_1, b_2, b_3\}.$$

The Cartesian product of the two sets is a set of 2×3 elements:

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}.$$

We can also list the elements in a 2×3 table:

(a_1, b_1)	(a_1, b_2)	(a_1, b_3)
(a_2, b_1)	(a_2, b_2)	(a_2, b_3)

Specify Cartesian product by properties. When dealing with sets of infinitely many elements, we cannot write their elements one by one. We specify each set by stating a property, not by listing its elements. Once we specify two sets, we generate all the elements in their Cartesian product, without writing the elements one by one.

For example, we define the set of rational numbers Q in terms of the Cartesian product of the set of integers Z and the set of positive integers N:

$$Q = \left\{ x \middle| x = m/n, m \in Z \text{ and } n \in N \right\}.$$

Cartesian product of more than two sets. We can similarly define Cartesian product of more than two sets. In particular, define the Cartesian product of three sets A, B, and C using the set-building notation:

$$A \times B \times C = \left\{ (a,b,c) \middle| a \in A, b \in B, \text{and } c \in C \right\}.$$

We can represent Cartesian product of the three sets by a stack. Let l be the number of elements in A, m be the number of elements in B, and n be the number of elements in C. The elements in the three sets generate a total of lmn ordered pairs. Thus, the Cartesian product $A\times B$ has lmn members, which can be represented by an $l\times m\times n$ stack.

We display visually the Cartesian product of two sets by a table, and the Cartesian product of three sets by a stack. We live in three dimensions, and we do not have a "natural" visual display of the Cartesian product of four or more sets. However, we can readily define Cartesian product of four or more sets. For example, we define the Cartesian product of four sets A, B, C and D as

$$A \times B \times C \times D = \left\{ (a,b,c,d) \middle| a \in A, b \in B, c \in C, \text{ and } d \in D \right\}.$$

Do not over-rate visualization. Here we learn a lesson: not all concepts are easy to display on a piece of paper. The piece of paper is two-dimensional, which play similar roles. For the Cartesian product of three sets, to display the three sets in an unbiased way, we need three dimensions. Furthermore, there is no "natural" (i.e., unbiased) way to display the Cartesian product of four or more sets.

By contrast, the concept of Cartesian product works well with any number of sets. When visualization confuses us, why bother with it? Visualization is much prized, but often over-rated. We will visualize as much as we can, but will not be upset if we cannot. In economics and engineering, we routinely deal with Cartesian products of huge numbers of sets.

Cartesian product of copies of a single set. We can, of course, form Cartesian product of a set and itself.

We constantly form the Cartesian product of a set of people and itself. We compare each pair of people by age, weight, and height. Are they related by blood or by marriage? Are they friends on Facebook or on Wechat? Were they born in the same country, or graduated from the same college?

We also form the Cartesian product of the set of all countries and itself. We compare each pair of counties by the gross domestic product, the population, and the form of government. Do they have a diplomatic relation? Are they enemies or friends?

Let S be a set. We write the Cartesian product $S\times S$ as S^2. For example, let S be the set of three types of vote:

$$S=\left\{\text{yes, no, abstention}\right\}.$$

The Cartesian product $S\times S$ represents the possible outcomes if two people cast votes:

$$S\times S=\left\{\begin{array}{ccc}(\text{yes, yes}) & (\text{yes, no}) & (\text{yes, abstention})\\ (\text{no, yes}) & (\text{no, no}) & (\text{no, absetention})\\ (\text{abstention, yes}) & (\text{abstention, no}) & (\text{abstention, abstention})\end{array}\right\}$$

As another example, let S be the set of ten single-digit numbers:

$$S=\left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\right\}.$$

The Cartesian product S^3 is a set of 10^3 elements, each being an ordered list of three digits. If the Cartesian product S^3 represents all three-digit numbers in the base 10 numerical system, we write

$$S^3=\left\{000, 001, 002, ..., 999\right\}.$$

This set has a total of 1000 elements.

In general, for any positive integer n, the notation S^n stands for the collection of all n-tuples of elements in the set A.

Let m be the number of elements in set S. Thus, the Cartesian product S^n has a total of m^n members, each being an n-tuple. Forming Cartesian product is a powerful way to generate new sets from old ones.

Cartesian Product of Sample Spaces*

Throw a coin and a die simultaneously. The sample space of tossing a coin is

$$\left\{\text{head, tail}\right\}.$$

The sample space of rolling a die is

$$\{a,b,c,d,e,f\}.$$

Throw a coin and a die simultaneously, and we obtain one of twelve possible outcomes:

$$\left\{\begin{matrix} (\text{head},a) & (\text{head},b) & (\text{head},c) & (\text{head},d) & (\text{head},e) & (\text{head},f) \\ (\text{tail},a) & (\text{tail},b) & (\text{tail},c) & (\text{tail},d) & (\text{tail},e) & (\text{tail},f) \end{matrix}\right\}.$$

Each outcome is an ordered pair; for example, the ordered pair (head,a) stands for the outcome that the coin falls head and the die stops with face a. The twelve possible outcomes come from the Cartesian product of the two sample spaces:

$$\{\text{head, tail}\}\times\{a,b,c,d,e,f\}.$$

Toss a coin twice. Toss a coin twice, and we have a total of four possible outcomes:

$$\{(\text{head, head}),(\text{head, tail}),(\text{tail, head}),(\text{tail, tail})\}.$$

The four outcomes constitute the sample space of the experiment "tossing a coin twice". This sample space comes from the Cartesian product of the sample space of the experiment "tossing a coin once" and itself:

$$\{\text{head, tail}\}\times\{\text{head, tail}\}.$$

In the experiment "tossing a coin twice", the event "getting a head and a tail" is

$$\{(\text{head, tail}),(\text{tail, head})\}.$$

We can similarly define other Cartesian products, such as tossing a coin 100 times, or throw three coins and five dice simultaneously.

Toss a coin three times. The experiment "tossing a coin three times" has a sample space of 2^3 samples:

$$\{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT\}.$$

Here we adopt a simple way to represent each triple. For example, the triple *HHT* states for the event that the coin falls head in the first toss, falls head in the second toss, and falls tail in the third toss.

The event "tossing a coin three times and getting one head and two tails" is

$$\{HTT,THT,TTH\}.$$

Counting the numbers of head and tail. When we toss a coin n times, how many outcomes are in the event of getting head k times and tail l times?

Tossing a coin n times is an experiment. Each outcome of the experiment is an n-tuple, an ordered list of n items, each of which is head or tail. The sample space of the experiment has a total of 2^n samples.

We now wish a count the number of n-tuples, each of which has k heads and l tails. Note that $n = k + l$. There are $n!$ number of permutations to place n distinct items to n slots. But items in this case are not all distinct. Of the n items, we have k heads and l tails, so that $k!l!$ number of permutations corresponds to the same n-tuple. Consequently, the number of outcomes in the event of getting k heads and l tails is

$$\frac{(k+l)!}{k!l!}.$$

Label the faces of a die by numbers. Usually we label the six faces of a die by numbers 1, 2, 3, 4, 5, and 6. This practice allows us to pose many questions.

When you roll a die twice, the sample space is

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

When you roll a die twice, what is the event of getting a sum of 7? Answer:

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

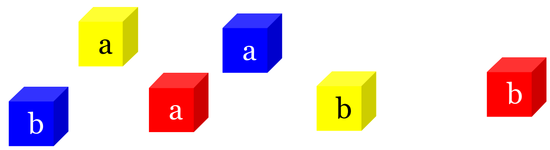
When you roll a die twice, what is the event of getting a product of 12? Answer:

$$\{(2,6),(3,4),(4,3),(6,2)\}.$$

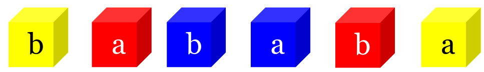
Mother, Son, and Cubes*

Cartesian product seems to be intuitive to us, but let us see how the concept may appear to a child.

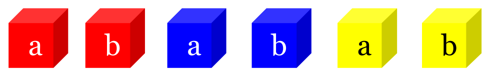
Cubes of two attributes. A mother gives her son six cubes. Each cube is painted with a color and marked with a letter. Usually the child scatters the six cubes on the floor, like this:



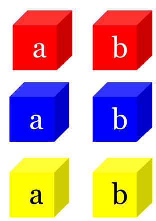
Sometimes he lines up the cubes in a single row, like this:



Other times he lines up the cubes in a row in some "natural order", like this:



On rare occasions, he places the blocks in a 3×2 table, like this:



The mother knows that each cube has two attributes, and each attribute defines a set. Let A be a set of three colors:

$$A = \{\text{red, blue, yellow}\}.$$

Let B be a set of two letters:

$$B = \{\text{a, b}\}.$$

For example, the ordered pair (red, a) represents the cube painted red and marked a. The Cartesian product of the two sets, $A \times B$, represents the set of six cubes: (red, a), (red, b), (blue, a), (blue, b), (yellow, a), (yellow, b).

The scattered cubes, the rows of cubes in one order or another all obscure a basic fact: the set of cubes is the Cartesian product of *two* sets of attributes. Only the table represents the nature of the Cartesian product well.

Cubes of three attributes. The mother now gives the child a set of twelve cubes. Each cube has a new attribute. Let C be a set of two textures:

$$C = \{\text{smooth, rough}\}.$$

The three-tuple (red, a, smooth) represents a cube painted red, marked by a, and having smooth faces. The Cartesian product of the three sets, $A \times B \times C$, describes the set of twelve cubes: (red, a, smooth), (red, b, smooth), (red, a, rough), (red, b, rough), (blue, a, smooth), (blue, b, smooth), (blue, a, rough), (blue, b, rough), $(\text{yellow, a, smooth})$, $(\text{yellow, b, smooth})$, $(\text{yellow, a, rough})$, $(\text{yellow, b, rough})$.

The child puts the twelve cubes in many ways. He usually scatters them on the floor. Sometimes he lines up the cubes in a single row in one order or another. Other times he lines up the cubes in a row in some "natural" order, like that listed above. The row is too long for us to fit in a single line on a piece of paper.

Occasionally he places the twelve cubes like a 6×2 table:

$$\left\{\begin{array}{cc} (\text{red, a, smooth}) & (\text{red, b, smooth}) \\ (\text{red, a, rough}) & (\text{red, b, rough}) \\ (\text{blue, a, smooth}) & (\text{blue, b, smooth}) \\ (\text{blue, a, rough}) & (\text{blue, b, rough}) \\ (\text{yellow, a, smooth}) & (\text{yellow,, b smooth}) \\ (\text{yellow, a, rough}) & (\text{yellow, b, rough}) \end{array}\right\}$$

On rare occasions, he piles up the blocks in a $3\times 2\times 2$ stack. His mother wishes to record his achievement by drawing the stack on a piece of paper, but doing so is cumbersome. Instead, she describes the stack in words. The stack has three rows, two columns, and two levels. Cubes in each row have the same color, cubes in each column have the same letter, and the cubes in each level have the same texture.

The scattered cubes, the single row, and the table all obscure a basic fact: the set of cubes is the Cartesian product of three sets of attributes. The stack represents the Cartesian product of three attributes well, but is not easy to draw on a piece of paper.

Cubes of four attributes. Now the mother challenges the child with a set of 60 cubes. Each cube has a fourth attribute. On each cube she also writes a number. Let *D* be a set of five numbers:

$$D = \{1,2,3,4,5\}.$$

For example, the four-tuple $(\text{red, a, rough, 5})$ represents a cube painted red, marked a, having rough faces, and marked 5. Now the Cartesian product $A\times B\times C\times D$ has a total of 60 cubes. The child lives in three dimensions. No matter how he places the cubes, he will always obscure the basic fact: the set of cubes is the Cartesian product of *four* sets.

Some arrangements get close to this basic fact. For example, the child can put the 60 cubes into five $3\times2\times2$ stacks, and cubes in each stack are marked with the same number. In the Cartesian product, the four attributes play similar roles, but this arrangement singles out one attribute, number. The number appears to stand out, whereas the other three attributes—color, letter, and texture—appear to be "integrated" within each stack.

Space, Time, and Spacetime*

Analogous to the cubes of four attributes, points in our world also have four attributes: north-south, east-west, up-down, and past-future.

We represent our world by a set of quadruples called spacetime, and represent each point in the world by a quadruple called an event. In this introduction to spacetime, we name various things.

Our algebraic world. The space in nature motivated Descartes (1637) to introduce the Cartesian product. He represented each point in a plane by two coordinates, an ordered pair of numbers. Thus, $(1,2)$ and $(2,1)$ represent distinct points in the plane. We must repeat a number twice to represent a point like $(3,3)$.

Einstein (1905) and Minkowski (1907) wove space and time together to describe our world. The Cartesian product of space and time is called spacetime.

Time, moment, duration, and lapse. A point in *time* is called a *moment*. In this sentence, the word "time" means a set, and the word "moment" means an element. We incessantly talk about elements in the Cartesian product of time and itself. Any two elements of time—that is, any two moments—is an ordered pair. We call the ordered pair a *duration*, and call the length of the duration a *lapse*.

When my wife asks me, "What's the time now?" She is not interested in the time, the set. Nor is she interested in the moment, an element in the set. Rather, she is interested in an element of the Cartesian product between the time and itself. Specifically, she wants to know the lapse between the moment now and the moment designated as 0 o'clock this morning, which serves as an origin in time for the day.

When an empress starts her reign, she sets the moment as an origin in time. The moment marks a new era in history, but does not affect laws of physics. On changing the origin in time from the birthdate of Confucius to the birthdate of Christ, we do not change laws of physics.

Our disregard for origin is built in our model of time. Any moment can serve as an origin, and laws of physics are indifferent to our choice of origin. That is, physics seeks laws that depend on the durations between moments, but not on the moments themselves. Here is a fundamental difference between physics and other disciplines, such as history, sociology, geology, biology, and cosmology.

Space, place, displacement, distance. What we have just said about time also applies to the one-dimensional space along a single line. Indeed, time and one-dimensional space have the identical algebraic structure. Each point in the *space* is called a *place*. In this sentence, the word "space" means a set, and the word "place" means an element. Any two places form an ordered pair. We call the ordered pair a *displacement*, and call the length of the displacement the *distance*.

Our space is a three-dimensional space, which we describe as a Cartesian product of three one-dimensional lines. We will describe this Cartesian product in depth later.

Spacetime, event, 4-displacement, interval. Let S be the set of all places in space, and let T be the set of all moments in time. In 1907, Minkowski introduced the Cartesian product of space and time:

$$S \times T = \left\{ (p,m) \middle| p \text{ is a place in space and } m \text{ is a moment in time} \right\}.$$

In his opening remark, Minkowski said, "From now onwards, space by itself and time by itself will recede completely to become mere shadows, and only a type of union of the two will still stand independently on its own."

It is good to name things. The Cartesian product $S \times T$ is called the *spacetime*. Each element of the spacetime, (p,m), is an ordered pair of a place in space and a moment in time, and is called an *event*.

The word "event" is not derived from the words "place" and "moment". We can, of course, name an ordered pair of place and moment a "placemoment". Minkowski called the Cartesian product of space and time the "world", and each ordered pair of a place in space and a moment in time a "world point". But the word "event" has won.

We often mark an event by something that happens. For instance, the place where, and the moment when, Jesus was born mark an event. The word "event" means an element in spacetime, not the happening "Jesus was born". We just use the happening to mark an element in spacetime.

As we mentioned before, each element in the *spacetime* is called an *event*. Any two events form an ordered pair. We call the ordered pair a *4-displacement*, and call the length of the displacement the *interval*.

Einstein vs. Minkowski. Our space is three-dimensional, and we measure length in each direction using a rod. Our time is one-dimensional, and we measure time using a clock. The Cartesian product of the readings of length in three directions, together with the reading of time, is called the spacetime. The spacetime is four-dimensional, and is analogous to the set of cubes of four attributes.

Einstein, the physicist, introduced a picture of the spacetime. In space, imagine a three-dimensional framework of rods, and imagine clocks placed everywhere on the rods. Just as the table obscures the nature of the cubes of three attributes, Einstein's picture obscures the four-dimensional nature of the spacetime.

The set of cubes of three attributes may have challenged the imagination of the child. The mom is amused when her child arranges the cubes of three attributes by a table, but is genuinely pleased when he piles the cubes into a stack.

The world is like a set of cubes of four attributes. The Mother Nature must be amused by Einstein's rods and clocks, but must be genuinely pleased by Minkowski's four-dimensional spacetime.

Things and their names. We tabulate things associated with the three sets—space, time and spacetime—along with their names. You and I of course disagree on names for these things, but we should agree on the types of things.

Set	*Point*	*Vector*	*Magnitude*
Space	place location point position	displacement separation	distance separation
Time	moment instant date location point position	duration separation lapse	duration separation lapse
Spacetime	event point world-point	4-displacment world-displacement separation	interval proper time proper distance

Our language is careless about time. We do not have a name for the Cartesian product of time and itself. Perhaps we just call it the set of durations. The name is clumsy, but serves the purpose. Furthermore, in everyday usage and in science, the word "time" can mean all these things: the set of all moments, a moment, the duration between two moments, and the length between two moments. These distinct meanings appear in the following sentences:

- Everything happens in space and time.
- He wounded himself at the worst time.
- He studied relativity during the time when he was healing.
- The healing took a long time.

Using the same word to mean different things does not confuse anyone. If asked, we know how to re-write this sentence: The time from this time to that time in time contains such a short time. The sentence uses the word "time" five times.

By contrast, we tend to use a more precise vocabulary for things associated with space: the displacement from this place to that place in space has a long distance. We even have a name for the Cartesian product of the space and itself: we call it the vector space. We will explain this idea in notes on vectors.