2.1 A SHALLOW TRUSS ELEMENT

We will now use the 'shallow truss theory' of Chapter 1 to derive the finite element equations for the shallow truss of Figure 2.1. The derivation will be closely related to the virtual work procedure of Section 1.3.2. Short-cuts could be used in the derivation but we will follow fairly conventional finite element procedures so that this example provides an introduction to the more complex finite element formulations that will follow. The element (Figure 2.1) has four degrees of freedom $u_1 = p_1$, $u_2 = p_2$, $w_1 = p_3$ and $w_2 = p_4$. Both the geometry and the displacements are defined with the aid of simple linear shape functions involving the non-dimensional coordinate, ξ, so

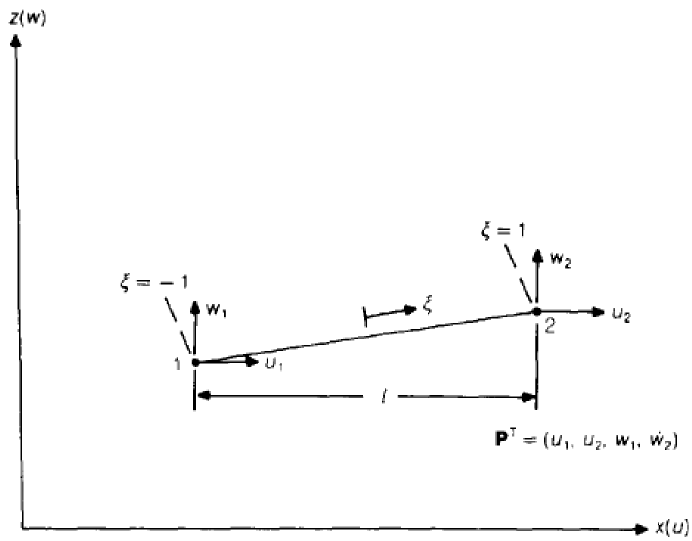

Figure 2.1 A shallow truss element.

that

$$x = \frac{1}{2}\begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad z = \frac{1}{2}\begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.1}$$

$$u = \frac{1}{2}\begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad w = \frac{1}{2}\begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{2.2}$$

Following from (1.51), the strain in the bar is

$$\varepsilon = \frac{du}{dx} + \left(\frac{dz}{dx}\right)\left(\frac{dw}{dx}\right) + \frac{1}{2}\left(\frac{dw}{dx}\right)^2. \tag{2.3}$$

From (2.1),

$$\frac{dx}{d\xi} = (x_2 - x_1)/2 = l/2 \tag{2.4}$$

while from (2.2),

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = (u_2 - u_1)/l = u_{21}/l \tag{2.5}$$

where the shorthand u_{21} has been used for $u_2 - u_1$. In a similar fashion,

$$\frac{dw}{dx} = w_{21}/l \qquad \frac{dz}{dx} = z_{21}/l. \tag{2.6}$$

Hence, from (2.3),

$$\varepsilon = \frac{u_{21}}{l} + \left(\frac{z_{21}}{l}\right)\left(\frac{w_{21}}{l}\right) + \frac{1}{2}\left(\frac{w_{21}}{l}\right)^2 \tag{2.7}$$

and the axial force in the bar is

$$N = EA\varepsilon \text{ (equation (2.7))}. \tag{2.8}$$

From (2.3), a change of strain, $\Delta\varepsilon$, corresponding to displacement changes Δu and Δw is given by

$$\Delta\varepsilon = \frac{d\Delta u}{dx} + \left(\frac{dz}{dx} + \frac{dw}{dx}\right)\frac{d\Delta w}{dx} + \frac{1}{2}\left(\frac{d\Delta w}{dx}\right)^2 \tag{2.9}$$

where the final (higher-order) term in (2.9) becomes negligible as Δw gets very small. Using (2.5) and (2.6):

$$\Delta\varepsilon = \frac{\Delta u_{21}}{l} + \frac{1}{l^2}(z_{21} + w_{21})\Delta w_{21} + \frac{1}{2l^2}\Delta w_{21}^2. \tag{2.10}$$

If a set of virtual nodal displacements†,

$$\delta\mathbf{p}_v^T = (\delta u_{v1}, \delta u_{v2}, \delta w_{v1}, \delta w_{v2}) \tag{2.11}$$

are applied, the resulting strain is, from (2.9),

$$\delta\varepsilon_v = \frac{1}{l}\delta u_{v21} + \frac{1}{l^2}(z_{21} + w_{21})\delta w_{v21} = \mathbf{b}^T\delta\mathbf{p}_v \tag{2.12}$$

where

$$\mathbf{b}^T = \frac{1}{l}(-1, 1, -\beta, \beta) \tag{2.13}$$

where

$$\mathbf{b}^{\mathrm{T}} = \frac{1}{l}(-1, 1, -\beta, \beta) \tag{2.13}$$

with

$$\beta = \frac{z_{21} + w_{21}}{l}. \tag{2.14}$$

In deriving (2.12) from (2.10), the quadratic terms involving δw_{v21}^2 have been considered negligible.

The virtual work equation can (see (1.77)) be expressed as

$$V = \int \sigma \delta \varepsilon_{\mathrm{V}} \, \mathrm{d}V - \mathbf{q}_e^{\mathrm{T}} \delta \mathbf{p}_v = 0 \tag{2.15}$$

where $\mathbf{q}_e$ are the external nodal forces corresponding to the nodal displacements, $\delta \mathbf{p}_v$. Because $\delta \varepsilon_v$ can be expressed, via (2.12), in terms of $\delta \mathbf{p}_v$, equation (2.15) can be re-written as

$$V = \delta \mathbf{p}_v^{\mathrm{T}} \mathbf{g} = \delta \mathbf{p}_v^{\mathrm{T}} (\mathbf{q}_i - \mathbf{q}_e) = \delta \mathbf{p}_v^{\mathrm{T}} \left(\int \sigma \mathbf{b} \, \mathrm{d}\mathrm{V} - \mathbf{q}_e \right) = 0 \tag{2.16}$$

where $\mathbf{q}_i$ is the internal force vector, given by

$$\mathbf{q}_i = \int \sigma \mathbf{b} \, \mathrm{d}V = N l \mathbf{b}. \tag{2.17}$$

[†] This ordering would not be the most convenient for element assembly, but the ordering could easily be altered prior to such assembly.

For equilibrium, (2.16) should be satisfied for any virtual displacements. $\delta\mathbf{p}_v$. Hence

$$\mathbf{g} = \mathbf{q}_i - \mathbf{q}_e = 0 \tag{2.18}$$

where $\mathbf{g}$ is the out-of-balance force vector.

From (1.80), $\mathbf{K}_t = \partial\mathbf{g}/\partial\mathbf{p}$ and a truncated Taylor expansion of $\mathbf{g}$, about an 'old' configuration, $\mathbf{g}_o$ gives

$$\mathbf{g}_n = \mathbf{g}_o + \frac{\partial\mathbf{g}}{\partial\mathbf{p}}\delta\mathbf{p} = \mathbf{g}_o + \mathbf{K}_t\delta\mathbf{p}. \tag{2.19}$$

p or b?

Hence, from equations (2.17)–(2.19),

$$\mathbf{K}_t = \frac{\partial\mathbf{g}}{\partial\mathbf{p}} = \frac{\partial\mathbf{q}_i}{\partial\mathbf{p}} = l\mathbf{b}\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}} + lN\frac{\partial\mathbf{p}}{\partial\mathbf{p}}. \tag{2.20}$$

From (2.8) and (2.12):

$$\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}} = \frac{\mathrm{d}N}{\mathrm{d}\varepsilon}\frac{\partial\varepsilon}{\partial\mathbf{p}} = EA\mathbf{b}^{\mathrm{T}}. \tag{2.21}$$

Hence,

$$\mathbf{K}_t = EAl\mathbf{b}\mathbf{b}^{\mathrm{T}} + lN\frac{\partial\mathbf{b}}{\partial\mathbf{p}} \tag{2.22}$$

or

$$\mathbf{K}_t = \frac{EA}{l}\begin{bmatrix} 1 & -1 & \beta & -\beta \\ -1 & 1 & -\beta & \beta \\ \beta & -\beta & \beta^2 & -\beta^2 \\ -\beta & \beta & -\beta^2 & \beta^2 \end{bmatrix} + \frac{N}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{2.23}$$

This is the matrix equivalent of (1.9) with the second matrix being the 'initial stress' matrix. Equation (1.9) can be recovered by setting

$$u_1 = w_1 = z_1 = u_2 = 0, \qquad w_2 = w \qquad \text{and} \qquad z_2 = z. \tag{2.24}$$