Complex variable methods

References

- Plane elasticity problems (http://imechanica.org/node/319)
- G.F. Carrier, M. Krook, C.E. Pearson, Functions of a complex variable.
- N.I. Muskhelishvili, Some basic problems of the mathematical theory of elasticity.
- A.N. Stroh, Dislocations and cracks in anisotropic elasticity. Philosophical Magazine 3, 625-646 (1958).
- A.N. Stroh, Steady-state problems in anisotropic elasticity. J. Math. Phys. 41, 77-103 (1962)

These notes are written to remind myself of what to say in class. Thus, the notes are not self-contained. Students are assumed to know about plane elasticity problems, and about functions of a complex variable. The book by Carrier, Krook and Pearson is good if you need to review ideas of functions of a complex variable.

I'll first illustrate some of these ideas by applying them to anti-plane shear problems. I'll then move on to in-plane deformation. The last topic will be two dimensional elasticity problems in anisotropic materials. If you found any errors in these notes, please kindly let me know (suo@seas.harvard.edu).

What type of PDEs can be solved using complex variable methods? In the lecture on plane elasticity problems (http://imechanica.org/node/319), we have seen that the governing equations in terms of the displacements have the following attributes:

- The equations are linear in displacements
- The equations are homogenous
- Every term in the equations has the same order of differentials
- Each function depends on two coordinates

Equations with such attributes may be solved using complex variable methods.

Anti-plane shear. Consider an isotropic, linearly elastic body in a state of anti-plane deformation. Examples: a crack, a hole. The field of the displacement takes the following form:

$$u = v = 0, \qquad w = w(x, y).$$

The nonzero components of the strain tensor are

$$\gamma_{xz} = \frac{\partial w}{\partial x}, \qquad \gamma_{yz} = \frac{\partial w}{\partial y}.$$

Hooke's law is specialized to

$$\tau_{xz} = \mu \gamma_{xz}, \quad \tau_{yz} = \mu \gamma_{yz}.$$

The equilibrium equation is

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

A combination of the above equations gives that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

This is the PDE that governs the displacement field.

On the surface of the body, two kinds of boundary conditions are commonly used:

- Prescribed displacement.
- Prescribed traction: $t_z = n_x \tau_{xz} + n_y \tau_{yz}$

The general solution to the PDE. Try a solution of the form

$$w(x,y) = f(z),$$

where

$$z = x + py.$$

Here p is a constant to be determined. Inserting $w(x,y) = f(z)$ into the PDE,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

we obtain that

$$\left(1+p^2\right)\frac{d^2 f(z)}{dz^2} = 0.$$

This equation is satisfied by an arbitrary function $f(z)$ if

$$1 + p^2 = 0.$$

This is an algebraic equation. The roots are $p = i$ and $p = -i$. Consequently, the general solution to the PDE is

$$w(x,y) = f(z) + g(\bar{z}),$$

where f and g are arbitrary functions; $z = x + iy$ and $\bar{z} = x - iy$.

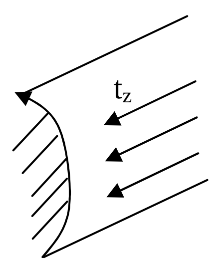
Conventions in writing complex conjugation. For example, let $f(z) = (z-a)^b$, where a and b are complex numbers. Thus, $f(\bar{z}) = (\bar{z}-a)^b$, $\bar{f}(z) = (z-\bar{a})^{\bar{b}}$, and $\bar{f}(\bar{z}) = (\bar{z}-\bar{a})^{\bar{b}}$.

Because the displacement w is real, the general solution is

$$w(x,y) = f(z) + \bar{f}(\bar{z}).$$

To be consistent with commonly used notation in the literature, we adopt another function $\omega(z) = 2i\mu f(z)$, so that

$$\mu w = \frac{1}{2i}\left[\omega(z) - \bar{\omega}(\bar{z})\right] = \mathrm{Im}\left[\omega(z)\right].$$



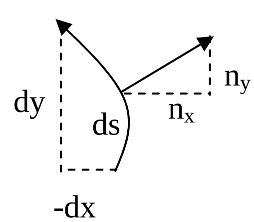
We next express stresses in terms of the complex function $\omega(z)$. Note that

$$\tau_{xz} = \mu\frac{\partial w}{\partial x} = \frac{1}{2i}\left[\frac{d\omega(z)}{dz} - \frac{d\omega(\bar{z})}{d\bar{z}}\right],$$

$$\tau_{yz} = \mu\frac{\partial w}{\partial y} = \frac{1}{2}\left[\frac{d\omega(z)}{dz} + \frac{d\omega(\bar{z})}{d\bar{z}}\right].$$

A combination of the above two expressions gives that

$$\tau_{yz} + i\tau_{xz} = \omega'(z).$$



We can also calculate the resultant force on an arc. Note that

$$n_x = \frac{dy}{ds}, \quad n_y = -\frac{dx}{ds}$$

Thus, the traction is

$$
\begin{aligned}
t_z &= n_x \tau_{xz} + n_y \tau_{yz} \\
&= \frac{1}{2i}\left[\omega'(z) - \bar{\omega}'(\bar{z})\right]\frac{dy}{ds} - \frac{1}{2}\left[\omega'(z) + \bar{\omega}'(\bar{z})\right]\frac{dx}{ds} \\
&= -\frac{1}{2}\omega'(z)\frac{dz}{ds} - \frac{1}{2}\bar{\omega}'(\bar{z})\frac{d\bar{z}}{ds} \\
&= -\frac{1}{2}\frac{d}{ds}\left[\omega(z) + \bar{\omega}(\bar{z})\right]
\end{aligned}
$$

The resultant force is

$$F_z = \int_{s_0}^{s} t_z ds = -\mathrm{Re}\left[\omega(z)\right] + \text{constant}$$

Summary of equations. The general solution to the anti-plane problem is given a function of a complex variable, $\omega(z)$, where $z = x + iy$, such that

$$\mu w = \frac{1}{2i}\left[\omega(z) - \bar{\omega}(\bar{z})\right] = \mathrm{Im}\left[\omega(z)\right]$$

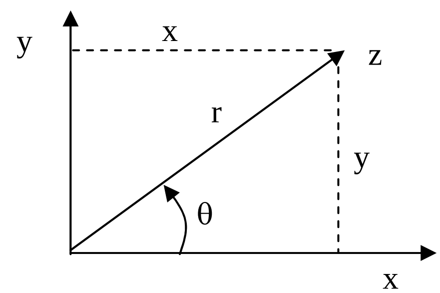
$$\tau_{yz} + i\tau_{xz} = \omega'(z)$$

$$F_z(s) = \int_{s_0}^{s} t_z ds = -\mathrm{Re}\left[\omega(z)\right] + \text{constant}$$

The PDE is satisfied by any function $\omega(z)$. No more PDE to solve. All we need to do is to select a function $\omega(z)$ to satisfy boundary conditions.

A point in a plane represents a complex number. A point in a plane is represented by the Cartesian coordinates (x, y), or by polar coordinates (r, θ). The two sets of coordinates are related by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$



The same point can also be represented by a single complex number, $z = x + iy$. Recall Euler's formula,

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

Thus we write

$$z = r\exp(i\theta).$$

We cal r the modulus and θ the argument of the complex number z.

Analytic functions. A function $f(z)$ of a complex variable z is a mapping from one complex number to another complex number. The function is differentiable if

$$\frac{f(\xi) - f(z)}{\xi - z}$$

approaches the same value for any direction of approach of ξ to z. This limit is called the derivative of the function $f(z)$, and is denoted by $f'(z)$ or $df(z)/dz$.

Example 1: $f(z) = z^3$. Calculate the ratio

$$\frac{\xi^3 - z^3}{\xi - z} = \xi^2 + \xi z + z^2.$$

However ξ approaches z, the above ratio approaches $3z^2$. Consequently, the function $f(z) = z^3$

is differentiable, and $f'(z) = 3z^2$.

Example 2: $f(z) = \bar{z}$. Let $\xi - z = \rho e^{i\alpha}$. Examine the ratio

$$\frac{\bar{\xi} - \bar{z}}{\xi - z} = e^{-2i\alpha}.$$

The value of the limit depends on how ξ approaches *z*. Consequently, the function $f(z) = \bar{z}$ is not differentiable.

Example 3: $f(z) = z\bar{z}$. Examine the ratio

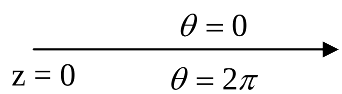
$$\frac{\xi\bar{\xi} - z\bar{z}}{\xi - z} = \frac{(z + \rho e^{i\alpha})(\bar{z} + \rho e^{-i\alpha}) - z\bar{z}}{\rho e^{i\alpha}} = (e^{i\alpha}\bar{z} + e^{-i\alpha}z) + \rho e^{-i\alpha}.$$

Unless $z = 0$, the value of the limit depends on how ξ approaches *z*. Consequently, the function $f(z) = z\bar{z}$ is differentiable at point *z* = 0, but is not differentiable at any other points.

If $f(z)$ is differentiable at z_0, and also at each point in some neighborhood of z_0, then $f(z)$ is said to be *analytic* at z_0. The terms *holomorphic*, *monogenic*, and *regular* are also used.

Multi-valued functions. Brunch cut. The function $\log z$ is defined as

$$\log z = \log r + i\theta.$$

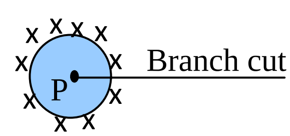


The function is multi-valued. That is, for the same point *z* on the plane, θ may take multiple values, so that $\log z$ takes multiple values.

To make the function single-valued, we need to restrict the range of θ. For example, we can restrict θ to be $0 \le \theta < 2\pi$. This restriction has a graphic interpretation on the plane: The plane is cut by a line $\theta = 0$, known as the brunch cut. When the point *z* moves in the plane, without crossing the brunch cut, the function $\log z$ is single-valued. Of course, for the function $\log z$, we can draw brunch cut in any direction, so long as we explicitly state how we cut and what is the range of the angle θ.

Line force in an infinite body. In an infinite body, a line force *P* (force per unit length) acts at *x* = 0 and *y* = 0. Linearity and dimensional consideration dictate that stress field at distance *r* scales as

$$\tau \propto P/r$$



We expect the solution takes the form

$$\omega(z) = A\log z,$$

where *A* is a constant to be determined.

We cut a small circular disk out.

Force balance: $F_z(r, 2\pi) - F(r, 0) = -P$

No dislocation: $w(r, 2\pi) - w(r, 0) = 0$

These conditions correspond to

$$\mathrm{Re}[2\pi i A] = P$$

$$\mathrm{Im}[2\pi i A] = 0$$

The solution is

$$A = \frac{P}{2\pi i},$$

so that the complex function is

$$\omega(z) = \frac{P}{2\pi i}\log z .$$

The displacement field is

$$\mu w = \mathrm{Im}[\omega(z)] = -\frac{P}{2\pi}\log r$$

The stress field is

$$\tau_{yz} + i\tau_{xz} = \omega'(z) = \frac{P}{2\pi i z} = \frac{P}{2\pi r}(\sin\theta - i\cos\theta)$$

or

$$\tau_{yz} = \frac{P}{2\pi r}\sin\theta, \tau_{xz} = -\frac{P}{2\pi r}\cos\theta .$$

Screw dislocation. Describe a screw dislocation. Dimensional consideration dictates that the stress field should take the form

$$\tau \sim \frac{b\mu}{r}.$$

Thus, we expect the complex function takes the form

$$\omega = B\log z = B(\log r + i\theta)$$

The constant B is determined by the following conditions:

- No resultant force $F(r,2\pi) - F(r,0) = 0$, or $\mathrm{Re}(2\pi i B) = 0$
- Burgurs vector: $w(r,2\pi) - w(r,0) = b$, to $\mathrm{Im}(2\pi i B) = b\mu$

The solution is

$$\omega = \frac{\mu b}{2\pi}\log z .$$

The displacement field is

$$\mu w = \mathrm{Im}[\omega(z)] = \frac{\mu b}{2\pi}\theta .$$

The stress field is

$$\tau_{yz} + i\tau_{xz} = \omega'(z) = \frac{\mu b}{2\pi z} = \frac{\mu b}{2\pi r}(\cos\theta - i\sin\theta).$$

Thus,

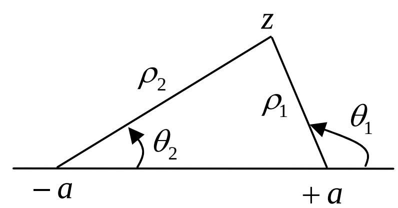
$$\tau_{yz} = \frac{\mu b}{2\pi r}\cos\theta, \tau_{xz} = -\frac{\mu b}{2\pi r}\sin\theta .$$

A crack in an infinite block. The following function

$$\omega'(z) = \frac{\tau z}{\sqrt{z^2 - a^2}}$$

satisfies the remote boundary condition

$$\tau_{yz} = \tau, \tau_{xz} = 0 \text{ as } |z| \to \infty .$$



The function also satisfies the traction-free condition on the crack faces. To see this, note that the function has a brunch cut at $-a < x < a$. Let

$$z - a = \rho_1 e^{i\theta_1}, \quad -\pi < \theta_1 < \pi$$

$$z + a = \rho_2 e^{i\theta_2}, \quad 0 < \theta_2 < 2\pi$$

so that

$$\frac{1}{\sqrt{z^2 - a^2}} = \frac{1}{\sqrt{\rho_1 \rho_2}} e^{-\frac{i}{2}(\theta_1 + \theta_2)}.$$

When z approaches the cut from above,

$$\rho_1 = a - x, \ \theta_1 = \pi, \ \ \rho_2 = a + x, \ \theta_2 = 0,$$

so that

$$\frac{1}{\sqrt{z^2 - a^2}} = \frac{1}{\sqrt{a^2 - x^2}} e^{-\frac{i\pi}{2}}.$$

When z approaches the cut from below,

$$\rho_1 = a - x, \ \theta_1 = -\pi, \ \ \rho_2 = a + x, \ \theta_2 = 0,$$

so that

$$\frac{1}{\sqrt{z^2 - a^2}} = \frac{1}{\sqrt{a^2 - x^2}} e^{+\frac{i\pi}{2}}.$$

In assigning the angles θ_1 and θ_2, we follow the following rule. In the figure, the vectors $z - a$ and $z + a$ are joined at point *z*. As we move *z* from one side of the branch cut to the other side, the two vectors should remain joined. Thus, if we rotate clockwise, $\theta_1 = -\pi$ and $\theta_2 = 0$, as given above. If we rotate counterclockwise, $\theta_1 = +\pi$ and $\theta_2 = 2\pi$. In this case,

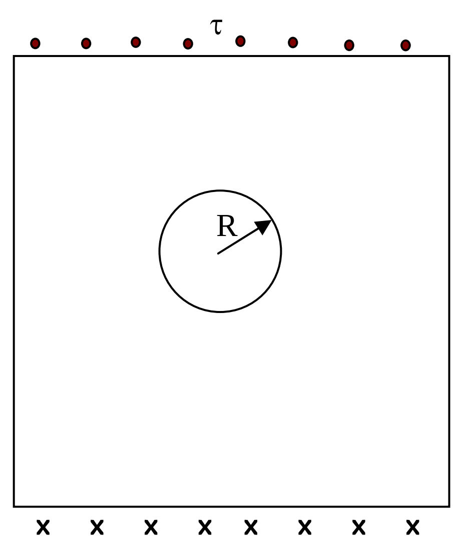
$$e^{-\frac{i}{2}(\theta_1 + \theta_2)} = e^{-\frac{3i\pi}{2}} = e^{+\frac{i\pi}{2}}.$$

That is, the end result is unchanged.

Recall that

$$\tau_{yz} + i\tau_{xz} = \omega'(z).$$

In both cases above, the $\mathrm{Re}[\omega'(z)] = 0$, so that $\tau_{yz} = 0$ on both faces of the crack.



A circular hole in an infinite block. Consider a circular hole, radius *R*, in an infinite block, subject to a remote anti-plane shear stress:

$$\tau_{yz} = \tau, \tau_{xz} = 0 \text{ as } |z| \to \infty.$$

Recall $\tau_{yz} + i\tau_{xz} = \omega'(z)$, so that

$$\omega(z) \to \tau z \text{ as } |z| \to \infty.$$

The hole is traction-free, so that

$$\mathrm{Re}[\omega(t)] = 0$$

when $t = R\exp(i\theta)$. To satisfy this boundary condition, we write

$$\omega(z) = \tau\left(z - \frac{R^2}{z}\right).$$

We can confirm that this function satisfies both the remote boundary condition and the traction free boundary condition on the surface of the hole.

The displacement field is

$$\mu w = \mathrm{Im}[\omega(z)] = \tau \sin\theta\left(r - \frac{R^2}{r}\right).$$

The stress field is

$$\tau_{yz} + i\tau_{xz} = \omega'(z) = \tau\left(1 + \frac{R^2}{z^2}\right),$$

so that

$$\tau_{yz} = \tau\left(1 + \frac{R^2}{r^2}\cos 2\theta\right), \tau_{xz} = -\tau\frac{R^2}{r^2}\sin 2\theta.$$

Contour integrals. Let $f(z)$ be an analytic function in a region, and C be a contour in this region. The following theorems hold:

(i) $\oint_C f(t)dt = 0$

(ii) $\frac{1}{2\pi i}\oint_C \frac{f(t)dt}{t-z} = f(z)$ if z is inside the contour.

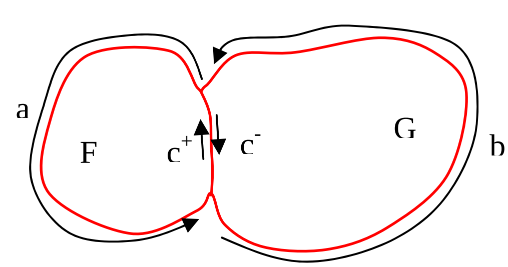
(iii) $\frac{n!}{2\pi i}\oint_C \frac{f(t)dt}{(t-z)^{n+1}} = \frac{d^n f(z)}{dz^n}$ if z is inside the contour.

Cauchy integral on a curve. Consider the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

where the path of integration, C, is some curve in the z plane, the integration variable t is a point on C, and $f(t)$ is a complex-valued function prescribed on C. The curve C need not be closed, and the function $f(t)$ need only be defined on the curve and need not be analytic in the plane. For reasonably behaved C and $f(t)$, when z is not on the curve C, the function $F(z)$ is analytic.

Analytic continuation. If function $f(z)$ is analytic in region F, and function $g(z)$ is analytic in region G. Region F and region G has some intersection, e.g., share part of their boundaries. If the two functions are equal in the intersection, there exists a function $\phi(z)$ analytic in the combined regions of F and G, such that



$$\phi(z) = \begin{cases} f(z), z \in F \\ g(z), z \in G \end{cases}$$

Proof. Note that $f(t) = g(t)$ when t is on $c \in F \cap G$. The function

$$\phi(z) = \frac{1}{2\pi i} \oint_{a+c^+} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i} \oint_{b+c^-} \frac{g(t)dt}{t-z}$$

is analytic in $F \cup G$, and has the property

$$\phi(z) = \begin{cases} f(z), z \in F \\ g(z), z \in G \end{cases}$$

A circular hole in an infinite block. Consider a circular hole, radius R, in an infinite block, subject to a remote anti-plane shear stress.

Let remote loading conditions be

$$\tau_{yz} = \tau, \tau_{xz} = 0 \text{ as } |z| \to \infty .$$

Recall $\tau_{yz} + i\tau_{xz} = \omega'(z)$, so that

$$\omega(z) \to \tau z \text{ as } |z| \to \infty .$$

Traction free on the surface of the circle:

$$\omega(t) + \overline{\omega}(\bar{t}) = 0 ,$$

where $t = R\exp(i\theta)$ is a point on the circle. Rewrite the above equation as

$$t\omega(t) = -t\overline{\omega}\left(\frac{R^2}{t}\right).$$

Observe that $z\omega(z)$ is analytic when z is outside the circle, and $z\overline{\omega}(R^2/z)$ is analytic when z is inside the circle. The above equality holds true on the circle.

There exists a function $f(z)$ analytic on the entire plane, such that

$$f(z) = \begin{cases} z\omega(z), \text{inside} \\ -z\overline{\omega}(R^2/z), \text{outside} \end{cases}$$

The only function that is analytic on the entire plane is a polynomial. Recall the remote boundary condition $\omega(z) \to \tau z$ as $|z| \to \infty$, so that

$$f(z) = \tau z^2 + Az + B .$$

Recall that as $z \to 0$, $\overline{\omega}(R^2/z) = \tau R^2/z$, so that $B = -R^2\tau$. The constant A does not affect stress distribution and is set to be zero. Thus, the solution is

$$\omega(z) = \tau\left(z - \frac{R^2}{z}\right).$$

We can confirm that this function satisfies both the remote boundary condition and the traction free boundary condition on the surface of the hole.

Conformal mapping. Let

$$z = \Gamma(\zeta)$$

be an analytic function that maps region R_ζ on the ζ-plane to region R_z on the z-plane. As an example, the function

$$z = \Gamma(\zeta) = \frac{a+b}{2}\zeta + \frac{a-b}{2}\frac{1}{\zeta}$$

maps the exterior of a unit circle on the ζ -plane to the ellipse on the z -plane.

Let $\omega(z)$ be a function analytic in region R_z. Then the composite function

$$\Omega(\zeta) = \omega(\Gamma(\zeta))$$

is analytic in R_ζ. In terms of the function $\Omega(\zeta)$, various physical fields are given by

$$\mu w(x,y) = \mathrm{Im}[\Omega(\zeta)]$$

$$F_z = -\mathrm{Re}[\Omega(\zeta)]$$

$$\tau_{yz} + i\tau_{xz} = \frac{d\omega(z)}{dz} = \frac{d\Omega(\zeta)/d\zeta}{d\Gamma(\zeta)/d\zeta}.$$

An elliptic hole in an infinite block. Consider an elliptic hole, semi-axes *a* and *b*, in an infinite block, subject to a remote anti-plane shear stress $\tau_{yz} = \tau$. The remote boundary condition can be written as

$$\tau_{yz} = \tau, \tau_{xz} = 0 \text{ as } |z| \to \infty.$$

Recall that

$$\tau_{yz} + i\tau_{xz} = \frac{d\omega(z)}{dz} = \frac{\Omega'(\zeta)}{\Gamma'(\zeta)},$$

so that

$$\frac{\Omega'(\zeta)}{\Gamma'(\zeta)} \to \tau \text{ as } |\zeta| \to \infty.$$

On the surface of the hole, there is no traction:

$$\mathrm{Re}[\Omega(t)] = 0 \text{ when } t = \exp(i\beta).$$

Both the remote boundary condition and the condition on the surface of the hole are satisfied by

$$\Omega(\zeta) = \frac{a+b}{2}\tau\left(\zeta - \frac{1}{\zeta}\right).$$

The stress field is

$$\tau_{yz} + i\tau_{xz} = \frac{\Omega'(\zeta)}{\Gamma'(\zeta)} = \frac{\frac{a+b}{2}\tau\left(1 + \frac{1}{\zeta^2}\right)}{\frac{a+b}{2} - \frac{a-b}{2\zeta^2}}.$$

At $\zeta = 1$, the stress is

$$\tau_{yz} = \tau\left(1 + \frac{a}{b}\right).$$

This gives the stress concentration factor.

Plemelj formulas (Carrier et al., p.413). *Cauchy integral long a curve*. Consider the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

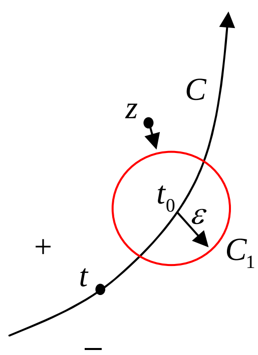
where the path of integration, *C*, is a curve in the *z* plane, the integration variable *t* is a point on *C*, and $f(t)$ is a complex-valued function prescribed on *C*. The function $f(t)$ is continuous and satisfies the Lipschitz condition

$$|f(t) - f(t_0)| < A|t - t_0|^\alpha$$

for all *t* on *C* in some neighborhood of t_0, where *A* and α are constants, with $0 < \alpha \le 1$. The curve *C* need not be a closed contour, and $f(t)$ need not be defined for any point off the curve.

Principal value of the Cauchy integral. When z is not on the curve *C*, the function $F(z)$ is clearly analytic. However, when z is a point on *C*, say $z = t_0$, the integral becomes unbounded. In this case, we can define

$$PV\int \frac{f(t)dt}{t-t_0},$$



where the integral extends on *C*, excluding the part of the curve in a circle of radius ε and centered at t_0. When $\varepsilon \to 0$, the above expression is known as the principal-value integral.

Example. The meaning of the principal value may be illustrated by an example. Consider an integral along the *x*-axis:

$$\int_{-1}^{4} \frac{f(x)dx}{x-1},$$

where $f(1) \ne 0$. This integral is undefined because of the singularity at $x = 1$. However, we can define the principal value of the integral as

$$\int_{-1}^{1-\varepsilon} \frac{f(x)dx}{x-1} + \int_{1+\varepsilon}^{4} \frac{f(x)dx}{x-1}, \qquad \varepsilon \to 0$$

Thus,

$$\int_{-1}^{4} \frac{f(x) - f(1)}{x-1}dx + f(1)PV\int_{-1}^{4}\frac{dx}{x-1}$$

The first integral is well defined if $f(x)$ satisfies the Lipschitz condition. The second integral is

$$PV\int_{-1}^{4}\frac{dx}{x-1} = \int_{-1}^{1-\varepsilon}\frac{dx}{x-1} + \int_{1+\varepsilon}^{4}\frac{dx}{x-1} = \log\left|\frac{1-\varepsilon-1}{-1-1}\right| + \log\left|\frac{4-1}{1+\varepsilon-1}\right| = \log\frac{3}{2}.$$

Note that the principal value is bounded because we specify that point *x* approaches 1 from two sides in a specific way. A different value will be obtained if we specify the approach in different ways. For example, let us consider

$$\int_{-1}^{1-\varepsilon_1}\frac{dx}{x-1} + \int_{1+\varepsilon_2}^{4}\frac{dx}{x-1} = \log\left|\frac{1-\varepsilon_1-1}{-1-1}\right| + \log\left|\frac{4-1}{1+\varepsilon_2-1}\right| = \log\frac{\varepsilon_1}{\varepsilon_2} + \log\frac{3}{2}$$

The function $F(z)$ is analytic when $z \notin C$. Now consider a point *z* on the + side of the plane. Let $z \to t_0$, and denote the limiting value of $F(z)$ by $F_+(t_0)$. We perturb the path *C* by removing the curve inside the circle and adding a semicircle C_1 Thus,

$$F_+(t_0) = \frac{1}{2\pi i}\int_{C-C_1} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)dt}{t-z}.$$

As $\varepsilon \to 0$, the above tends to

$$F_+(t_0) = \frac{1}{2\pi i} PV\int_C \frac{f(t)dt}{t-t_0} + \frac{1}{2} f(t_0).$$

Similarly, when z approaches t_0 from the – side of the plane, we have

$$F_-(t_0) = \frac{1}{2\pi i} PV\int_C \frac{f(t)dt}{t-t_0} - \frac{1}{2} f(t_0).$$

These two formulas are known as the Plemelj formulas.

Subtracting or adding the two formulas, we obtain that

$$F_+(t_0) - F_-(t_0) = f(t_0),$$

and

$$F_+(t_0) + F_-(t_0) = \frac{1}{\pi i} PV\int \frac{f(t)dt}{t-t_0}.$$

A boundary value problem. *Statement of the problem*. Let C be a curve in a region R, and $f(t)$ be a known function prescribed on C. Find a function $F(z)$ that is analytic in R except on C, and satisfies the boundary condition

$$F_+(t_0) - F_-(t_0) = f(t_0)$$

for any point t_0 on C.

Solution. We write the solution in the form

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} + P(z),$$

where $P(z)$ is a function analytic in R, but not necessarily on C. According to the Plemelj formulas, we obtain that

$$P_+(t_0) - P_-(t_0) = 0.$$

Consequently, $P(z)$ is analytic in R. We have found the complete solution to the boundary value problem.

A singular integral equation. Given a curve C on a plane, and two functions $a(t_0)$ and $b(t_0)$ prescribed on C. Find a function $f(t)$ that satisfies

$$a(t_0) f(t_0) + PV\int_C \frac{f(t)dt}{t-t_0} = b(t_0).$$

We can convert this integral equation for a function defined on a curve into a boundary value problem for a function in a region. Define

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}.$$

Using the Plemelj formulas, we can rewrite the singular integral equation as

$$a(t_0)[F_+(t_0) - F_-(t_0)] + \pi i[F_+(t_0) + F_-(t_0)] = b(t_0).$$

Thus, the singular integral equation is equivalent to finding a function analytic in the plane (except on C) and satisfies the above boundary condition. Problem like this leads us to the

Riemann-Hilbert problem.

Riemann-Hilbert problem (Carrier et al., p.418). Let *C* be curve in a region *R*, and $f(t)$ and $g(t)$ be known functions prescribed on *C*. Find a function $W(z)$ that is analytic in *R* except on *C*, and satisfies the boundary condition

$$W_+(t) - g(t)W_-(t) = f(t)$$

for any point *t* on *C*.

First, let us find a solution to the homogeneous equation:

$$\chi_+(t) - g(t)\chi_-(t) = 0,$$

or

$$\chi_+(t) = g(t)\chi_-(t).$$

Taking logarithm on both sides of the equation, we obtain that

$$\log \chi_+(t) - \log \chi_-(t) = \log g(t).$$

This boundary value problem has been solved before. The general solution is

$$\log \chi(z) = \frac{1}{2\pi i}\int_C \frac{\log g(t)dt}{t-z} + Q(z),$$

where $Q(z)$ is any function analytic in *R*. The need for $Q(z)$ is evident when we expect the equation $\chi_+(t) = g(t)\chi_-(t)$. Multiplying any solution with an analytic solution gives another solution.

Second, in the original equation $W_+(t) - g(t)W_-(t) = f(t)$, replace $g(t)$ by $\chi_+(t)/\chi_-(t)$, and we obtain that

$$\frac{W_+(t)}{\chi_+(t)} - \frac{W_-(t)}{\chi_-(t)} = \frac{f(t)}{\chi_+(t)}.$$

Once again, this problem has been solved before. The general solution is

$$\frac{W(z)}{\chi(z)} = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{\chi_+(t)(t-z)} + P(z),$$

where $P(z)$ is any function analytic in *R*.

A special case: $g(t)$ is a constant *g*, the curve *C* is $-a < x < a$, and the region *R* is the infinite plane. A homogenous solution is

$$\log \chi(z) = \frac{\log g}{2\pi i}\int_{-a}^{a} \frac{dx}{x-z} = \frac{\log g}{2\pi i}\log\frac{z-a}{z+a}.$$

For example, we will encounter the case

$$W_+(t) + W_-(t) = f(t).$$

In this case, $g = -1$. Thus, $\log g = i\pi$, so that a homogenous solution is

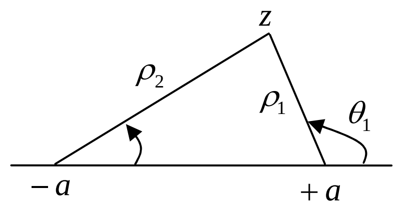
$$\chi(z) = \left(\frac{z-a}{z+a}\right)^{1/2}.$$

We can also obtain another homogeneous solution by multiplying an analytic function. Later we will use this solution to solve crack problems. To ensure the square-root singularity, we divide the above solution by $z-a$, so that

$$\chi(z) = \frac{1}{\sqrt{z^2-a^2}}.$$

To see that this is indeed a homogenous solution, note that the function has a brunch cut at $-a < x < a$. Let

$$z - a = \rho_1 e^{i\theta_1}, \quad -\pi < \theta_1 < \pi$$
$$z + a = \rho_2 e^{i\theta_2}, \quad 0 < \theta_2 < 2\pi$$



so that

$$\chi(z) = \frac{1}{\sqrt{\rho_1 \rho_2}} e^{-\frac{i}{2}(\theta_1 + \theta_2)}.$$

When z approaches the cut from above,

$$\rho_1 = a - x, \ \theta_1 = \pi, \ \rho_1 = a + x, \ \theta_2 = 0,$$

so that

$$\chi_+(x) = \frac{1}{\sqrt{a^2 - x^2}} e^{-\frac{i\pi}{2}}.$$

When z approaches the cut from below,

$$\rho_1 = a - x, \ \theta_1 = -\pi, \ \rho_2 = a + x, \ \theta_2 = 0,$$

so that

$$\chi_-(x) = \frac{1}{\sqrt{a^2 - x^2}} e^{+\frac{i\pi}{2}}.$$

Consequently,

$$\chi_+(x) + \chi_-(x) = 0.$$

The solution to the inhomogeneous boundary value problem is

$$\frac{W(z)}{\chi(z)} = \frac{1}{2\pi i} \int_{-a}^{a} \frac{f(x)dx}{\chi_+(x)(x - z)} + P(z),$$

where $P(z)$ is a polynomial.

A crack in an infinite block. We have guessed the solution to this problem before. Now let us see how this solution can be found by using the above formal procedure. The solution is a function $\omega(z)$ analytic in the entire plane, except on the cut $-a < x < a$. Recall that $\tau_{yz} + i\tau_{xz} = \omega'(z)$, the remote applied stress $\tau_{yz} = \tau$ requires that

$$\omega(z) \to \tau z \text{ as } |z| \to \infty.$$

Because τ_{yz} is the same on the two faces of the crack. Consequently, for $-a < x < a$, we have

$$\omega_+'(x) + \overline{\omega}_-'(x) = \omega_-'(x) + \overline{\omega}_+'(x).$$

or

$$\omega_+'(x) - \overline{\omega}_+'(x) = \omega_-'(x) - \overline{\omega}_-'(x).$$

By analytic continuation, the function $\omega'(z) - \overline{\omega}'(z)$ must be analytic in the entire plane. Thus, the function must be a polynomial. Recall the remote boundary condition, $\omega(z) \to \tau z$ as $|z| \to \infty$. We conclude that

$$\omega'(z) - \overline{\omega}'(z) = 0$$

in the entire plane.

We next consider the traction-free boundary condition $\tau_{yz} = 0$ on the crack faces, $-a < x < a$. Thus, on the top crack face, the traction vanishes:

$$\omega_+'(x) + \overline{\omega}_-'(x) = 0 .$$

This is rewritten as

$$\omega_+'(x) + \omega_-'(x) = 0 .$$

The solution to this homogenous problem is

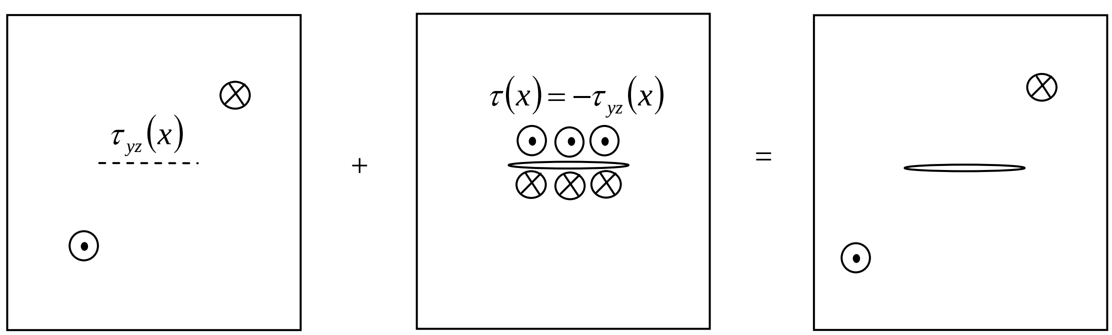
$$\omega'(z) = \frac{Q(z)}{\sqrt{z^2 - a^2}} .$$

where $Q(z)$ is a polynomial. The remote boundary condition, $\omega(z) \to \tau z$ as $|z| \to \infty$, dictates that

$$\omega'(z) = \frac{\tau z}{\sqrt{z^2 - a^2}} .$$

A crack in an infinite body subject to arbitrary loads. A traction-free crack, length 2*a*, is in an infinite body. The body is loaded by a set of loads, such that the body is equilibrated in a state of anti-plane deformation. We would like to determine the stress field in the body.

The problem can be viewed as a linear superposition of two problems. In problem A, there is no crack, and the same set of loads is applied in the uncracked body. We can determine the distribution of the stress in the body. In particular, the stress component on the *x* axis is $\tau_{yz}(x)$. In problem B, there is a crack, but there is no load inside the body. Both faces of the crack is prescribed with the traction on the on the surfaces by a distribution of stress $\tau(x) = -\tau_{yz}(x)$. A superposition of the two problems reproduces the original problem, with a traction free crack and a set of loads. Problem has no crack, and does not contribute to the stress intensity factor. We next focus on solving Problem B.



Problem A Problem B Original Problem

The solution to Problem B is given by a function $\omega(z)$ analytic in the entire plane, except on the cut $-a < x < a$. Because τ_{yz} is continuous across the entire *x*-axis, we obtain that

$$\omega_+'(x) + \overline{\omega}_-'(x) = \omega_-'(x) + \overline{\omega}_+'(x).$$

or

$$\omega_+'(x) - \overline{\omega}_+'(x) = \omega_-'(x) - \overline{\omega}_-'(x).$$

The left-hand side is an analytic function in the upper half plane, and the right hand side is an analytic function in the lower half plane. By analytic continuation, the function $\omega'(z)-\bar{\omega}'(z)$ must be analytic in the entire plane. Thus, the function must be a polynomial. Because no stress is applied remotely, so that the stress field vanishes remote from the crack. Consequently,

$$\omega'(z)-\bar{\omega}'(z)=0$$

in the entire plane.

On the top crack face, the traction is prescribed:

$$\omega_+'(x)+\bar{\omega}_-'(x)=-2\tau(x).$$

This is rewritten as

$$\omega_+'(x)+\omega_-'(x)=-2\tau(x).$$

The solution is

$$\frac{\omega'(z)}{\chi(z)}=\frac{1}{2\pi i}\int_{-a}^{a}\frac{-2\tau(x)dx}{\chi_+(x)(x-z)}+P(z).$$

Let the polynomial be $P(z)=p_0+p_1z+\ldots$ As $|z|\to\infty$,

$$\frac{P(z)}{\sqrt{z^2-a^2}}\to\frac{p_0}{z}+p_1+\ldots$$

Because there is no resultant force and dislocation, $p_0=0$. Because there is no remote stress, $p_1=0$. We also cannot have higher order terms. Thus, $P(z)=0$.

When $z\to a$, we have

$$\omega'(z)\to\frac{1}{\pi\sqrt{2a(z-a)}}\int_{-a}^{a}\sqrt{\frac{a+x}{a-x}}\tau(x)dx.$$

At distance r ahead of the crack, the stress is square root singular, and the stress intensity factor K is defined as

$$\tau_{yz}=\frac{K}{\sqrt{2\pi r}}.$$

A comparison of the two expressions gives the stress intensity factor:

$$K=\frac{1}{\sqrt{\pi a}}\int_{-a}^{a}\sqrt{\frac{a+x}{a-x}}\tau(x)dx.$$

A crack interacting with a singularity. Now we return to the superposition. In problem A, the stress field is taken to be induced by singularities like a dislocation, a line force, etc. Assume that the complex function for problem A is $\omega_A(z)$. Thus, the traction τ_{yz} on the x-axis is calculated from

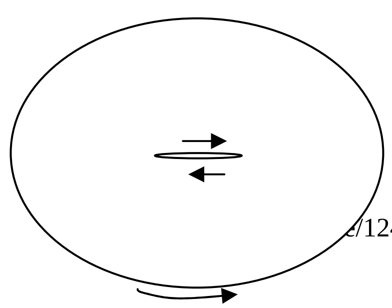
$$2\tau(x)=\omega_A'(x)+\bar{\omega}_A'(x).$$

The negative of this traction is applied to the crack in Problem B. The complex function of problem B is

$$\frac{\omega'(z)}{\chi(z)}=-\frac{1}{2\pi i}\int_{-a}^{a}\frac{\omega_A'(x)+\bar{\omega}_A'(x)}{\chi_+(x)(x-z)}dx.$$

We need to calculate the integral

$$I=\frac{1}{2\pi i}\int_{-a}^{a}\frac{\omega'_A(x)+\bar{\omega}'_A(x)}{\chi_+(x)(x-z)}dx$$



We look at a similar integral

$$J = \frac{1}{2\pi i}\oint \frac{\omega'_A(t) + \overline{\omega}'_A(t)}{\chi(t)(t-z)}dt$$

Along the contour specified in the figure. We can confirm that

$$J = J_\infty + 2I ,$$

where J_∞ is integrated over remote contour, and J can be calculated using the residue theorm.

For example, consider a crack interacting with a dislocation. Thus, Problem A consists of a dislocation at $z = s$ in an infinite block, so that

$$\omega'_A(z) = \frac{\mu b}{2\pi(z-s)} .$$

Thus,

$$J_\infty = \frac{1}{2\pi i}\oint \frac{\omega'_A(t) + \overline{\omega}'_A(t)}{\chi(t)(t-z)}dt$$

$$= \left(\frac{\mu b}{2\pi}\right)\frac{1}{2\pi i}\oint \frac{\sqrt{t^2 - a^2}}{t-z}\left(\frac{1}{t-s} + \frac{1}{t-\overline{s}}\right)dt$$

$$= \left(\frac{\mu b}{2\pi}\right)\frac{1}{2\pi i}\oint \frac{2}{t}dt = \frac{\mu b}{\pi}$$

and

$$J = \frac{1}{2\pi i}\oint \frac{\omega'_A(t) + \overline{\omega}'_A(t)}{\chi(t)(t-z)}dt$$

$$= \left(\frac{\mu b}{2\pi}\right)\frac{1}{2\pi i}\oint \frac{1}{\chi(t)(t-z)}\left(\frac{1}{t-s} + \frac{1}{t-\overline{s}}\right)dt$$

$$= \left(\frac{\mu b}{2\pi}\right)\left[\frac{1}{\chi(z)}\left(\frac{1}{z-s} + \frac{1}{z-\overline{s}}\right) + \frac{1}{\chi(s)(s-z)} + \frac{1}{\chi(\overline{s})(\overline{s}-z)}\right]$$

Here we have used the residue theorem. We can then find $\omega'(z)$ and determine the stress intensity factor of the crack tip due to a dislocation.

In-plane deformation in terms of complex functions. We now consider the plane elasticity problems. Recall that the governing equation for Airy's stress function is

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0 .$$

The PDE has the attributes listed in the beginning of these notes, and the solution is of the form

$$U(x,y) = f(x + py).$$

Letting $\xi = x + py$ and inserting into the PDE, we find that

$$\left(1 + 2p^2 + p^4\right)\frac{d^4 f(\xi)}{d\xi^4} = 0 .$$

The fourth order algebraic equation

$$1 + 2p^2 + p^4 = 0$$

has roots *i* and –*i*. Thus, $f(x+iy)$ and $g(x-iy)$ are solution to the PDE. In addition, we need to consider the degeneracy. Here is a general way to handle degeneracy. Suppose we modify the PDE somewhat so that the degeneracy is lifted. That is, a degenerated root p splits into p_1 and p_2. Thus, both $f_1(x+p_1y)$ and $f_2(x+p_2y)$ are solutions. Any linear combination is also a solution. Consider the following linear combination:

$$\frac{f(x+p_1y)-f(x+p_2y)}{p_1-p_2}.$$

This is the same as

$$\frac{\partial}{\partial p}f(x+py)=yf'(z).$$

The general solution to the PDE is

$$U=\mathrm{Re}[f(z)+yh(z)]$$

Recall that $y=(z-\bar{z})/(2i)$. Using the notation of Muskhelishvili, we write the general solution as

$$U=\mathrm{Re}\left[\int\psi(z)dz+\bar{z}\phi(z)\right].$$

A direct calculation gives the stresses, resultant forces, and displacements as

$$\frac{\sigma_x+\sigma_y}{2}=2\,\mathrm{Re}[\phi'(z)]$$

$$\frac{\sigma_y-\sigma_x}{2}+i\tau_{xy}=\bar{z}\phi''(z)+\psi'(z)$$

$$f_x+if_y=-i\left[\phi(z)+z\bar{\phi}'(\bar{z})+\bar{\psi}(\bar{z})\right]+\text{costant}$$

$$2\mu(u_x+iu_y)=\kappa\phi(z)-z\bar{\phi}'(\bar{z})-\bar{\psi}(\bar{z})$$

where $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$.

Translation of coordinates. Consider a translation of coordinates $z_*=z-s$. In terms of the complex variable z_*, Airy's function is

$$U=\mathrm{Re}\left[\int\psi_*(z_*)dz_*+(\bar{z}-\bar{s})\phi_*(z_*)\right].$$

A comparison with the expression in terms of the complex variable z leads to

$$\phi(z)=\phi_*(z_*),\quad \psi(z)=\psi_*(z_*)-\bar{s}\phi_*'(z_*).$$

This unusual transformation is clearly a consequence of the degeneracy.

Traction prescribed boundary value problems. If a boundary value problem only involves tractions on the surface, then the stress distribution is independent of material constants.

Next consider a problem involving two bonded materials, with traction provides on various surfaces. The traction prescribed surfaces does not lead to dependence on material constants. Along the interface between the two materials, the traction is continuous:

$$\phi_1(z)+z\bar{\phi}_1'(\bar{z})+\bar{\psi}_1(\bar{z})=\phi_2(z)+z\bar{\phi}_2'(\bar{z})+\bar{\psi}_2(\bar{z}),$$

and the displacement is continuous:

$$\kappa_1\phi_1(z)-z\bar{\phi}_1'(\bar{z})-\bar{\psi}_1(\bar{z})=\Gamma\left[\kappa_2\phi_2(z)-z\bar{\phi}_2'(\bar{z})-\bar{\psi}_2(\bar{z})\right],$$

where $\Gamma=\mu_1/\mu_2$.

Adding the two equations, we obtain that

$$(\kappa_1+1)\phi_1(z)=(\Gamma\kappa_2+1)\phi_2(z)+(1-\Gamma)\left[z\bar{\phi}_2'(\bar{z})+\bar{\psi}_2(\bar{z})\right]$$

$$\phi_1(z)=\frac{\Gamma\kappa_2+1}{\kappa_1+1}\phi_2(z)+\frac{1-\Gamma}{\kappa_1+1}\left[z\bar{\phi}_2'(\bar{z})+\bar{\psi}_2(\bar{z})\right].$$

Thus, the material dependence is expressed in two parameters. Dundurs (1968) was the first to noticed this fact. He wrote the two parameters as

$$\alpha = \frac{\Gamma(\kappa_2+1)-(\kappa_1+1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}, \quad \beta = \frac{\Gamma(\kappa_2-1)-(\kappa_1-1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}.$$

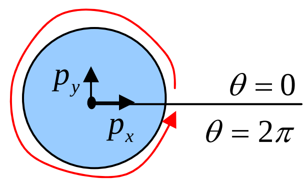
Once a student has learned how to apply complex variable methods to anti-plane deformation, she can readily use the same methods to in-plane deformation. The following is a collection of sample problems. They will not be discussed in class in detail. The student is expected to go over them as homework.

Lines forces. An infinite block is subject to a line force. Let p_x and p_y be the components of the force per unit length. First assume that the force acts at the origin, $z=0$. We expect that the solution takes the form

$$\phi(z) = A\log z, \quad \psi(z) = B\log z$$

where A and B are complex-valued constants. They are determined by two conditions.

Force balance. If we draw a free-body diagram of a small disk, the line force must be balanced by the resultant force acting on the perimeter of the disk:



$$(f_x + if_y) + (p_x + ip_y) = 0.$$

This leads to

$$-i\left[\phi(z) + z\overline{\phi}'(\bar{z}) + \overline{\psi}(\bar{z})\right]_0^{2\pi} + p_x + ip_y = 0,$$

or

$$A2\pi - \overline{B}2\pi + p_x + ip_y = 0.$$

No dislocation. If we go fully around the perimeter of the disk, displacement is continuous:

$$\left[\kappa\phi(z) - z\overline{\phi}'(\bar{z}) - \overline{\psi}(\bar{z})\right]_0^{2\pi} = 0,$$

or

$$\kappa A2\pi i + \overline{B}2\pi i = 0.$$

These two conditions give

$$B = -\kappa\overline{A}, \quad A = -\frac{p_x + ip_y}{2\pi(\kappa+1)}.$$

If the line force acts at point $z = s$, using the coordinate transformation, we obtain the solution

$$\phi(z) = A\log(z-s), \quad \psi(z) = -\kappa\overline{A}\log(z-s) - \frac{A\bar{s}}{z-s}, \quad A = -\frac{p_x + ip_y}{2\pi(\kappa+1)}$$

Edge dislocation. Consider an edge dislocation at $z = s$ with components b_x and b_y. Following similar steps, we find the solution:

$$\phi(z) = A\log(z-s), \quad \psi(z) = \overline{A}\log(z-s) - \frac{A\bar{s}}{z-s}, \quad A = \frac{\mu(b_x + ib_y)}{\pi i(\kappa+1)}.$$

A circular hole in an infinite body subject to remote tension. Boundary conditions are

- As $|z| \to \infty$, $\sigma_x = \tau_{xy} = 0$, $\sigma_y = S$
- When $|z| = a$, $f_x = f_y = 0$.

 Recall that

$$\frac{\sigma_x + \sigma_y}{2} = 2\operatorname{Re}[\phi'(z)]$$

$$\frac{\sigma_y - \sigma_x}{2} + i\tau_{xy} = \bar{z}\phi''(z) + \psi'(z)$$

The remote boundary conditions requires that as $|z| \to \infty$

$$\frac{S}{2} \leftarrow 2\operatorname{Re}[\phi'(z)]$$

$$\frac{S}{2} \leftarrow \bar{z}\phi''(z) + \psi'(z)$$

Thus

$$\phi'(z) \to \frac{S}{4}, \quad \psi'(z) \to \frac{S}{2}$$

or

$$\phi(z) \to \frac{S}{4}z, \quad \psi(z) \to \frac{S}{2}z.$$

We have dropped constants that do not affect stress field.

We next look at the traction-free condition on the surface of the hole:

$$\phi(t) + t\bar{\phi}'(\bar{t}) + \bar{\psi}(\bar{t}) = 0,$$

for any point on the circle, $t = a\exp(i\theta)$. Note that $\bar{t} = a^2/t$, and we rewrite the above equation as

$$t\phi(t) + t^2\bar{\phi}'(a^2/t) + t\bar{\psi}(a^2/t) = 0$$

In the above equation, the first function is analytic outside the circle, and the other two functions are analytic inside the circle. Analytic continuation requires that there exist a function $f(z)$ analytic in the entire plane, such that

$$f(z) = \begin{cases} z\phi(z), |z| > a \\ -z^2\bar{\phi}'(a^2/z) - z\bar{\psi}(a^2/z), |z| < a \end{cases}$$

Because as $|z| \to \infty$, $\phi(z) \to \frac{S}{4}z$, so that

$$f(z) = \frac{S}{4}z^2 + Az + B.$$

Consequently

$$\phi(z) = \frac{S}{4}z + A + \frac{B}{z}, \quad \psi(z) = -\frac{\bar{B}}{a^2}z + \bar{A} - \frac{a^2S}{2z} + \frac{a^2B}{z^3}.$$

We will drop constant *A*. Because as $|z| \to \infty$, $\psi(z) \to \frac{S}{2}z$, so that $B = -a^2S/2$. The solution is

$$\phi(z)=\frac{Sa}{2}\left(\frac{z}{2a}-\frac{a}{z}\right),\quad \psi(z)=\frac{Sa}{2}\left(\frac{z}{a}-\frac{a}{z}-\left(\frac{a}{z}\right)^3\right).$$

An elliptic hole in an infinite body subject to remote tension. We will use an analytic function to map the exterior of a unit circle to the exterior of the ellipse:

$$z=\Gamma(\zeta),$$

where

$$\Gamma(\zeta)=R\zeta+\frac{Rm}{\zeta},\quad R=\frac{a+b}{2},\quad m=\frac{a-b}{a+b}.$$

Write

$$\phi(z)=\phi(\Gamma(\zeta)),$$

so that

$$\frac{d\phi}{dz}=\frac{d\phi/d\zeta}{d\Gamma/d\zeta}.$$

The traction-free condition on the surface of the hole becomes

$$\phi(t)+\frac{\Gamma(t)}{\overline{\Gamma}'(\bar t)}\overline{\phi}'(\bar t)+\overline{\psi}(\bar t)=0,$$

for any point on the circle, $t=a\exp(i\theta)$. The solution is

$$\phi(\zeta)=\frac{SR}{4}\left(\zeta-\frac{m+2}{\zeta}\right),\quad \psi(\zeta)=\frac{SR}{4}\left[(2+m)\zeta-\frac{1}{\zeta}-\left(\frac{\zeta+m\zeta^2}{\zeta^2-m}\right)\left(1+\frac{2+m}{\zeta^2}\right)\right].$$

Recall that

$$\sigma_x+\sigma_y=4\operatorname{Re}\left[\frac{d\phi}{dz}\right]=4\operatorname{Re}\left[\frac{d\phi/d\zeta}{d\Gamma/d\zeta}\right]$$

$$=S\operatorname{Re}\left[\frac{\zeta^2+m+2}{\zeta^2-m}\right]$$

On the surface of hole, traction vanishes, so that at $z=a$ or $\zeta=1$, $\sigma_x=0$, and

$$\sigma_y=S\left(\frac{3+m}{1-m}\right)=S\left(1+2\frac{a}{b}\right).$$

This gives the well known stress concentration factor.

Colinear cracks in an infinite body. An array of cracks lies on the *x*-axis. Following the superposition procedure, we need to consider the problem of traction applied on the faces of the crack. In this problem, $\phi(z)$ and $\psi(z)$ are analytic in the entire plane except on the cracks. On the two faces of the crack, the tractions are identical. Recall that

$$\sigma_y+i\tau_{xy}=\phi'(z)+\overline{\phi}'(\bar z)+\bar z\phi''(z)+\psi'(z).$$

It is convenient to define

$$\Phi(z)=\phi'(z),\qquad \Omega(z)=\frac{d}{dz}\left[z\phi'(z)+\psi(z)\right].$$

Thus,

$$\sigma_y+i\tau_{xy}=\overline{\Phi}(\bar z)+\Omega(z)+(\bar z-z)\Phi'(z).$$

Equating the traction on the top and bottom faces of the cracks, we obtain that

$$\overline{\Phi}_-(x) + \Omega_+(x) = \overline{\Phi}_+(x) + \Omega_-(x),$$

for x on the crack faces. Rewrite the above as

$$\Omega_+(x) - \overline{\Phi}_+(x) = \Omega_-(x) - \overline{\Phi}_-(x).$$

Because $\Phi(z)$ and $\Omega(z)$ are analytic in the plane except on the cracks, the above continuity condition requires that the function $\Omega(z) - \overline{\Phi}(z)$ be analytic in the entire plane. Because no remote stress is applied, $\Omega(z) - \overline{\Phi}(z) = 0$ everywhere in the plane.

Let us apply a traction vector $\sigma_y + i\tau_{xy} = -T(x)$ on the crack faces. The boundary condition is written as

$$\Omega_-(x) + \Omega_+(x) = -T(x).$$

This problem has been solved before. Thus, the solution is

$$\frac{\Omega(z)}{\chi(z)} = \frac{1}{2\pi i}\int_{-a}^{a} \frac{-2T(x)dx}{\chi_+(x)(x-z)} + P(z).$$

where $P(z)$ is a polynomial, to be determined to ensure that there is no net force and dislocation from each crack. For a single crack at $|x| < a$, $P(z) = 0$. The stress intensity factor is

$$K_I + iK_{II} = \frac{1}{\sqrt{\pi a}}\int_{-a}^{a}\sqrt{\frac{a+x}{a-x}}\,T(x)dx.$$

Stress field around a tip of an interfacial crack. Consider a semi-infinite crack ($x < 0$) on an interface. The complex functions in the half plane above the x-axis are $\Phi^a(z)$ and $\Omega^a(z)$, and the complex functions below are $\Phi^b(z)$ and $\Omega^b(z)$. Traction is continuous across the entire x-axis, so that

$$\overline{\Phi}^a(x) + \Omega^a(x) = \overline{\Phi}^b(x) + \Omega^b(x).$$

Thus,

$$\Omega^a(x) - \overline{\Phi}^b(x) = \Omega^b(x) - \overline{\Phi}^a(x)$$

holds true for the entire x-axis. The left side is a function analytic in the half plane above, and the right side is a function analytic in the half plane below. Also, the stress vanishes remote from the crack. Analytic continuation requires that

$$\Omega^a(z) = \overline{\Phi}^b(z) \text{ for z above the x-axis}$$

$$\Omega^b(z) = \overline{\Phi}^a(z) \text{ for z below the x-axis}$$

Define the crack opening displacement by

$$\delta_x(x) = u_x^a(x) - u_x^b(x), \quad \delta_y(x) = u_y^a(x) - u_y^b(x)$$

A direct calculation shows that

$$-i\frac{d}{dx}\left(\delta_y + i\delta_x\right) = \frac{4}{\overline{E}}\left[(1-\beta)\Omega^a(x) - (1+\beta)\Omega^b(x)\right]$$

where

$$\frac{2}{\overline{E}} = \left(\frac{1}{\overline{E}_1} + \frac{1}{\overline{E}_2}\right)$$

The displacement is continuous across the bonded interface, so that one can define a function $f(z)$ which is analytic in the whole plane except on the crack, such that

$$\Omega^a(z) = (1-\beta)f(z), \text{ when } z \text{ is in the half plane above}$$

$$\Omega^a(z) = (1+\beta) f(z), \text{ when } z \text{ is in the half plane above}$$

One the crack faces, we assume that the traction vanishes

$$(1-\beta) f_+(x) + (1+\beta) f_-(x) = 0, \quad \text{for } -\infty < x < 0.$$

This is a homogenous equation. A solution is

$$\chi(z) = z^{-1/2+i\varepsilon},$$

where

$$\varepsilon = \frac{1}{2\pi} \log \frac{1-\beta}{1+\beta}.$$

ahead the tip of the crack, the stress field takes the form

$$\sigma_{yy} + \tau_{xy} = \frac{K r^{i\varepsilon}}{\sqrt{2\pi r}}.$$

Here K is the complex-valued stress intensity factor.

For more examples of interfacial cracks, see Z. Suo, "Singularities interacting with interfaces and cracks," *Int. J. Solids and Structures*, **25**, 1133-1142 (1989).

For physical significance of K see J.R. Rice. Elastic fracture mechanics concepts for interfacial cracks. *J. Appl. Mech.* 1988, **55**, 98-103. (http://esag.harvard.edu/rice/139_Ri_ElFracMechInterf_JAM88.pdf)

Two-dimensional problems for anisotropic materials. **Stroh formalism**. Inserting the stress-strain relations,

$$\sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l},$$

into the equilibrium equations,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0,$$

one obtains the PDEs for the displacement field:

$$C_{ijkl} \frac{\partial u_k}{\partial x_l \partial x_j} = 0.$$

Consider solutions of the form

$$u_k(x_1, x_2) = a_k f(x + py).$$

Inserting into the PDEs, one obtains that

$$\left(C_{i1k1} + p\left(C_{i1k2} + C_{i2k1}\right) + p^2 C_{i2k2}\right) a_k = 0.$$

This is an eigenvalue problem. Go over examples in Z. Suo, "Singularities, interfaces and cracks in dissimilar anisotropic media," *Proc. R. Soc. Lond*. **A427**, 331-358 (1990).