

Differential Equation to solve by Eigenvalue and Eigenvector approach

We have the equation,

$$\begin{aligned}\frac{d^2y}{dx^2} - \omega^2y &= 0, \\ y &= f(x).\end{aligned}\tag{1}$$

We can write the solution of Equation (1) as

$$y(x) = c_1e^{-\omega x} + c_2e^{\omega x}\tag{2}$$

The other 2nd order ODE we have is

$$\frac{d^2y}{dx^2} - (bx + \omega^2)y = 0\tag{3}$$

the solution of which is given by

$$y = c_1\text{AiryAi}\left(\frac{bx + \omega^2}{b^{2/3}}\right) + c_2\text{AiryBi}\left(\frac{bx + \omega^2}{b^{2/3}}\right)\tag{4}$$

Equation (3) will be same as Equation (1) when $b = 0$. But in the solution, we do not get (2) by putting $b = 0$ in (4).

The calculations what we have done are as follows.

$$\text{Let, } Y = \left[\frac{bx + \omega^2}{b^{2/3}} \right],$$

$$\lim_{b \rightarrow 0} [Y] = \infty,$$

$$\text{AiryAi}(\infty) = 0$$

$$\text{AiryBi}(\infty) = \infty$$

$$\text{AiryAi}(Y) = \frac{1}{2\sqrt{\pi}} \frac{\exp(-\frac{2}{3}Y^{3/2})}{Y^{1/4}}$$

$$\text{AiryBi}(Y) = \frac{\exp(\frac{2}{3}Y^{3/2})}{\sqrt{\pi}Y^{1/4}}$$

$$\frac{2}{3}Y^{3/2} = \frac{2}{3} \left[\frac{bx + \omega^2}{b^{2/3}} \right]^{3/2},$$

$$= \frac{2\omega^3}{3b} \left[1 + \frac{3}{2} \frac{bx}{\omega^2} \right],$$

$$= \left[\frac{2\omega^3}{3b} + \omega x \right]$$

$$[Y]^{-1/4} = \left[\frac{bx + \omega^2}{b^{2/3}} \right]^{-1/4},$$

$$= \frac{\omega^{-1/2}}{b^{-1/6}} \left[1 + \frac{bx}{\omega^2} \right]^{-1/4},$$

$$= \frac{b^{1/6}}{\sqrt{\omega}} \left[1 + \frac{bx}{\omega^2} \right]^{-1/4},$$

$$= \frac{b^{1/6}}{\sqrt{\omega}} \left[1 + \frac{-1}{4} \frac{bx}{\omega^2} + \frac{1}{2} \frac{-1}{4} \left(\frac{-1}{4} - 1 \right) \left(\frac{bx}{\omega^2} \right)^2 \dots + A_6 \left(\frac{bx}{\omega^2} \right)^6 + A_7 \left(\frac{bx}{\omega^2} \right)^7 + \dots \right]$$

(5)

So, in the limit $b \rightarrow 0$ the only existing term is

$$[Y]^{-1/4} = A_6 \frac{x^6}{\omega^{25/2}} \tag{6}$$

Then, the solution is of the form :

$$y(x) = C_1 x^6 e^{-\left[\frac{2\omega^3}{3b} + \omega x\right]} + C_2 x^6 e^{\left[\frac{2\omega^3}{3b} + \omega x\right]} \quad (7)$$

As mentioned earlier, Equation (4) and hence Equation (7) should conform to the Equation (2) when $b \rightarrow 0$. But, we cannot match them. May be, we have missed something and want help on this.

To find the gap we want to solve the Equation (1) by Eigenvalue, Eigenvector approach.

Equation (3) can be written as

$$\begin{Bmatrix} y' \\ y_1' \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ bx + \omega^2 & 0 \end{bmatrix} \begin{Bmatrix} y \\ y_1 \end{Bmatrix} \text{ where, } y_1 = y' \quad (8)$$

We know the procedure to solve the system when the matrix is of constant elements but here it contains elements function of x .

I think boundary conditions will not be required to find the solution. We actually want the general solution.