Deformation and polarization

Elastic dielectrics. All materials contain electrons and protons. These charged particles form bonds in a dielectric, and move relative to one another by short distances in response to an electric potential or a force. That is, all dielectrics can deform and polarize. The two processes, deformation and polarization, are inherently coupled.

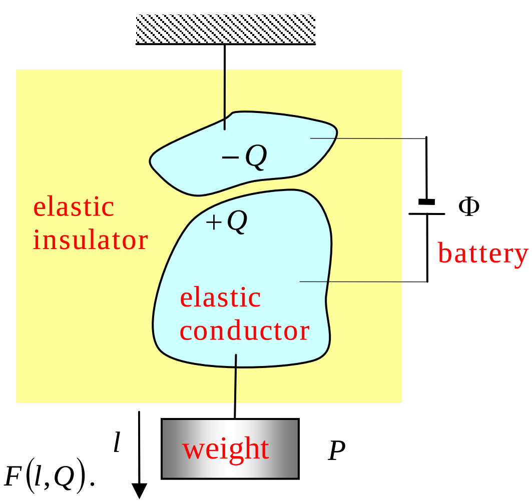
In practice, however, we may choose to include none, or one, or both of these processes by using idealized models, such as

- Rigid body
- Rigid dielectric
- Elastic solid
- Elastic dielectric

The following notes focus on elastic dielectrics. Our discussion will be motivated by dielectric elastomers, materials that have been developed intensely in recent years for applications as fast, lightweight, and large-displacement actuators. See Pelrine, R., Kornbluh, E., Pei, Q. and Joseph, J. 2000. High-speed electrically actuated elastomers with strain greater than 100%. Science 287, 836-839.

Equilibrating an elastic dielectric, a weight, and a battery. In the notes on electric potential (http://imechanica.org/node/914), we have discussed the equilibrium between a capacitor, a weight, and a battery. The discussion remains valid even if the capacitor deforms. Let us summarize the essential ideas.

The capacitor consists of two conductors separated by an insulator. Both the conductors and the insulators can deform. The capacitor is connected to a weight and a battery.



The capacitor is a system with two independent variables: the displacement l of the weight, and the electric charge Q drawn from battery. We will analyze isothermal processes, and drop temperature from explicit consideration. The thermodynamics of the capacitor is characterized by the Helmholtz free energy, F, as a function of the two independent variables:

$$F = F(l, Q).$$

Associated with small changes in the independent variables, the free energy of the capacitor changes by

$$\delta F = \frac{\partial F(l,Q)}{\partial l}\delta l + \frac{\partial F(l,Q)}{\partial Q}\delta Q .$$

The two partial derivatives can be interpreted as follows.

Let P be the force applied by the weight. Upon moving by a small displacement, δl, the weight does work $P\delta l$. Let Φ be the electric potential of the battery. Upon charging the capacitor with a small amount of electricity, δQ, the battery does work $\Phi\delta Q$. When the capacitor, the weight, and the battery equilibrate, the change in the free energy of the capacitor equals the sum of the work done by the weight and the work done by the battery, namely,

$$\delta F = P\delta l + \Phi\delta Q .$$

This equation is a representation of the condition of equilibrium.

A comparison of the above two expressions for δF gives that

$$\left[P - \frac{\partial F(l,Q)}{\partial l}\right]\delta l + \left[\Phi - \frac{\partial F(l,Q)}{\partial Q}\right]\delta Q = 0 .$$

This equation holds in the state of equilibrium for arbitrary small variations δl and δQ. Consequently, this condition of equilibrium implies two distinct equations:

$$P = \frac{\partial F(l,Q)}{\partial l}, \quad \Phi = \frac{\partial F(l,Q)}{\partial Q} .$$

These equations constitute an alternative representation of the condition of equilibrium.

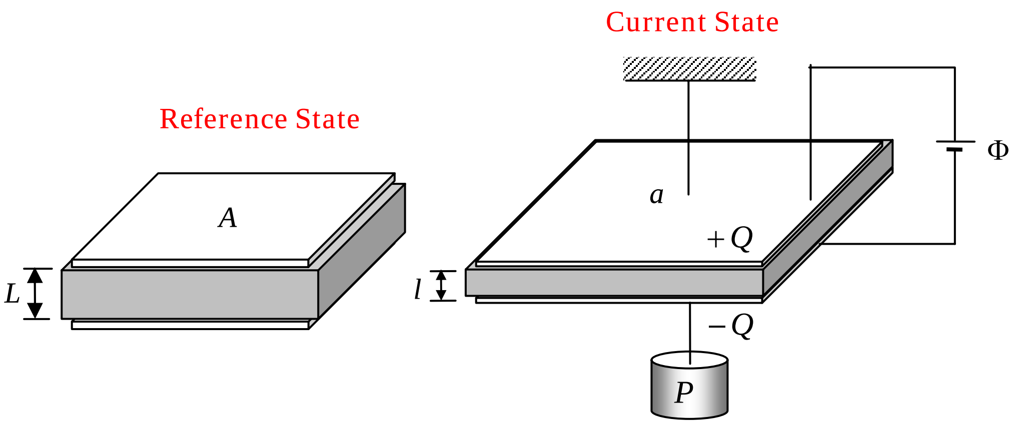
The above prescription is applicable to a single molecule, a cell, or a large system. On the basis of the prescription, these notes will develop a field theory of elastic dielectrics, following the paper by Suo, Zhao and Greene (JMPS 56, 467, 2008; http://www.seas.harvard.edu/suo/papers/188.pdf). The references in the paper will lead you to the literature. The notes assume working knowledge of finite deformation (http://imechanica.org/node/538).

A homogeneous field in a parallel-plate capacitor. The field theory is less general than the above prescription, because the field theory assumes that material can be divided into small pieces, each piece undergoing a homogeneous deformation. But the field theory will enable the finite element method to analyze diverse designs of actuators.

To exhibit the essentials of the field theory, we first analyze a parallel-plate capacitor. We assume that the capacitor is so constructed that the field in the capacitor is homogenous, an assumption that enables us to readily define quantities per unit length, per unit area, and per unit volume. Furthermore, the electrodes are so complaint that they do not constrain the deformation of the dielectric.

Take any state of the dielectric as the reference state, in which the dielectric has area A and thickness L. In the current state, the dielectric deforms to area a and thickness l. Define the stretch λ as the thickness of the dielectric in the current state divided by the thickness of the dielectric in the reference state:

$$\lambda = \frac{l}{L} .$$



Define the nominal stress s as the force supplied by the weight in the current state divided by the area in the reference state:

$$s = \frac{P}{A}.$$

Define the nominal electric field $\tilde{E}$ as the electric potential of the battery in the current state divided by the thickness of the dielectric in the reference state:

$$\tilde{E} = \frac{\Phi}{L}.$$

Define the nominal electric displacement $\tilde{D}$ by the charge Q on an electrode in the current state divided by the area in the reference state:

$$\tilde{D} = \frac{Q}{A}.$$

Define the nominal density of the Helmholtz free energy, W, by the Helmholtz free energy of the capacitor, F, divided by its volume in the reference state:

$$W = \frac{F}{AL}.$$

When the capacitor, the weight, and the battery equilibrate, the change in the free energy of the capacitor, δF, equals the sum of the work done by the weight, $P\delta l$, and the work done by the battery, $\Phi \delta l$, namely,

$$\delta F = P\delta l + \Phi \delta Q.$$

Divide this condition of equilibrium in the current state by the volume of the capacitor in the reference state, and we obtain that

$$\delta W = s\delta\lambda + \tilde{E}\delta\tilde{D}.$$

This equation is a representation of the condition of equilibrium between the dielectric, the weight, and the battery.

As a material model, the nominal density of free energy is prescribed as a function:

$$W = W\left(\lambda, \tilde{D}\right).$$

Associated with small changes in the independent variables, the nominal density of the free energy changes by

$$\delta W = \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \lambda}\delta\lambda + \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \tilde{D}}\delta\tilde{D} .$$

When the dielectric, the weight, and the battery equilibrate, we can combine the two expressions for δW , and obtain that

$$\left[s - \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \lambda} \right]\delta\lambda + \left[\tilde{E} - \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \tilde{D}} \right]\delta\tilde{D} = 0 .$$

Because λ and $\tilde{D}$ can vary independently, the above condition of equilibrium leads to two distinct equations:

$$s = \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \lambda}, \qquad \tilde{E} = \frac{\partial W\left(\lambda, \tilde{D}\right)}{\partial \tilde{D}} .$$

This pair of equation constitutes an alternative representation of the condition of equilibrium between the dielectric, the weight, and the battery. Once the function $W\left(\lambda, \tilde{D}\right)$ is prescribed as a material model of the dielectric, the above equations gives the values of the force due to the weight and the electric potential of the battery needed to equilibrate with the dielectric.

On the definition of stress. In the above, we have defined the stress as the force due to the weight in the current state divided by the area of the dielectric in the reference state, namely,

$$s = \frac{P}{A} .$$

In particular, if we do not hang the weight to the dielectric, $P = 0$, and the stress in the dielectric vanishes. This is true even when the dielectric is connected to the battery, which will cause the dielectric to deform.

Thus, when the battery applies an electric potential to the dielectric, the positive charge on one electrode and the negative charge on the other electrode cause the dielectric to thin down. We simply report what we have observed in experiments: the electric potential causes the dielectric to deform. We do not jump to the conclusion that the electric potential causes a compressive stress. Indeed, in the absence of the force due to the weight, we simply say that the stress is zero.

We view the deformation caused by the electric potential in the same way as we view the deformation caused by a change in the temperature: both are stress-free deformation when the material is unconstrained.

In case you are puzzled by this definition of stress, it might make you feel better that the same issue has also occupied great minds. I quote Richard Feynman (The Feynman Lectures on Physics, Volume II, p.10-8, 1964):

> "This is a very difficult problem which has not been solved, because it is, in a sense, indeterminate. If you put charges inside a

> dielectric solid, there are many kinds of pressures and strains. You cannot deal with virtual work without including also the mechanical energy required to compress the solid, and it is a difficult matter, generally speaking, to make a unique distinction between the electrical forces and mechanical forces due to solid material itself. Fortunately, no one ever really needs to know the answer to the question proposed. He may sometimes want to know how much strain there is going to be in a solid, and that can be worked out. But it is much more complicated than the simple result we got for liquids."

I also quote Maxwell (A Treatise on Electricity & Magnetism, 1873, Article 111):

> "I have not been able to make the next step, namely, to account by mechanical considerations for these stresses in the dielectric. I therefore leave the theory at this point..."

Nominal vs. true quantities. When a dielectric deforms, the nominal electric field and the nominal electric displacement are work-conjugate. This statement is understood as follows. The work done by the battery, $\Phi \delta Q$, is given by

$$\Phi \delta Q = \left(L\tilde{E} \right) \delta \left(A\tilde{D} \right) = AL\tilde{E}\delta\tilde{D} .$$

By contrast, the true electric field and the true electric displacement are *not* work-conjugate. This statement can be seen readily as follows. Define the true electric field by the voltage supplied by the battery in the current state divided by the thickness in the current state:

$$E = \frac{\Phi}{l} ,$$

Define the true electric displacement D by the charge Q in the current state divided by the area in the current state:

$$D = \frac{Q}{a} .$$

In terms of the true electric field and the true electric displacement, the work done by the battery is

$$\Phi \delta Q = \left(lE \right) \delta \left(aD \right) = lED\delta a + laE\delta D .$$

For a solid dielectric, $\delta a \neq 0$, so that the true electric displacement is *not* work-conjugate to the true electric field.

Homogeneous deformation in three dimensions. Consider a block of a dielectric stretched in three directions, λ_1, λ_2, and λ_3. The block is subject to an electric field in direction 3. The nominal density of the Helmholtz free energy is a function of the four variables:

$$W = W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)$$

In equilibrium, the change in the free energy equals the work done by the applied forces and by the battery:

$$\delta W = s_1 \delta\lambda_1 + s_2 \delta\lambda_2 + s_3 \delta\lambda_3 + \tilde{E}\delta\tilde{D}.$$

The nominal stresses and the nominal electric field relate to the partial derivatives:

$$s_1 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_1},$$

$$s_2 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_2},$$

$$s_3 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_3},$$

$$\tilde{E} = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \tilde{D}}.$$

Maxwell stresses in a vacuum. We can view a vacuum as a dielectric with vanishing elasticity. Consequently, only the electric field contributes to the energy. In the presence of electric field E, the energy per unit volume is

$$\frac{1}{2}\varepsilon_0 E^2,$$

where $\varepsilon_0 = 8.85\times 10^{-12} CV^{-1}m^{-1}$ is the permittivity of the vacuum. Using in coordinates, we assume that this electric field coincides with direction 3.

To use the above formulas, we think of the vacuum as a vanishingly compliant elastic dielectric, undergoing a homogenous deformation λ_1, λ_2, and λ_3. The above formula is the energy divided by the volume in the current state. Consequently, the nominal density of free energy is

$$W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right) = \frac{1}{2}\varepsilon_0 E^2 \lambda_1 \lambda_2 \lambda_3.$$

This equation is not expressed in terms of proper variables yet. Recall that

$$D = \varepsilon_0 E,$$

and

$$D = \frac{\tilde{D}}{\lambda_1 \lambda_2}.$$

We obtain that

$$W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right) = \frac{\tilde{D}^2 \lambda_3}{2\varepsilon_0 \lambda_1 \lambda_2}.$$

Taking partial derivatives, we obtain that

$$s_1 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_1} = -\frac{\tilde{D}^2 \lambda_3}{2\varepsilon_0 \lambda_1^2 \lambda_2},$$
$$s_2 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_2} = -\frac{\tilde{D}^2 \lambda_3}{2\varepsilon_0 \lambda_2^2 \lambda_1},$$
$$s_3 = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \lambda_3} = \frac{\tilde{D}^2}{2\varepsilon_0 \lambda_1 \lambda_2},$$
$$\tilde{E} = \frac{\partial W\left(\lambda_1, \lambda_2, \lambda_3, \tilde{D}\right)}{\partial \tilde{D}} = \frac{\tilde{D} \lambda_3}{\varepsilon_0 \lambda_1 \lambda_2}.$$

These relations can be expressed in terms of the true quantities as

$$\sigma_1 = \sigma_2 = -\frac{1}{2}\varepsilon_0 E^2, \quad \sigma_3 = \frac{1}{2}\varepsilon_0 E^2, \quad D = \varepsilon_0 E .$$

These equations recover the stresses obtained by Maxwell.

Incompressible elastic dielectric. For a material capable of large deformation, the change in the shape of the material is typically much larger than the change in the volume of the material. Consequently, the material is often idealized as being incompressible, namely,

$$\lambda_1 \lambda_2 \lambda_3 = 1 .$$

For an incompressible elastic dielectric, the nominal density of the free energy is a function of three independent variables:

$$W = W\left(\lambda_1, \lambda_2, \tilde{D}\right).$$

Associated with small changes in the independent variables, the function varies by

$$\delta W = \frac{\partial W\left(\lambda_1, \lambda_2, \tilde{D}\right)}{\partial \lambda_1}\delta\lambda_1 + \frac{\partial W\left(\lambda_1, \lambda_2, \tilde{D}\right)}{\partial \lambda_2}\delta\lambda_2 + \frac{\partial W\left(\lambda_1, \lambda_2, \tilde{D}\right)}{\partial \tilde{D}}\delta\tilde{D} .$$

When the dielectric equilibrates to the weights and the battery, the change in the free energy equals to the sum of the work done by the weights and the battery, namely,

$$\delta W = s_1 \delta\lambda_1 + s_2 \delta\lambda_2 + s_3 \delta\lambda_3 + \tilde{E}\delta\tilde{D} .$$

Recall the condition of incompressibility, $\lambda_1 \lambda_2 \lambda_3 = 1$, and the condition of equilibrium is written in terms of independent variations:

$$\delta W = \left(s_1 - s_3 \frac{1}{\lambda_1^2 \lambda_2}\right)\delta\lambda_1 + \left(s_2 - s_3 \frac{1}{\lambda_2^2 \lambda_1}\right)\delta\lambda_2 + \tilde{E}\delta\tilde{D} .$$

Thus

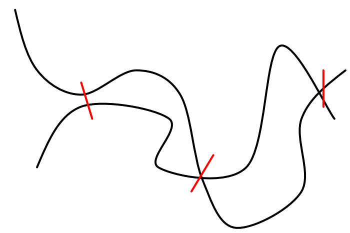
$$s_1 - s_3 \frac{1}{\lambda_1^2 \lambda_2} = \frac{\partial W\left(\lambda_1, \lambda_2, \tilde{D}\right)}{\partial \lambda_1},$$

$$s_2 - s_3 \frac{1}{\lambda_2^2 \lambda_1} = \frac{\partial W(\lambda_1, \lambda_2, \tilde{D})}{\partial \lambda_2},$$

$$\tilde{E} = \frac{\partial W(\lambda_1, \lambda_2, \tilde{D})}{\partial \tilde{D}}.$$

These three equations constitute an alternative representation of the condition of equilibrium.

Ideal dielectric elastomers. An elastomer is a crosslinked network of long and flexible polymers. When the degree of crosslink is low and the deformation is well below the extension limit, the molecular units in the polymers can polarize as freely as in a liquid polymer. Under these conditions, the permittivity is nearly unaffected by deformation. The material is also taken to be incompressible, $\lambda_1 \lambda_2 \lambda_3 = 1$. Consequently, the nominal density of free energy function takes the form



$$W(\lambda_1, \lambda_2, \tilde{D}) = W_s(\lambda_1, \lambda_2) + \frac{1}{2}\varepsilon E^2.$$

Here E is the true electric field, ε is the permittivity, and $W_s(\lambda_1, \lambda_2)$ is the free energy associated with the stretching of the crosslinked network. This material model is called the ideal dielectric elastomer (Hong, Zhao, and Suo, PRB 76, 134113, 2007; http://www.seas.harvard.edu/suo/papers/196.pdf).

We need to convert the above expression in terms of the nominal electric displacement. The true electric field relates to the true electric displacement by $E = D/\varepsilon$. The true electric displacement is defined by $D = Q/a$, while the nominal electric displacement is defined by $\tilde{D} = Q/A$. They are related as

$$D = \frac{\tilde{D}}{\lambda_1 \lambda_2}.$$

Consequently, in terms the nominal electric displacement, the free-energy function is

$$W(\lambda_1, \lambda_2, \tilde{D}) = W_s(\lambda_1, \lambda_2) + \frac{1}{2\varepsilon}\left(\frac{\tilde{D}}{\lambda_1 \lambda_2}\right)^2.$$

Recall the relations concerning the partial derivatives:

$$s_1 - s_3 \frac{1}{\lambda_1^2 \lambda_2} = \frac{\partial W(\lambda_1, \lambda_2, \tilde{D})}{\partial \lambda_1},$$

$$s_2 - s_3 \frac{1}{\lambda_2^2 \lambda_1} = \frac{\partial W(\lambda_1, \lambda_2, \tilde{D})}{\partial \lambda_2},$$

$$\tilde{E} = \frac{\partial W(\lambda_1, \lambda_2, \tilde{D})}{\partial \tilde{D}}.$$

These relations can be expressed in terms of the true quantities:

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W_s(\lambda_1, \lambda_2)}{\partial \lambda_1} - \varepsilon E^2,$$

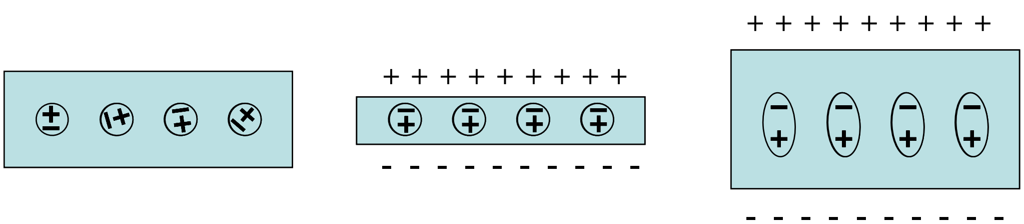
$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W_s(\lambda_1, \lambda_2)}{\partial \lambda_2} - \varepsilon E^2,$$

$$D = \varepsilon E.$$

In many applications of dielectric elastomers, the dielectrics take the form of thin membranes, with no stress applied through the thickness, $\sigma_3 = 0$. The above expression shows that a through-thickness electric field adds a compressive stress of magnitude εE^2 in the two in-plane direction. This magnitude is twice the Maxwell stress.

This set of equations has been used in much of the literature on dielectric elastomers. The equations are usually justified in terms of the Maxwell stress. Now we have interpreted these equations using the model of ideal dielectric elastomers.

Electrostriction. A dielectric deforms when subject to a voltage. The voltage may cause some dielectrics to become thinner, but other dielectrics to become thicker. The Maxwell stress is unable to account for dielectrics that thicken under a voltage. Indeed, as evident above, the Maxwell stress can account for voltage-induced deformation only for a very special type of materials, the ideal dielectric elastomers, where the permittivity is deformation-independent.



(a) (b) (c)

In an ideal dielectric elastomer, the dielectric behavior is taken to be liquid-like, unaffected by deformation. This idealization is expected to fail when the degree of crosslink is high, or when the deformation approaches the extension limit. In such a case, deformation will affect dielectric behavior. Models of full coupling between deformation and polarization are not well developed. As an example, consider a model known as the quasi-linear dielectrics (Zhao and Suo, JAP 104, 123530, 2008). In this model, the permittivity is taken to be a function of the stretches:

$$\varepsilon = \varepsilon(\lambda_1, \lambda_2).$$

The free energy function is written as

$$W(\lambda_1, \lambda_2, \tilde{D}) = W_s(\lambda_1, \lambda_2) + \frac{\varepsilon(\lambda_1, \lambda_2)}{2} E^2.$$

The same procedure as described above leads to

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W_s}{\partial \lambda_1} - \varepsilon E^2 - \frac{1}{2}\frac{\partial \varepsilon}{\partial \lambda_1}\lambda_1 E^2 ,$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W_s}{\partial \lambda_2} - \varepsilon E^2 - \frac{1}{2}\frac{\partial \varepsilon}{\partial \lambda_2}\lambda_2 E^2 .$$

Notice the additional terms due to the function $\varepsilon(\lambda_1, \lambda_2)$.

Exercise. Derive the above equations.

Pull-in instability. When a battery applies an electric potential to a membrane of a dielectric elastomer, the membrane reduces its thickness. Consequently, the same electric potential induces a high electric field. This positive feedback may cause the membrane to thin down drastically, resulting in electrical breakdown. We now model this phenomenon.

In the reference state, the electric potential is not applied, and the membrane has thickness L and area A. When the electric potential Φ is applied, the membrane reduces its thickness to l and increases its area to a. The stretch of the membrane in the direction of the thickness is

$$\lambda = l / L .$$

We assume that the material is an incompressible. Consequently, in the current direction, the stretches in the two directions in the plane of the membrane are both equal to $\lambda^{-1/2}$, and the area of the membrane is

$$a = A / \lambda .$$

In the current state, the true electric field is

$$E = \Phi / l .$$

We further assume that the membrane is an ideal dielectric elastomer, so that its dielectric behavior is liquid like, unaffected by deformation. Let ε be the permittivity, which is taken to be independent of the stretch. In the current state, the true electric displacement is

$$D = \varepsilon E ,$$

The electric charge on either electrode is

$$Q = aD .$$

The membrane and the battery constitute a composite. The free energy of the composite, $\hat{F}$, consists of three parts: the elastic energy of the membrane, the electric energy of the membrane, and the electric energy of the battery:

$$\hat{F} = \frac{\mu}{2}\left(\lambda^2 + 2\lambda^{-1} - 3\right)AL + \frac{\varepsilon}{2}E^2 AL - Q\Phi .$$

Write this free energy as a function of the stretch and the electric potential:

$$\hat{F}(\lambda, \Phi) = \frac{\mu}{2}\left(\lambda^2 + 2\lambda^{-1} - 3\right)AL - \frac{\varepsilon}{2\lambda^2}\left(\frac{\Phi}{L}\right)^2 AL .$$

When the electric potential is prescribed by the battery, the stretch λ reaches the equilibrium value when the free energy is minimal.

Setting

$$\frac{\partial \hat{F}\left(\lambda, \tilde{D}\right)}{\partial \lambda} = 0 ,$$

we obtain the condition of equilibrium

$$\left(\frac{\Phi}{L}\right)^2 = \frac{\mu}{\varepsilon}\left(\lambda - \lambda^4\right).$$

This equation determines the stretch λ when the membrane equilibrates with the battery.

The above function $\Phi(\lambda)$ reaches a peak at

$$\lambda_c = 4^{-1/3} \approx 0.63 .$$

This peak corresponds to the onset of instability. The critical electric potential is

$$\frac{\Phi_c}{L} \sim \sqrt{\frac{\mu}{\varepsilon}} \sim \sqrt{\frac{10^6\ N/m}{10^{-10}\ F/m}} = 10^8 V/m .$$

For further discussion of the pull-in instability, see the paper by Zhao and Suo, APL 91, 061921 (2007).

Exercise. Calculate the critical electric potential when the membrane is subject to a biaxial force in the plane of the membrane.

Inhomogeneous field. A theory along the same line has also been worked out for inhomogeneous fields; for a review of literature, see the paper by Suo, Zhao, Greene, JMPS 56, 467 (2008). Here we summarize basic equations.

The stretch is generalized to the deformation gradient, $\mathbf{F}$, and the electric displacement is generalized to a vector, $\tilde{\mathbf{D}}$. The thermodynamics of a dielectric elastomer is characterized by prescribing the nominal density of the Helmholtz free energy as a function of the deformation gradient and the electric displacement:

$$W = W\left(\mathbf{F}, \tilde{\mathbf{D}}\right).$$

When the body undergoes a rigid body motion, the free energy is invariant. Consequently, the function depends on the deformation gradient through the Green deformation tensor:

$$C_{KL} = F_{iK} F_{iL} .$$

The condition for local equilibrium relates the tensor of nominal stress and the vector of nominal electric field to the partial derivatives.

$$s_{iK} = \frac{\partial W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)}{\partial F_{iK}}, \quad \tilde{E}_K = \frac{\partial W\left(\mathbf{F}, \tilde{\mathbf{D}}\right)}{\partial \tilde{D}_K} .$$

We describe a body by a field of material particles, and name each particle by its coordinate, $\mathbf{X}$, when the body in a reference state. In the current state at time t, the particle $\mathbf{X}$ moves to a place with coordinate $\mathbf{x}$. The function

$$\mathbf{x} = \mathbf{x}\left(\mathbf{X}, t\right)$$

describes the history of deformation of the body. The deformation gradient is defined as

$$F_{iK} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K}.$$

On each material element of volume, we prescribe mass $\rho(\mathbf{X})dV$, electric charge $q(\mathbf{X},t)dV$ and mechanical force $\mathbf{B}(\mathbf{X},t)dV$. The conservation of linear momentum requires that

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = \rho(\mathbf{X})\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2}$$

in the volume and

$$\left(s_{iK}^- - s_{iK}^+\right)N_K = T_i$$

on an interface.

In the current state at time t, the electric potential at particle $\mathbf{X}$ is denoted as $\Phi(\mathbf{X},t)$. Define the nominal electric field as

$$\tilde{E}_K = -\frac{\partial \Phi(\mathbf{X},t)}{\partial X_K}.$$

Electric charge is prescribed as $q(\mathbf{X},t)dV$ on each material element of volume, and as $\omega(\mathbf{X},t)dA$ on each material element of interface. Gauss's law requires that

$$\frac{\partial \tilde{D}_K(\mathbf{X},t)}{\partial X_K} = q(\mathbf{X},t)$$

in the volume of the body, and

$$\left(\tilde{D}_K^+ - \tilde{D}_K^-\right)N_K = \omega(\mathbf{X},t)$$

on the interfaces.

Finite element method. Several groups have implemented finite element methods. See

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