

Short Introduction to Finite Elements in One Dimension

N. Sukumar

Department of Civil and Environmental Engineering

University of California

Davis, CA 95616

September 9, 2007

1 Introduction

The finite element method (FEM) is a function/basis-based approach to solve partial differential equations (PDEs). Finite elements are widely used in diverse fields to solve static and dynamic problems—solid mechanics, fluid mechanics, electromagnetics, biomechanics, etc. There are many textbooks on FEM, among which, the books of Zienkiewicz and Taylor, Bathe, Hughes, and Reddy provide a sound introduction to the subject. In contrast to the finite-difference method, some of the distinguishing features of the FEM are as follows:

- use of the *weak* or *variational* form of the governing equations so as to reduce the continuity requirements on admissible functions. This allows lower-order C^0 approximations to be used for second-order partial differential equations;
- ability to model complex geometries (mesh generation tools are used to discretize the domain) and to treat essential (Dirichlet), natural (Neumann), or mixed (Robin) boundary conditions are noteworthy;
- spatially varying coefficients that arise in the PDE (for example, tensors such as the conductivity, diffusivity, or material moduli) can be readily accommodated in the formulation;
- one-sided convergence since the variational form ensures that convergence is from above (for example, convergence of eigenvalues in eigenproblems); and
- adaptivity and error estimators are available to ensure systematic improvements in accuracy.

For nonlinear PDEs in bounded domains ($\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$), where material nonlinearities (body force or heat source are functions of position \mathbf{x}) or when geometric nonlinearities (large stretch and rotations) are present, finite elements are a very powerful, robust, and reliable simulation tool.

In these notes, I present the main ingredients of the development of finite element basis functions and their use to solve linear boundary-value problems. For clarity, I only present the derivation of the finite element equations for a model Dirichlet one-dimensional (Poisson) problem. The notion of basis functions, strong and weak form, essential and natural boundary conditions, development of the discrete linear system of equations, element-based assembly, etc., are touched upon. The basis principles extend to higher-dimensions, but in \mathbf{R}^2 or \mathbf{R}^3 , integral theorems need to be invoked and the book-keeping (vectors, matrices) gets to be a bit more involved. Just the most essential ingredients are mentioned; the exposition is by no means complete, so please ask me questions so that I can further elaborate on some of the concepts.

2 One-Dimensional Poisson Equation

2.1 Strong form

Consider the following one-dimensional linear boundary-value problem:

$$-(a(x)u'(x))' = b(x) \quad \text{in } \Omega = (0, 1), \quad (1a)$$

$$u(0) = 1, \quad u(1) = 2, \quad (1b)$$

where $(\cdot)' \equiv d(\cdot)/dx$, u is the dependent variable (e.g., temperature in steady-state heat conduction, concentration in the diffusion of a specie, or the displacement in one-dimensional elasticity), a is the conductivity coefficient, and $b(x)$ is a source term. The boundary conditions are both essential (Dirichlet) since u is specified at the end points; if u' is prescribed at one of the end-points it is known as a natural (Neumann) boundary condition. Note that Ω does not include the end-points; $\partial\Omega$ is used to denote the boundary (here $\partial\Omega = \{0, 1\}$). Equation (1) is referred to as the strong form of the boundary-value problem.

2.2 Weak form

In the finite element method, the weak form of the boundary-value problem is used. To this end, we define $r(x) = -(a(x)u'(x))' - b(x)$ to be the residual. If $u(x)$ is the exact solution, then $r(x) \equiv 0$, but for any other $u(x)$ (e.g., a numerical approximation), $r(x)$ will in general not be zero at all points $x \in \Omega$. To develop a suitable numerical scheme, we multiply $r(x)$ by an arbitrary (for now) *test or weighting function* $w(x)$ and integrate over the domain. Now, $r(x) = 0$ is met only in a weighted-integral (average) sense and not point-wise:

$$\int_0^1 r(x)w(x) dx = 0 \quad \Rightarrow \quad \int_0^1 [-(a(x)u'(x))' - b(x)]w(x) dx = 0 \quad \forall w, \quad (2)$$

where w is at least a piece-wise continuous function ($w \in C^0(\Omega)$ will suffice). A function u lies in $C^k(\Omega)$ if all its derivatives up to order k are continuous. On integrating the first term by parts, we obtain

$$[-a(x)u'(x)w(x)]_0^1 + \int_0^1 a(x)u'(x)w'(x) dx - \int_0^1 b(x)w(x) dx = 0 \quad \forall w. \quad (3)$$

Now, we require that $w(0) = 0$ and $w(1) = 0$, i.e., the test function must vanish where essential boundary conditions are prescribed. Hence, the first term (boundary) vanishes. The test functions are analogous to a first variation in u ($w \sim \delta u$) and hence if u is specified at a point its variation at that point must be zero. On keeping terms with integrands that involve u and w on the left-hand side and moving the term with the integrand that has just w to the right-hand side, we obtain

$$\int_0^1 a(x)u'(x)w'(x) dx = \int_0^1 b(x)w(x) dx \quad \forall w, \quad (4)$$

which is the weak form. Formally, the weak form is written as: find the trial function $u(x) \in \mathcal{S}$ such that

$$\int_0^1 a(x)u'(x)w'(x) dx = \int_0^1 b(x)w(x) dx \quad \forall w \in \mathcal{V}, \quad (5a)$$

where the trial and test function spaces are

$$\mathcal{S} = \{u : u \in H^1(\Omega), u(0) = 1, u(1) = 2\}, \quad (5b)$$

$$\mathcal{V} = \{w : w \in H^1(\Omega), w(0) = 0, w(1) = 0\}, \quad (5c)$$

where $H^k(\Omega)$ is the Sobolev space that contains functions that are square-integrable (L^2) upto order k . It is a bit less restrictive than $C^0(\Omega)$ in higher-dimensions, but if it might ease the understanding you can place $C^0(\Omega)$ in the above definition of the spaces. So, all admissible trial functions must lie in \mathcal{S} and all admissible test/weighting functions (variations) must lie in \mathcal{V} . The weak form given in Eq. (5) is equivalent to the strong form in Eq. (1). The important observation is that in the weak form, only first derivative terms in u and w appear and hence as long as these integrals are well-defined the weak form is computable. Hence, by reducing the order of the derivatives (one each on u and w), we can now choose functions u and w that are just continuous but not continuously differentiable (u, w need not lie in $C^1(\Omega)$). This gives us more freedom (there are many more C^0 functions than C^1 functions) to pick our trial and test approximations. Note that for Eq. (1) to make sense in the classical sense, $u \in C^2(\Omega)$ (u must be continuously twice-differentiable). The functions a and b can now be even discontinuous (C^{-1}). Furthermore, now PDEs that admit only solutions in the weak-sense can be solved— b can be the δ -function (point heat sources or point loads) since for a unit point heat source located at $x = x_0$ ($x_0 \in (0, 1)$), the corresponding term on the right-hand side of the weak form is $\int_0^1 \delta(x - x_0)w(x) dx = w(x_0)$, which is finite.

For some class of problems (especially linear), a variational form, which is identical to the weak form is also easily derived. The variational form rests on the existence of a potential energy functional $\Pi[u]$, such that $\delta\Pi[u] = 0$ leads to the variational form. If such a $\Pi[u]$ exists, then the variational and weak form are identical. The weak form approach is more general since it is applicable to any PDE. For the Poisson problem posed in Eq. (1), we can write $\Pi[u]$ as

$$\Pi[u] = \frac{1}{2} \int_0^1 a(x)(u'(x))^2 - \int_0^1 b(x)u(x) dx, \quad (6)$$

and the variational form is derived by setting

$$\delta\Pi[u] = 0, \quad u(x) \in \mathcal{S}, \quad (7)$$

and hence we obtain

$$\int_0^1 a(x)u'(x)\delta u'(x) dx = \int_0^1 b(x)\delta u(x) dx \quad \forall \delta u \in \mathcal{V}, \quad (8)$$

which is identical to the weak form given in Eq. (5) if one lets $\delta u \equiv w$.

One-dimensional linear elasticity: We present the strong form and the weak or variational form for one-dimensional elasticity. Consider a bar that is fixed at the left end ($x = 0$) and is subjected to a traction σ_0 at the right end ($x = L$). The bar has a non-uniform cross-sectional area $A(x)$, and the Young's modulus $E(x)$ of the bar is assumed to be a function of position. The body force per unit length acting along the bar is $b(x)$. The strong form is:

$$(\sigma(x)A(x))' + b(x) = 0 \quad \text{in } (0, L) \quad \text{(Static equilibrium)} \quad (9a)$$

$$\sigma = E(x)\varepsilon \quad \text{(Constitutive law)} \quad (9b)$$

$$\varepsilon = u' \quad \text{(Strain-displacement relation)} \quad (9c)$$

$$u(0) = 0 \quad \text{(Essential boundary condition)} \quad (9d)$$

$$u'(L) = \frac{\sigma_0}{E(L)} \quad \text{(Natural boundary condition).} \quad (9e)$$

For the one-dimensional elasticity problem, the potential energy functional $\Pi[u]$ is:

$$\Pi[u] = \underbrace{\frac{1}{2} \int_0^L E(x)A(x)(u'(x))^2 dx}_{W^{\text{int}}} - \underbrace{\left[\int_0^L b(x)u(x) dx + \sigma_0 A(L)u(L) \right]}_{W^{\text{ext}}}, \quad (10)$$

and the weak/variational form is: find the trial function $u(x) \in \mathcal{S}$ such that

$$\int_0^L E(x)A(x)u'(x)\delta u'(x) dx = \int_0^L b(x)\delta u(x) dx + \sigma_0 A(L)\delta u(L) \quad \forall \delta u \in \mathcal{V}, \quad (11a)$$

where the trial and test function spaces (identical for this problem) are

$$\mathcal{S} = \{u : u \in H^1(\Omega), u(0) = 0\}, \quad (11b)$$

$$\mathcal{V} = \{\delta u : \delta u \in H^1(\Omega), \delta u(0) = 0\}. \quad (11c)$$

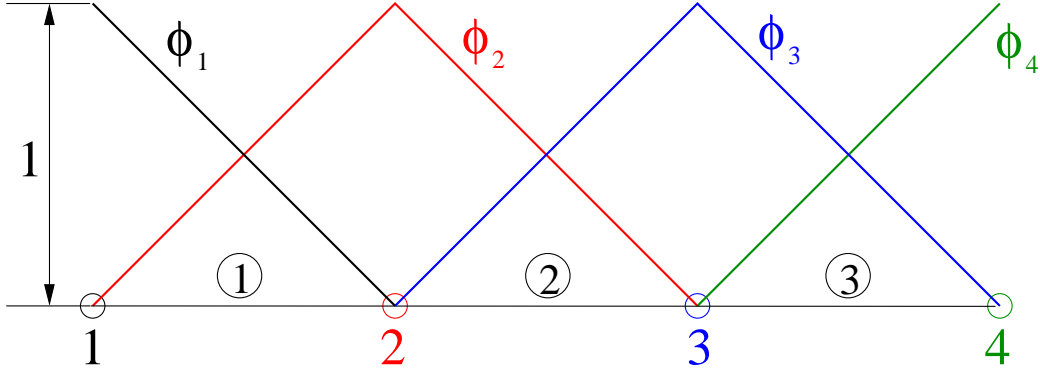


Figure 1: One-dimensional finite element basis functions.

2.3 Finite element approximation

Unit vectors are fundamental ‘building blocks’ to mathematically represent any vector in 3-space. Along similar lines, basis functions that are independent (Fourier bases are well-known in infinite-dimensional space) can be used to approximate a function in a finite-dimensional space. In the finite element method, a domain Ω is partitioned into m non-overlapping subdomains (elements) such that $\Omega = \cup_{e=1}^m \Omega_e$, where Ω_e is a line segment in 1D. The vertex of every element is said to be a node, and the coordinate x_j of node j is referred to as the nodal coordinate. So, if the domain is described by n nodes, we approximate $u(x)$ by $u^h(x)$ (h is a measure of the nodal-spacing) with

$$u^h(x) = \sum_{j=1}^n \phi_j(x) u_j, \quad (12)$$

where $\phi_j(x)$ are known as basis functions and u_j are nodal values (value of the temperature/potential at x_j). Piece-wise linear basis functions on a finite element mesh are shown in Fig. 1. The basis function for each node is non-zero only over the elements it is attached to. This provides locality (basis functions are said to have compact support), which leads to a sparse stiffness (coefficient) matrix. Hence, for any point x , at most two basis functions contribute in the summation given in Eq. (12). This leads to the standard FE formulation of developing basis functions that are restricted to elements (known as shape functions), and as a result *element-based assembly* is carried out to form the stiffness matrix and external force vector (linear system of equations is $\mathbf{Kd} = \mathbf{f}$).

2.4 Discrete weak form and linear system of equations

From hereon, in the interest of clarity and conciseness, I’ll suppress the dependence on x for the functions that appear in the weak form. On substituting u^h instead of u in Eq. (5), we obtain the discrete weak form:

$$\int_0^1 a(u^h)'(w^h)' dx = \int_0^1 b w^h dx \quad \forall w^h \in \mathcal{V}^h \subset \mathcal{V}, \quad (13)$$

where \mathcal{V}^h is the finite-dimensional finite element space. The weak form in Eq. (5) is in infinite-dimensional space since u and w can assume any function that meet the requirements of \mathcal{S} and \mathcal{V} , respectively; however, on using the finite element approximation we restrict our attention to functions that are of the form given in Eq. (12). On substituting u^h from Eq. (12) in the above equation, we obtain

$$\int_0^1 \sum_{j=1}^n (w^h)' a \phi_j' u_j dx = \int_0^1 b w^h dx \quad \forall w^h \in \mathcal{V}^h \subset \mathcal{V}, \quad (14)$$

and since we have n unknowns, we need n independent equations to obtain a unique solution for the u_j 's. By virtue of the Dirichlet boundary conditions and the properties of ϕ_j (interpolatory $\Rightarrow \phi_j(x_i) = \delta_{ji}$), we already know that $u_1 = 1$ and $u_n = 2$. Hence, an obvious choice is to pick $w^h = \phi_i$ ($i = 2, \dots, n-1$). Note that this choice is admissible (test functions must vanish on the essential boundary) since the ϕ_i ($i = 2, \dots, n-1$) are zero at $x = 0$ and $x = 1$. So, the discrete weak form is:

$$\int_0^1 \left[\sum_{j=1}^n a \phi_i' \phi_j' dx \right] u_j = \int_0^1 b \phi_i dx \quad (i = 2, 3, \dots, n-1), \quad (15)$$

which leads to a linear system of equations:

$$\mathbf{K} \mathbf{d} = \mathbf{f}, \quad \mathbf{K}_{ij} = \int_0^1 a \phi_i' \phi_j' dx, \quad \mathbf{f}_i = \int_0^1 b \phi_i dx, \quad \mathbf{d} = \{u_1, u_2, \dots, u_n\}^T. \quad (16)$$

where for convenience the first ($i = 1$) and the last row ($i = n$) are also formed. The essential boundary conditions ($u_1 = 1$, $u_n = 2$) are imposed by modifying \mathbf{K} and \mathbf{f} , and then the linear system of equations is solved.

In a FE implementation, an element-centric viewpoint is adopted. The global stiffness matrix and force vector are assembled from element-level constructions (element stiffness matrix and force vectors; use of *stiffness* and *force* has historical roots since the FE method was initially developed for structural/solid mechanics applications). Some of the main steps in a FE implementation are as follows:

1. Read mesh, nodal coordinates, and form element connectivity matrix for each element. The element connectivity matrix indicates the nodes that are connected to an element; for linear elements in 1D this is a $m \times 2$ matrix.
2. The shape functions N_I ($I = 1, 2$), which are the local restriction of the basis functions to an element, are constructed in a reference element $\xi \in \Omega_0 = [-1, 1]$: $N_1(\xi) = (1 - \xi)/2$ and $N_2(\xi) = (1 + \xi)/2$ (Fig. 2).

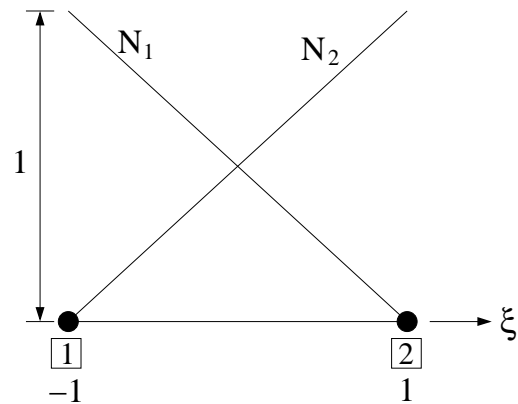


Figure 2: Linear finite element shape functions.

3. An isoparametric map $x \equiv x(\xi) = \sum_{I=1}^2 N_I(\xi)x_I$ is used to map points from ξ to x or vice-versa (map is linear).
4. In a finite element code, \mathbf{K} and \mathbf{f} are assembled on an element-basis: $\mathbf{K} = \sum_e \mathbf{k}^e$ and $\mathbf{f} = \sum_e \mathbf{f}^e$, where \sum denotes an assembly operation and not a direct sum. Expressions for \mathbf{k}^e and \mathbf{f}^e are:

$$\mathbf{k}_{IJ}^e = \int_{\Omega_e} a N_I' N_J' dx, \quad \mathbf{f}_I^e = \int_{\Omega_e} b N_I dx. \quad (17)$$

5. All integrals over Ω_e (x -space) are transformed to Ω_0 using the isoparametric transformation and the chain-rule is used to compute the derivatives of N_I . In addition, $dx = J d\xi$ where $J = dx/d\xi = L_e/2$ for a linear element (L_e is the length of the element). In 2D and 3D, the inverse of the Jacobian matrix $\mathbf{J} = d\mathbf{x}/d\xi$ will appear when derivatives are transformed and $d\Omega = (\det \mathbf{J})d\Omega_0$. For example in one dimension,

$$\int_{\Omega_e} b N_I dx = \int_{-1}^1 b N_I J d\xi = \frac{1}{2} \int_{-1}^1 b N_I L_e d\xi. \quad (18)$$

6. Numerical integration (Gauss quadrature) is used to evaluate the integrals on Ω_0 . Such rules approximate integrals over the ξ -domain:

$$\int_{-1}^1 g(\xi) d\xi \approx \sum_{k=1}^{nsp} w_k g(\xi_k), \quad (19)$$

where nsp is the number of Gauss points, w_k are Gauss weights associated with the Gauss points ξ_k . The Gauss points and weights are tabulated in most FE books. For example, $\xi_1 = 0$, $w_1 = 2$ (1-point quadrature) and $\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$, $w_1 = w_2 = 1$ (2-point quadrature).

7. After assembling \mathbf{K} and \mathbf{f} , the matrix \mathbf{K} and the vector \mathbf{f} are modified to incorporate the essential boundary conditions.
8. The linear system $\mathbf{Kd} = \mathbf{f}$ is solved to obtain \mathbf{d} .
9. On obtaining $\mathbf{d} = \{u_1, u_2, \dots, u_n\}^T$, the FE solution is computed using Eq. (12).

3 Finite Elements in Two-Dimensions

We describe finite element shapes and nodal basis functions in 2D, so that the extension to 3D is seamless. In two-dimensional finite element computations, the domain Ω is partitioned into m non-overlapping subdomains (elements), $\Omega = \cup_{e=1}^M \mathcal{T}_e$, where \mathcal{T}_e is either a triangle or a quadrilateral (isoparametric transformation of the elements

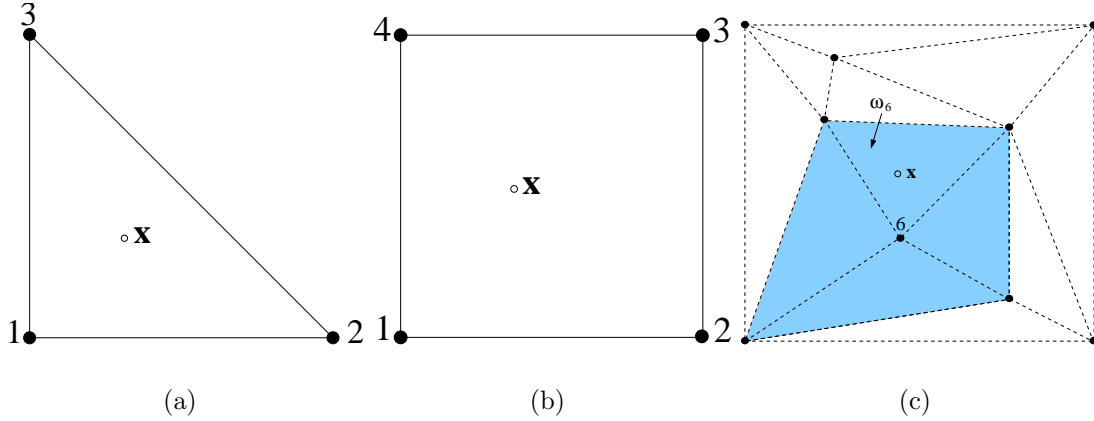


Figure 3: (a) Triangle; (b) Square; and (c) Nodal basis function support (ω_6) on a Delaunay mesh.

shown in Figs. 3a and 3b). As indicated earlier, basis functions are associated with each vertex (node) of \mathcal{T}_e , and within an element, the local restriction of a basis function is known as a shape function. In Fig. 3c, the nodal basis function support ω_6 is shown by the shaded area; the basis function for node a is positive in ω_a and zero on $\partial\omega_a$. For a finite element mesh in 2D, contour and 3D plots of the nodal basis (hat) function for node A are depicted in Fig. 4.

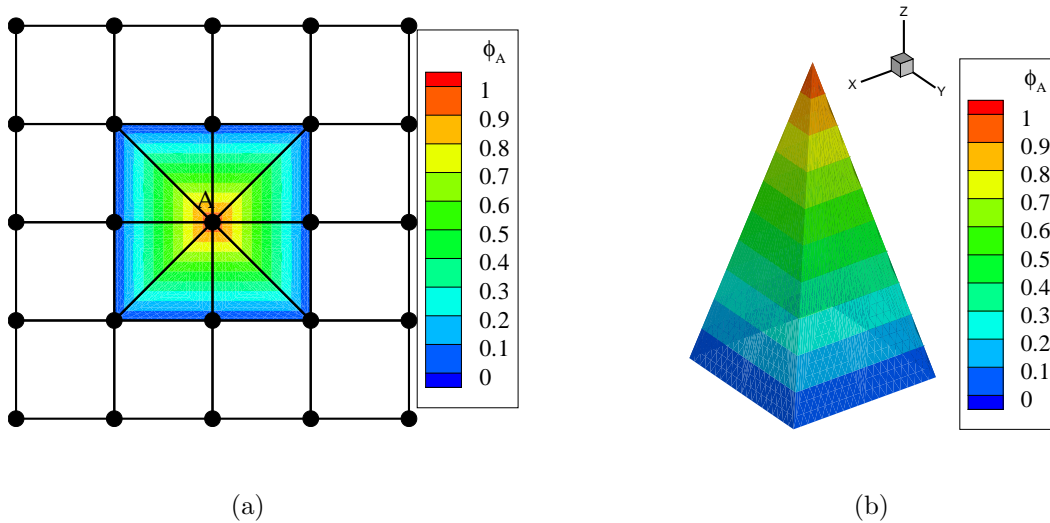


Figure 4: Two-dimensional finite element basis (hat) function for node A .

Visualizing basis functions is challenging in \mathbf{R}^3 , and hence the above examples in two-dimensions serve to illustrate the natural extension of the properties of basis functions within three-dimensional finite element shapes (tetrahedra, prisms, cubes).