Fractals in tribology

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“I have spent half my life having hard time to get my idea published, and the other half having hard time with people copying my ideas”

«I am having hard time completing my 8 volumes authobiography….»

BB. Mandelbrot, personal communications after ICF11 dinner, Torino 2005
BB Mandelbrot (1924-2010)

• Papers in geometry, finance, physics, image creation and compression, turbulence, fracture, hydraulics, medicine and many more. Which SSD is he in? Would he get one or more ASN?
• has today in GS 114808 citations and h-index = 92
• The oldest prof to get tenure @Yale (Economics), after long IBM career
• Possible ground for collaboration…
In the most basic sense, fractals are objects that display self-similarity over a wide range of scales.

A fractal is formed when pulling apart two glue-covered acrylic sheets.
<table>
<thead>
<tr>
<th>(0 &lt; D &lt; 1)</th>
<th>Generalized Cantor set</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.538</td>
<td>Feigenbaum attractor</td>
</tr>
<tr>
<td>D</td>
<td>Fractal</td>
</tr>
<tr>
<td>------</td>
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<tr>
<td>1.2</td>
<td>Dendrite Julia set</td>
</tr>
<tr>
<td>1.2619</td>
<td>2D Cantor dust</td>
</tr>
<tr>
<td>1.2619</td>
<td>Koch curve</td>
</tr>
</tbody>
</table>

$1 < D < 1.3$
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<tr>
<th>1.5000</th>
<th>a <a href="https://en.wikipedia.org/wiki/Weierstrass_function">Weierstrass function</a></th>
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<tr>
<td>1.5849</td>
<td><a href="https://en.wikipedia.org/wiki/Sierpinski_triangle">Sierpinski triangle</a></td>
</tr>
<tr>
<td>1.8272</td>
<td>A self-<a href="https://en.wikipedia.org/wiki/Affine_transformation">affine</a> fractal set</td>
</tr>
<tr>
<td>1.8928</td>
<td>3D <a href="https://en.wikipedia.org/wiki/Cantor_set">Cantor dust</a></td>
</tr>
<tr>
<td>$2&lt;D&lt;2.3$</td>
<td>Pyramid surface</td>
</tr>
<tr>
<td>$2.06 \pm 0.01$</td>
<td><strong>Lorenz attractor</strong></td>
</tr>
<tr>
<td>$2.3219$</td>
<td>Fractal pyramid</td>
</tr>
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Many earlier oversimplified conclusions by Mandelbrot have not resisted the test of time. Fracture mechanics essentially dissipates energy both on a surface and on a volume (plastic deformation), so one can artificially say that it dissipates over a fractal surface and then the fractal dimension needs to change (multifractal…)

Never heard again of D as measure of toughness in metals…
There was also a bitter debate at the end of 1990's about too much literature claiming «fractal» scaling when in fact geometry had at most 1–2 decades of selfsimilarity.

**Limited scaling range.** The number of decades (factors of 10) spanned by experimentally led to the labeling of the studied systems as fractal (4).
Fractals & chaos

- Can be used in non linear dynamics to classify the dimension of strange attractors and hence the «degree» of chaos
Nothing new under the sun?

• Pure mathematicians tend to dismiss Mandelbrot as a mere salesman.
• Mandelbrot claims that even if the objects he brings forth have been known to pure mathematicians, they tended to be disgusted by them as mere pathological monsters, and it is he who showed how natural and useful they really are for the study of nature.
Turbulence

Mandelbrot (1982) ‘turbulence involves many fractal facets’ – claims geometric aspects of turbulence have been ignored. But his own investigations of 1974, 1975 ‘they involve suggestions with few hard results as yet.’ (Mandelbrot 1982)
Finance

• another example is the modeling of commodity prices, which he claimed did not follow the standard Brownian motion with Gaussian distributions, but hyperbolic ones and not independently but showing some traces of memory. In particular this led to models with much bigger fluctuations, more in accordance with observations.
Mandelbrot set $z_{n+1} = z_n^2 + c$

a vindication of the Platonic view of mathematics
Application to tribology

- Archard 1957 was a fractal ante litteram (magnification-dependent solution)

- Fractal dimension was introduced in tribology by Majumdbdar and Bhushan in 1990, but the contact area was arbitrarily defined as non fractal by a geometrical intersection of the rough surface with a plane, leading to a power law distribution of contact spot diameters (Korckak’s law).
Linear elastic contact of the Weierstrass profile†

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Received 12 January 1999; revised 18 May 1999; accepted 2 June 1999

A contact problem is considered in which an elastic half-plane is pressed against a rigid fractally rough surface, whose profile is defined by a Weierstrass series. It is shown that no applied mean pressure is sufficiently large to ensure full contact and indeed there are not even any contact areas of finite dimension—the contact area consists of a set of fractal character for all values of the geometric and loading parameters.

The Weierstrass function: 
\[ z(x) = g_0 \sum_{n=0}^{\infty} \gamma^{(D-2)n} \cos(2\pi \gamma^n x/\lambda_0). \]
The Archard load redistribution process showed that what looks full contact eventually is split at smaller scales into partial contact, and so on. For some combination of parameters (in particular, for a nearly continuous spectrum), there is convergence towards full contact: despite the contact area tends to zero, the actual areas of contact are in full contact. This has profound implications as later on we cannot simplify the full Persson’s theory.

Figure 2. Evolution of the contact pressure distribution for $\gamma^{D-1} < 2$. A full contact region (1) evolves to (2) at the next scale. With one further reduction of scale, regions (2A) evolve once again to (2), while regions (2B) evolve to (3).
Weiestrass model

- Ciavarella Demelio Barber & Jang in 2000 showed instead contact area is itself a fractal of dimension less than a surface.
- The elastic model for the tribological problem is essentially *ill-posed* as in the limit that the load $P$ is shared on an infinite number of points, where the local pressure is infinite! $P=0 \times \infty$
- Bowden-Tabor old plastic model is more reasonable, but is forgotten.........
Persson (2001) introduced a very elegant theory for Gaussian random (fractal) surfaces which has evolved over the years to this day, permitting to solve the elastic contact problem in great details.

- It tries to solve the problem as a function of «magnification», and does not worry that the fractal limit is bizzare.
- How to fix «magnification» then?
Curiously, we have seen a long debate and “contact challenges” between “asperity models (GW, 1966)” and “Persson theory (2001)” mainly about the correct expression of contact area.

Most interesting and relevant part of Persson’s theory (2007, 2008) is perhaps that force vs mean separation converges in the fractal limit, and so does electrical resistance or elastic stiffness, which depend on macroscopic quantities not on tail of the PSD spectrum.
Predicting Friction?

- Almost all models predict linearity of real contact area with load, which is then used as argument to justify Amonton’s law --- this implies that there is a constant shear strength, which seems to suggest a plastic deformation. The argument is really circular!

- We have no elastic contact model today that can predict the friction coefficient, based on the rough surface details.

- Perhaps some effects of roughness on friction in viscoelastic materials are understood qualitatively, but extremely sensitive to the so-called large wavevector cutoff, which remains rather arbitrary (Persson suggests to truncate to $h'_{\text{rms}}=1.3$).

Adhesion

- Adhesion started with Bradley 1932 for rigid sphere, to JKR 1971 for elastic sphere, starting a long discussion for the case of a single sphere with the DMT «semi-rigid» solution, still ongoing.

- David Tabor clarified in 1979 the controversy between the Cambridge School with the energetic approach (JKR) and the Russian school of DMT (semirigid superposition of repulsive solution with forces in the gaps), introducing a parameter $\mu$ which is the ratio between length scale of the singular «fracture mechanics» field, and the contact area or sphere radius.
Van der Waals forces are quite strong and they should lead to theoretical strength for a perfect crystal (adhesion paradox of Kendall’s “sticky Universe”). This doesn’t happen for inevitable roughness.

![Lennard-Jones traction law](image)

Figure 1: The Lennard-Jones traction law between two half spaces. The interface energy $\Delta \gamma$ corresponds to the shaded area.

\[ 3^{1/6}\varepsilon \text{ and is } \sigma_0 = 16\Delta \gamma/9\sqrt{3}\varepsilon. \]
From DMT to JKR

The DMT limit becomes non-hysteretic, and is captured well with the Maugis solution simplifying the force gap relationship.

The original DMT solution is not good unless Tabor parameter is extremely close to 0.

Fig. 2 - Solutions of JKR, DMT and Maugis intermediate Tabor parameter range $\mu = 0, 0.05, 0.25, 1.5$. JKR is obtained very closely at positive indentations for $\mu \approx 1$. 
Generalization of JKR approach

\[ P_2 = P_1 - (\Delta_1 - \Delta_2) \left( \frac{\partial F}{\partial \Delta} \right)_{\Delta_1} \] (4)

Figure 3: Two-step loading scenario. (i) “repulsive” loading without adhesive forces until a given contact area is reached (point A in the figure); (ii) Unloading at constant total contact area up point B.

JKR original energy calculation can be generalized approximately (Ciavarella, JMPS, 2018), but does not work in the fractal limit
Adhesion – FT, PR, PS theories

- Much less is known for rough surfaces:

- Fuller and Tabor 1975 using the GW asperity model and JKR theory seemed to fit results for rubber spheres in contact with rough perspex plates: adhesion was destroyed for very low amplitude of roughness

Adhesion Paradoxes in the fractal limit

• Fuller and Tabor suggest stickiness is always zero (regardless of any other feature)!

• Pastewka and Robbins suggest stickiness is always infinite for all surfaces having $D>2.4$ (which include most of the natural and even man made surfaces)!
The main idea of «BAM» model is purely geometrical.  
1) the entire DMT solution for the sphere in the form reported by Maugis (not given by DMT), is obtained also by considering a Maugis constant force for separations up to a characteristic distance $dr$ and force of attraction the product of theoret strength and overlap area $A(d+dr)-A(d)$.  
2) This is superposed to Hertz theory for the repulsive force $F(d)$.
large $\sigma_{rep}$ (Persson 2007, eqn. 20)$^4$

\[
\frac{\sigma_{rep}}{E^*} \approx \frac{0.5 \times 0.75}{2} q_0 h_{rms} \exp \left( \frac{-u}{\gamma h_{rms}} \right)
\]

where $\gamma \approx 0.4$, $q_0$ is the small wavevector in the self-affine process, and $h_{rms}$ is the rms amplitude of roughness.

Therefore, using a simple Maugis model for adhesion, and the bearing area estimate for the area of attraction for a rough contact, we compute the difference of the bearing area at separation $u_{att} = u - \Delta r$, and $u$, and the attractive pressure can be estimated in a single line as

\[
\frac{\sigma_{att}}{\sigma_{th}} = \frac{1}{2k} \left[ \text{Erfc} \left( \frac{u_{att}}{\sqrt{2}h_{rms}} \right) - \text{Erfc} \left( \frac{u}{\sqrt{2}h_{rms}} \right) \right]
\]

for not too
Full-BAM refers to the version including high fractal dimensions where there is dependence also on slopes. Comparison with PR data for pull-off shows reasonable agreement.
In DMT theory, using a Maugis potential the adhesive contact area is easily defined integrating the probability distribution of gaps $P(u)$ from 0 to the Maugis range of attraction. We showed that $P(u)$ converges with the magnification $\zeta$, thus any criterion on stickiness within the DMT assumptions cannot depend on the PSD large wavevector components.
On stickiness of multiscale randomly rough surfaces

The attractive contact area can be written as

$$\frac{A_{ad}}{A_{nom}} = \frac{3}{2} \alpha_V \hat{p}_{rep} \left( \frac{\epsilon}{h_{rms}} \right)^{2/3}$$

where $\hat{p}_{rep}$ is the dimensionless average repulsive pressure, $\epsilon$ is the Maugis range, $h_{rms}$ is the surface rms and $\alpha_V$ is a coefficient that can be obtained from the $\frac{A_{ad}}{A_{nom}}$ vs $\left( \frac{\epsilon}{h_{rms}} \right)^{2/3}$ curve.
On stickiness of multiscale randomly rough surfaces

The external pressure is the sum between the repulsive and the adhesive contribution

\[
\frac{p_{\text{ext}}}{E^*} = \frac{p_{\text{rep}}}{E^*} - \frac{\sigma_0}{E^*} \frac{A_{\text{ad}}}{A_{\text{nom}}} = \frac{A_{\text{rep}}}{A_{\text{nom}}} \frac{\sqrt{2m_2}}{2} \left[ 1 - \frac{l_a}{\epsilon} \frac{3}{2} \frac{a_V}{q_0 h_{\text{rms}}} \left( \frac{\epsilon}{h_{\text{rms}}} \right)^{2/3} \right]
\]

Stickiness is obtained when the slope becomes negative

\[
\frac{1}{\kappa} = \frac{p_{\text{ext}}/(E^* \sqrt{2m_2})}{A_{\text{rep}}/A_{\text{nom}}}
\]

In particular for low fractal dimension, high magnification and assuming \(a_V \approx 3\) (neglecting the weak dependence on pressure) simple criterion for stickiness is obtained

\[
\frac{h_{\text{rms}}}{\epsilon} < \left( \frac{9 l_a/\epsilon}{4 \epsilon q_0} \right)^{3/5}
\]

Which depends only on the shortest wavevector \(q_0\) and on the surface RMS. Both are converging quantities with magnification. This contrast with the currently available stickiness criterion from Fuller and Tabor (1966) and Pastewka and Robbins (2014), which depend on the PSD truncation, thus are also difficult to check experimentally.
On stickiness of multiscale randomly rough surfaces

Our predictions has been checked against full numerical results obtained using the Contact-App from Lars Pastewka over a wide range of parameters.
On stickiness of multiscale randomly rough surfaces

Using our criterion for stickiness we can define a sticky and region where surfaces are expected to naturally snap into contact after a gentle approach. From JTB's results, we know that complex instabilities and patterns form at very low RMS amplitude of roughness, and hence in the sticky range, DMT type of analysis can be expected to hold only above the dashed line in panel A.
On stickiness of multiscale randomly rough surfaces

A self-affine randomly rough surface is defined by its PSD, $C(q) = C_0 q^{-2(H+1)}$. Our criterion can be written directly as a function of $C_0$. Assuming reasonable estimates for the rest of parameters we obtain that real surfaces should stick for $E^* < 0.3$ MPa, which is in agreement with the very well known Dahlquist criterion for adhesives.
Comparison with BAM

Fig. 4. Comparison of the attractive area $\frac{A_{ad}}{A_{nom}}$ estimated by Persson’s theory (black, blue, and red line respectively for $\zeta = 10, 100, 1000$) as improved by Afferrante et al. (2018). Red dashed line shows BAM (Ciavarella, 2017) prediction, and dashed black line indicates a guide to the eye with $\left(\frac{\epsilon}{h_{rms}}\right)^{2/3}$. Case of Fig. 2
Comparison with BAM

Fig. 8. Estimates of the limit RMS amplitude of roughness for the BAM model (blue solid line) and the present one (black solid line).
Wear

Wear remains one of the least scientifically understood tribological processes. The most common approach in wear refers to Archard [2] as wear volume $V$ is proportional to the sliding length, the normal force, and inverse with the hardness $H$ of the material. Hence, the wear rate is proportional to pressure $p$:

$$V' = \frac{kp}{H}$$
Critical Length Scale In Wear

- One of the interesting ideas for adhesive wear (one of the most prevalent types of wear) was suggested in 1958 by Rabinowicz [12] in *Wear* and was later forgotten (the paper has 12 citations in Google Scholar!), except rediscovered recently in numerical experiments by Molinari’s group at EPFL.
- *There is a critical size (of contact radius), such that smaller fragments remain adherent while larger fragments come off in loose form.*

\[ a^* = \lambda G \frac{W}{\sigma_j^2} \]

Predicting Archard Wear Coefficient?

• wear coefficients vary by more than 7 orders of magnitude.

• Archard "We postulate: worn volume $\sim a^3$ and effective sliding distance $\sim a$, therefore, the contribution of this contact to the wear per cm of sliding $\sim a^2$; also load supported by contact $\sim a^2$. Therefore, for this contact, the contribution to the wear rate is proportional to the load supported by it. A similar argument applies to all other contacts, and the total wear rate is proportional to the load".

• Unconvincing: constant density of asperities with height, all asperities are wearing in his model, in contrast with the Rabinowicz Aghababaei critical size concept.
Upscaling The Concept Of Rabinowicz Length Scale In Multiscale Contact.

• Frérot et al. [16] for example inserted the Rabinowicz-Aghababaei critical scale in the Archard model, i.e. assumed $K$ is the probability that contact area $A$ is larger than $A^*$ (where $A^*$ is the area of critical length scale $a^*$) for a given load $W$

$$K = P(A > A^*) = \int_{A^*}^{\infty} p(A, W) dA$$

• More precisely, starting from analytical predictions, using the quasi-realistic Greenwood-Williamson model with exponential distribution of asperities $\phi = (C/\sigma_s)\exp(-z_s/\sigma_s)$ for $z_s > 0$, and $\sigma_s$ being a scale parameter of the order of RMS amplitude, $p(A, W)$ turns out a negative exponential independent of the applied load $W$

$$p(A, W) = \frac{1}{b\sigma_s}\exp\left(-\frac{A}{b\sigma_s}\right)$$

Wear Paradox

- Therefore, (i) the wear coefficient $K$ is also independent of the load,

$$K = \exp\left(-\frac{A^*}{b\sigma_s}\right)$$

- (ii) there is a linear relationship between the contact area and the applied load in both elastic and plastic cases.
- However, the sensitivity to radius to resolution of the instrument (or to truncation of the fractal process) is extremely high, this would mean that in the fractal limit the elastic model predicts always infinitesimally small wear! Notice this is not just consequence of the asperity model
Simple Way Out Of Paradox?

- Pei et al. show that plasticity produces distribution of contact clusters closer to the very simple overlap model than to the elastic model.

Fig. 2 An example of contact area prediction for: a) elastic; b) elasto-plastic and c) rigid overlap model (adapted from [19])

Interesting that for fractal limit, friction coefficient tends to a universal value 0.25 (which is curiously enough, the value predicted by Leonardo 500 years ago!) Strangely instead, the low plasticity index and large R leads to strange infinite friction...
Intermediate conclusions - 1

- The contact of rough surfaces has been studied for a long time (from 1957), with the hope to explain friction, adhesion, wear and other tribological problems.

- Some considerable progress has been made in the area of elastic contact, where we know in details the solution for nominally flat, infinite surfaces. The choice of elasticity is mainly for mathematical convenience!

- However, friction has been studied since the times of Leonardo and in 500 years, no predictive model has emerged, nor significant improvement from rough contact models. Indeed, plastic models of friction today are more well posed and recognize $f=0.25$ like Leonardo predicted in his notebooks!
Intermediate conclusions - 2

• For adhesion, we described progress with BAM by Ciavarella (2017) and with the criterion of Violano, Afferrante, Papangelo & Ciavarella (2018) who have found early claims to be incorrect: stickiness does not depend on truncation of the PSD spectrum.

• Recently, rough contact models have also been attempted in the hope to predict the coefficient of proportionality between wear loss and friction dissipation which was observed already by Reye in 1860, and then Archard in the 1950’s. (both papers were ignored in the new papers in top journals!)

• Resolution-dependence of contact area make the models very ill-defined, and many predictions quite hard.
Conclusions

• the contact “sport” of simulating elastic multiscale contact with fractal accurate models that has dominated the specialized literature (including my own contributions!), is mostly a little remote still from real tribological problems.

• With fractals, it is easy to end up with paradoxical limit conclusions. We made the example of two important theories in adhesion. The way out of the paradox requires very significant effort (Persson’s solution) or simple but clever ideas (BAM).

• And after this effort, we are just about to seeing perhaps some really quantitative applications…