Extension of A Composite Time Integration Scheme for Dynamic Problems

Sandeep Kumar [1*], Sachin S. Gautam[2†]
[1] Department of Mechanical Engineering, IIT Guwahati, Guwahati Guwahati, Assam, 781039, India
[2] Department of Mechanical Engineering, IIT Guwahati, Guwahati Guwahati, Assam, 781039, India

Abstract

This paper proposes a simple extension to a collocation based composite time integration proposed by Silva and Bezerra [16]. In this scheme, each time step is divided further into two substeps which may not be necessarily equal. In the first substep, the Newmark scheme is employed an d the three point backward Euler scheme is used in the second substep. The proposed scheme is applied to non-linear problems to study the transient response solution under large deformations and long time durations. The influence of Newmark parameters and substep sizes on conservation of energy and momentum is studied through a numerical example. It is found that the numerical dissipation increases as the Newmark parameters are changed. Also, as the substep size corresponding to Newmark scheme increases the numerical dissipation increases. The proposed scheme can be used to study of nonlinear transient response of structures with required dissipation.

Keyword: Nonlinear dynamic analysis, implicit time integration scheme; composite time integration; numerical stability; numerical dissipation; energy conserving/decaying algorithms.

1 Introduction

The distinctive nature between static and dynamic problem is the presence of inertia forces which opposes the motion generated by the applied dynamic loading. The

* Master's Student, sandeepkumar9995@gmail.com
† Asst. Professor, ssg@iitg.ernet.in

dynamic nature of a problem is dominant if the inertia forces are large compared to the total applied forces [14]. In order to investigate the characteristics of transient dynamic problems, the resulting motion of a structural dynamic problem is studied for a given load distribution in space and time. Transient response analysis is used to compute the dynamic response of structure subjected to time-varying excitation. In order to obtain the time history of transient response time integration schemes are widely used. Time integration schemes are broadly classified into two categories: explicit and implicit [1, 5, 6, 14]. In an explicit scheme, the displacement and the velocity at the current time step are found using the values from the previous time step. The acceleration is then calculated by substituting these values into dynamic equation and solving system of simultaneous linear equations [8, 7, 15]. If lumped matrices are used, then no iterations are required to solve the system of equations. Some of the examples of explicit schemes are central difference method, Runge-Kutta method etc. A prominent disadvantage of the explicit schemes is that they are only conditionally stable. This means that the time step size has to be below a critical value. Explicit schemes are widely used for fast transient analysis, for example, in the analysis of crash problems. On the other hand in the implicit schemes, the displacement and velocity at the current time step are expressed not only interms of the values of the previous time step but also of the current time step. Hence, the solution of system of resulting equations requires an iterative scheme, usually Newton-Rapshon method, to obtain the solution. This allows for larger time step size to be used during the analysis. Some of the examples of implicit schemes are Newmark scheme [12], Bathe composite scheme [3, 2, 4] etc.

The implicit trapezoidal scheme is unconditonally stable for the linear dynamic problems. However, for nonlinear dynamic problems, the trapezoidal scheme does not guarantee the conservation of energy and momentum as time progresses [10, 17, 3, 11]. It fails to provide high frequency dissipation in nonlinear analysis. Even if smaller time step is considered, convergence is not guaranteed as it may lead to excitation of even higher frequencies which lead to instability. In linear dynamic analysis the spectral stability is sufficient condition for unconditional stability of the time integration scheme [13]. However, for nonlinear dynamic analysis spectral stability is required but it is only a necessary condition [10].

Recently, Bathe and coworkers have proposed an implicit composite time integration scheme for dynamic analysis [2, 4, 3]. The scheme is usually referred as *Bathe scheme*. In the Bathe scheme, a highly dissipative time integration scheme is combined with a non-dissipative time integration scheme. For conservation of energy and momentum, trapezoidal scheme is combined to three-point backward Euler scheme. Trapezoidal scheme ensures second order accuracy and the back-ward Euler scheme ensures high-frequency numerical dissipation. Numerical dissipation is considered to be advantageous as it ensures better numerical stability for time integration schemes. Silva and Bezerra [16] proposed a scheme which is based on the Bathe scheme [2] but with generalised substep sizes instead of equal substep size used in the Bathe scheme. It was shown that for too large time step, the scheme remains stable but numerical dissipations are also large. Klarmann and Wagner [9] have further analyzed the Bathe scheme for variable step sizes and have shown that at a particular value of the step size, the period elongation is minimum and the numerical dissipation is maximum.

In the present work, an extension to the implicit composite scheme of Silva and Bezerra [16] is proposed. In this composite scheme, Newmark scheme [12] has been coupled with three-point backward euler scheme thus making it a three parameter based composite time integration scheme. The proposed scheme is applied to a nonlinear dynamic problem. The rest of the paper is structured as follows. In section 2 the proposed scheme is explained. Numerical example is presented in section 3. Section 4 concludes this paper.

2 Proposed Implicit Composite Scheme

The scheme proposed in the present work extends the composite scheme proposed by Silva and Bezerra [16] where the variable time substep sizes are used. The proposed implicit composite scheme is a parameter based time integration scheme in which the Newmark scheme [12] is applied in the first substep and three-point backward Euler method for the second substep. The composite scheme is shown schematically in Figure (1).

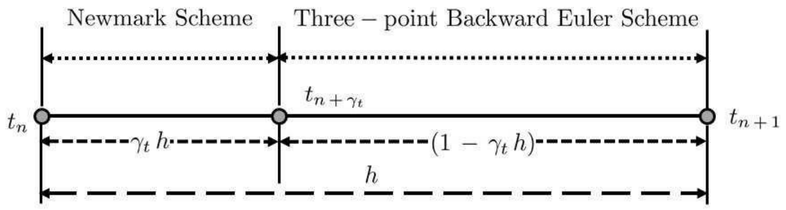


Figure 1: Proposed Composite Scheme. The time step is denoted by $t_{n+1} - t_n = h$.

The governing equations of equilibrium for nonlinear transient structural dynamic problems is expressed as follows:

$$M\ddot{u} + C\dot{u} + N(u,\ t) = F(t) \qquad (1)$$

where M is the mass matrix, C is the damping matrix, $N(u, t)$ is the internal force vector which is, in general, a function of displacement vector u and time t and $F(t)$ is the external force vector. The vectors of velocity and acceleration are represented by $\dot{u}$, and $\ddot{u}$ respectively. Note that for linear dynamic analysis.

is, the internal force vector $N(u, t)$ can be written as $K\,u$ where K is the stiffness matrix. Next, the proposed scheme is explained in detail by applying it to Eq.(1).

Considering $t_{n+\gamma t} = t_n + \gamma_t\, h$ (where h is the time step size) as an instance of time between t_n and t_{n+1} for $\gamma_t \in (0,1)$, Newmark scheme is applied over the first substep, $\gamma_t\, h$ (see Fig. 1). The approximations for displacement and velocity at time $t_{n+\gamma t}$ for Newmark scheme are given by

$$\begin{aligned} \dot{u}_{n+\gamma_t} &= \dot{u}_n + \gamma_t\, h\left[(1-\gamma)\,\ddot{u}_n + \gamma\,\ddot{u}_{n+\gamma_t}\right] \\ u_{n+\gamma_t} &= u_n + (\gamma_t\, h)\,\dot{u}_n + \frac{(\gamma_t\, h)^2}{2}\left[(1-2\beta)\,\ddot{u}_n + (2\beta)\,\ddot{u}_{n+\gamma_t}\right] \end{aligned} \qquad (2)$$

where β, γ are Newmark scheme parameters.

In the second substep the three point backward Euler scheme is applied over the second substep $(1 - \gamma_t)\, h$. The approximation for velocity and acceleration at time t_{n+1} for the three point backward Euler scheme is given by

$$\begin{aligned} \dot{u}_{n+1} &= c_1\, u_n + c_2\, u_{n+\gamma_t} + c_3\, u_{n+1}\,, \\ \ddot{u}_{n+1} &= c_1\, \dot{u}_n + c_2\, \dot{u}_{n+\gamma_t} + c_3\, \dot{u}_{n+1}\,, \end{aligned} \qquad (3)$$

where the constants are given expressed as

$$\begin{aligned} c_1 &= \frac{(1-\gamma_t)}{\gamma_t\, h}\,, \\ c_2 &= \frac{-1}{(1-\gamma_t)\,\gamma_t\, h}\,, \\ c_3 &= \frac{(2-\gamma_t)}{(1-\gamma_t)\, h}\,. \end{aligned} \qquad (4)$$

3 Numerical examples

Table 1: Newmark parameters.

$S.No$	β	γ
1	0.25	0.5
2	0.3025	0.6
3	0.36	0.7
4	0.4225	0.8
5	0.49	0.9

In section 2, formulations of the proposed scheme was presented. In this section, the proposed scheme is implemented to solve the flexible pendulum problem and to examine the performance of the proposed scheme. The flexible pendulum problem is a classical geometrical nonlinear example which involves large displacements and rotations. It is used to study the ability of a time integration scheme for solving nonlinear problems. Formulations of this problem has been analyzed by Kuhl and Crisfield [10]. A set of Newmark parameters (Refer Table: 1) and γ_t values have been chosen to test the performance of the proposed scheme. Various values of Newmark parameters (β,γ) are selected (Refer Table 1) according to the following formula [1]:

$$\begin{aligned} \beta &\geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2, \\ \gamma &\geq \frac{1}{2}. \end{aligned} \tag{5}$$

The geometrical and physical characteristics of the elastic pendulum, the initial conditions, the boundary conditions and other data are shown in Figure 2. For elastic pendulum, the stiffness $E\,A$ is taken as 10^4 N and the initial radial acceleration $\ddot{u}$ as 0 m/s^2 [10]. Elastic pendulum possess both high and low frequency responses. For this two degree-of-freedom model, the first mode is represented by the pendulum motion. The second mode, which contains high frequency responses, is represented by axial motion [13]. Due to modified initial conditions, the pendulum will be loaded with centrifugal force which induces high frequency vibration along the pendulum length. To capture the high axial frequency, time steps considered are h = 0.01 seconds and h = 0.05 seconds. The substep sizes taken are : γ_t = 0.2 and 0.9. The transient analysis is done for a total time of 30 seconds.

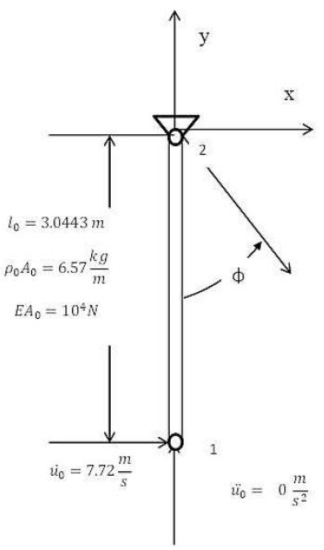


Figure 2: Flexible pendulum. Data and initial conditions.

Figures (3(a)- 3(b)) and Figures (4(a)- 4(b)) show the variation of total energy and angular momentum with time for h=0.01s and γ_t=0.2 and 0.9 respectively. It is observed that for same time step, as the value of γ_t increases, numerical dissipation in total energy and angular momentum also increases. This same behavior has been observed when h is changed to 0.05, see Figures (5(a), 5(b)) and Figures (6(a), 6(b)). Also, as the value of Newmark parameters increases, numerical dissipation increases. Maximum dissipation is for the newmark parameters (β,γ)=(0.49,0.9). No growth in energy and momentum of the system has been observed. Hence, through higher numerical dissipation of the proposed scheme, better numerical stability for the given non-linear problem can be obtained compared to Bathe scheme [2].

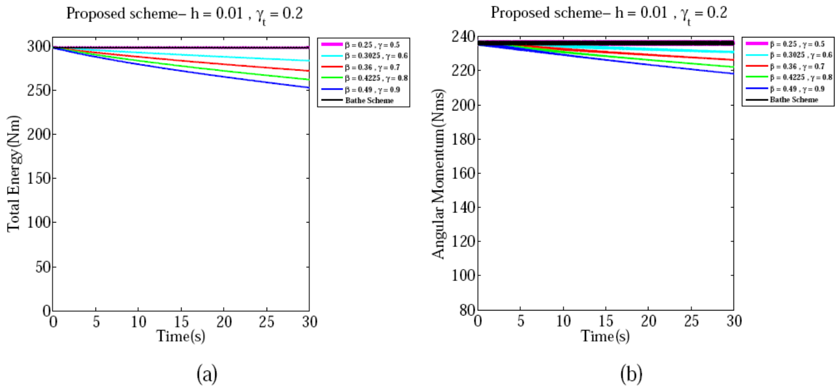


Figure 3: Variation of energy-momentum with time for h=0.01 s and γ_t=0.2.

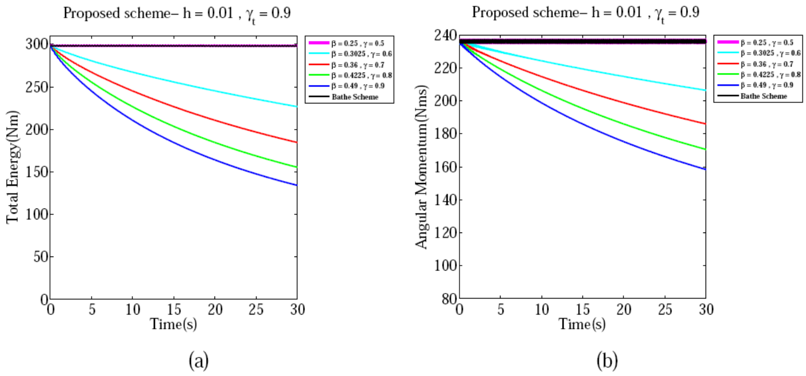
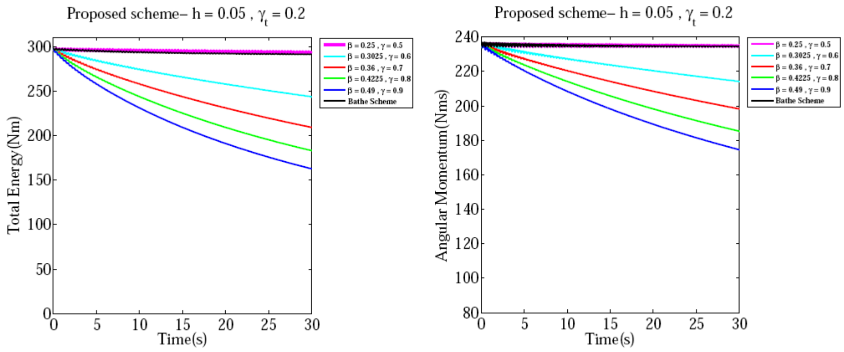
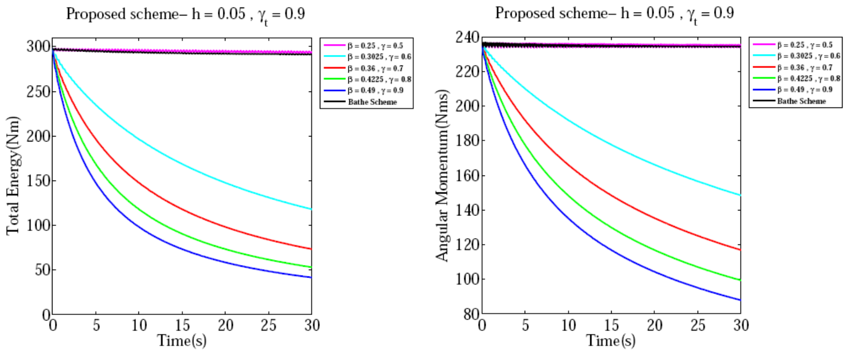


Figure 4: Variation of energy-momentum with time for h=0.01 s and γ_t=0.9.



(a) (b)

Figure 5: Variation of energy-momentum with time for h=0.05 s and γ_t=0.2.



(a) (b)

Figure 6: Variation of energy-momentum with time for h=0.05 s and γ_t=0.9.

4 Conclusion

The proposed composite scheme is applied to a nonlinear dynamic problem. This scheme gives more flexibility to vary the dissipation aspect by choosing different combinations of Newmark parameters (Refer: Table 1) and γ_t values. The proposed scheme gives more numerical stability compared to Bathe scheme [2]. The performance of the scheme is studied for different values of γ_t on energy and momentum conservation. For a particular time step, as the value of γ_t increases, numerical dissipation also increases. Numerical dissipation also increases with the increase of Newmark parameters (Refer: Table 1). It can also be concluded from the present study that use of too large time step leads to excessive numerical dissipation.

Acknowledgement: The senior author would like to thank the Aachen Institute for Advanced Study in Computational Engineering Science (AICES), RWTH Aachen University, Aachen, Germany for providing travel assistance to carry out some part of this work during summer of 2014.

References

[1] K. J. Bathe. *Finite Element Procedures*. PHI Learning Private Limited, 1996.

[2] K. J. Bathe. Conserving energy and momentum in nonlinear dynamics: A simple implicit time integration scheme. *Computers and Structures*, 85:437– 445, 2007.

[3] K. J. Bathe and M. M. I. Baig. On a composite implicit time integration procedure for nonlinear dynamics. *Computers and Structures*, 85:2513–2524, 2005.

[4] K. J. Bathe and G. Noh. Insight into an implicit time integration scheme for structural dynamics.

Computers and Structures, 98-99:1–6, 2012.

[5] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts And Applications of Finite Element Analysis*. John Wiley and Sons, Singapore, 2003.

[6] J. Har and K. K. Tamma. *Advances In Computational Dynamics of Particles, Materials And Structures*. John Wiley and Sons, Singapore, 2012.

[7] G. Hulbert and J. Chung. Explicit time integration algorithm for structural dynamics with optimal numerical dissipation. *Computer Methods in Applied Mechanics and Engineering*, 137:175–188, 1996.

[8] G. M. Hulbert. A unified set of single-step asymptotic annihilation algorithms for structural dynamics. *Computer Methods in Applied Mechanics and Engineering*, 113:1–9, 1994.

[9] S. Klarmann and W. Wagner. Enhanced studies on a composite time integration scheme in linear and non-linear dynamics. *Computational Mechanics*, 55:455–468, 2015.

[10] D. Kuhl and M. A. Crisfield. Energy-conserving and decaying algorithms in non-linear structural dynamics. *International Journal for Numerical Methods in Engineering*, 45:569–599, 1999.

[11] T. Y. Liu, Q. B. Li, and C. B. Zhao. An efficient time-integration method for nonlinear dynamic analysis of solids and structures. *Science China- Physics, Mechanics and Astronomy*, 56:798–804, 2013.

[12] N. M. Newmark. A method of computation for structural dynamics. *Journal of the Engineering Mechanics Division, American Society of Civil Engineers*, 85:67–94, 1959.

[13] M. R. Pajand and M. K. Rad. More accurate and stable time integration scheme. *Engineering with Computers*, 2014.

[14] P. Paultre. *Dynamics of Structures*. Wiley,New Delhi, 2013.

[15] S. N. Penry and W. L. Wood. Comparison of some single-step methods for the numerical solution of the structural dynamic equation. *International Journal for Numerical Methods in Engineering*, 21:1941–1955, 1985.

[16] W. T. M. Silva and L. M. Bezerra. Performance of composite implicit time integration scheme for nonlinear dynamic analysis. *Mathematical Problems in Engineering*, 2008.

[17] J. C. Simo, N. Tarnow, and K. K. Wong. Exact energy-momentum conserving algorithms and symplectic schemes for nonlinear dynamics. *Computer Methods in Applied Mechanics and Engineering*, 100:63–116, 1992.

[18] P. Wriggers. *Nonlinear Finite Element methods*. Springer-Verlag Berlin Heidelberg, 2008.