Composites: A Myriad of Microstructure Independent Relations

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Typically, the elastic properties of composite materials are strongly microstructure dependent. So it comes as a pleasant surprise to come across exact formulae for (or linking) effective moduli that are universally valid no matter how complicated the microstructure. Such exact formulae provide useful benchmarks for testing numerical and actual experimental data, and for evaluating the merit of various approximation schemes. This paper presents a sampling of results in the field.

1. INTRODUCTION

The word myriad has its origin as the Greek word for 10,000. It would be a real challenge to present that many microstructure independent relations amongst the effective properties of composites, especially in a short paper. Instead this article presents an appetizer of results pertaining mostly to elasticity and thermoelasticity. The sampling is sufficiently diverse to encompass the main ideas used to generate microstructure independent relations in many different contexts, including thermoelectricity and piezoelectricity, where a host of microstructure independent results have been obtained: see [1–5] and references therein.

2. UNIFORM FIELDS

Consider a bimetal strip. When the temperature is raised the strip bends. This is due to the difference in thermal expansion of the two metals. Now consider the bimetal strip immersed in water. As the pressure in the water is increased the strip bends due to the difference in bulk moduli of the two metals. Now one can imagine applying just the right combination of temperature increase and water pressure increase or decrease so both phases expand at exactly the same rate and there is no distortion. Of course this same argument applies not just to bimetal strips but to any geometric configuration of two isotropic phases and in particular to a two-phase composite.

Suppose the two-phases are isotropic so that a block of phase 1 or phase 2 immersed in a fluid heat bath at temperature T and pressure p expands or contracts isotropically as T and p are varied. Let $\rho_1(T,p)$ and $\rho_2(T,p)$ denote the mass density of phase 1 or

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phase 2 relative to some base temperature T_0 and base pressure p_0: thus $1/\rho_1(T,p)$ and $1/\rho_2(T,p)$ measure the relative change in the volume of each phase as the temperature and pressure changes from (T_0,p_0) to (T,p). According to this definition we have

$$\rho_1(T_0,p_0)=\rho_2(T_0,p_0)=1. \tag{1}$$

So the two surfaces $\rho_1(T,p)$ and $\rho_2(T,p)$ intersect at $(T,p)=(T_0,p_0)$. Unless the surfaces are tangent at this point they will intersect along a trajectory passing through (T_0,p_0). Along this trajectory $(T(h),p(h))$ parameterized by h both phases expand or contract at an equal rate.

Now suppose a composite is manufactured at the base temperature T_0 and pressure p_0 with no internal residual stress. When this composite is placed in the heat bath at temperature T and pressure p there is no reason to suppose the composite will expand or contract isotropically as T and p are varied. Indeed by considering the example of the bimetal strip it is clear that internal shear stresses and warping can occur. However along the trajectory $(T(h),p(h))$ the composite will expand isotropically and its density relative to its density at the base temperature and pressure will be

$$\rho_*(T(h),p(h))=\rho_1(T(h),p(h))=\rho_2(T(h),p(h))\quad\forall h. \tag{2}$$

Rewriting this relation as $1/\rho_*(T(h),p(h))=1/\rho_1(T(h),p(h))=1/\rho_2(T(h),p(h))$ and differentiating with respect to h, gives

$$3\alpha_*\frac{dT(h)}{dh}-\frac{1}{\kappa_*}\frac{dp(h)}{dh}=3\alpha_1\frac{dT(h)}{dh}-\frac{1}{\kappa_1}\frac{dp(h)}{dh}=3\alpha_2\frac{dT(h)}{dh}-\frac{1}{\kappa_2}\frac{dp(h)}{dh}, \tag{3}$$

where

$$\kappa_a(T,p)=-\Big\{\frac{\partial(1/\rho_a)}{\partial p}\Big\}^{-1},\qquad \alpha_a(T,p)=\frac{1}{3}\frac{\partial(1/\rho_a)}{\partial T},\qquad a=1,2\text{ or }*, \tag{4}$$

are the tangent bulk moduli and thermal expansion constants of the phases and composite along the trajectory. Provided the trajectory has been suitably parameterized dp/dh and dT/dh will not be both zero. Hence the determinant of the system of equations (3) must vanish which gives the well-known relation

$$\alpha_*=\frac{\alpha_1(1/\kappa_*-1/\kappa_2)-\alpha_2(1/\kappa_*-1/\kappa_1)}{1/\kappa_1-1/\kappa_2}, \tag{5}$$

between effective bulk moduli and effective thermal expansion coefficients due to Levin [6]. Thus (2) is a non-linear generalization of Levin's formula. This simple observation is joint but unpublished work with J. Berryman presented in Pittsburgh in 1994 at the SIAM Meeting on Mathematics and Computation in the Materials Sciences.

There is another viewpoint which sheds light on Levin's work. Let us begin with a result that applies to elasticity, and not just to thermoelasticity. Suppose the elasticity tensor field of a composite has the property that there exist symmetric matrices $\boldsymbol{v}$ and $\boldsymbol{w}$ with $\boldsymbol{\mathcal{C}}(\boldsymbol{x})\boldsymbol{v}=\boldsymbol{w}$ for all $\boldsymbol{x}$. Then the uniform strain field which equals $\boldsymbol{v}$ everywhere and the uniform stress field which equals $\boldsymbol{w}$ everywhere are solutions of the elasticity equations. Consequently the effective tensor $\boldsymbol{\mathcal{C}}_*$ must satisfy $\boldsymbol{\mathcal{C}}_*\boldsymbol{v}=\boldsymbol{w}$. (This remark was made to me by A. Cherkaev, although in the context of thermal expansion it dates back to work of Cribb [7]; see also Dvorak [8] who extended the idea.) In an isotropic polycrystal where the pure crystal has cubic symmetry this condition is satisfied, with $\boldsymbol{v} = \boldsymbol{I}$, and the result implies Hill's microstructure independent formula [9] for the effective bulk modulus of such a polycrystal.

A corollary is that if in a two phase composite the tensor $\boldsymbol{\mathcal{C}}_1 - \boldsymbol{\mathcal{C}}_2$ is singular with

$$(\boldsymbol{\mathcal{C}}_1 - \boldsymbol{\mathcal{C}}_2)\boldsymbol{v} = 0 \quad \text{then} \quad (\boldsymbol{\mathcal{C}}_* - \boldsymbol{\mathcal{C}}_2)\boldsymbol{v} = 0. \tag{6}$$

Now consider the equations of thermoelasticity. These take the form

$$\begin{pmatrix} \boldsymbol{\epsilon}(\boldsymbol{x}) \\ \varsigma(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{S}}(\boldsymbol{x}) & \boldsymbol{\alpha}(\boldsymbol{x}) \\ \boldsymbol{\alpha}(\boldsymbol{x})^T & c_p(\boldsymbol{x})/T_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}(\boldsymbol{x}) \\ \theta \end{pmatrix} \quad \text{with} \quad \nabla \cdot \boldsymbol{\sigma} = 0, \; \boldsymbol{\epsilon} = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]/2, \tag{7}$$

where $\theta = T - T_0$ is the change in temperature T measured from some constant base temperature T_0, $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\sigma}(\boldsymbol{x})$ are the strain and stress fields, $\varsigma(\boldsymbol{x})$ is the local increase in entropy per unit volume over the entropy of the state where $\boldsymbol{\sigma} = \theta = 0$, $\boldsymbol{\mathcal{S}}(\boldsymbol{x})$ is the compliance tensor, $\boldsymbol{\alpha}(\boldsymbol{x})$ is the tensor of thermal expansion and $c_p(\boldsymbol{x})$ is specific heat per unit volume at constant stress. Macroscopically the average fields satisfy

$$\begin{pmatrix} \langle \boldsymbol{\epsilon} \rangle \\ \langle \varsigma \rangle \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{S}}_* & \boldsymbol{\alpha}_* \\ \boldsymbol{\alpha}_*^T & c_{*p}/T_0 \end{pmatrix} \begin{pmatrix} \langle \boldsymbol{\sigma} \rangle \\ \theta \end{pmatrix}, \tag{8}$$

and this serves to define the effective elasticity tensor $\boldsymbol{\mathcal{S}}_*$, the effective tensor of thermal expansion $\boldsymbol{\mathcal{S}}_*$, and the effective constant of specific heat at constant stress c_{*p}.

The important observation is that because the entropy field $\varsigma(\boldsymbol{x})$ is not subject to any differential constraints, we can ignore it completely when computing the effective compliance tensor and effective thermal expansion tensor. In other words it suffices to work with the reduced set of equations

$$\boldsymbol{\epsilon}(\boldsymbol{x}) = \boldsymbol{M}(\boldsymbol{x}) \begin{pmatrix} \boldsymbol{\sigma}(\boldsymbol{x}) \\ \theta \end{pmatrix} \quad \text{with} \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\epsilon} = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]/2, \tag{9}$$

where for three-dimensional thermoelasticity $\boldsymbol{M}(\boldsymbol{x}) = (\, \boldsymbol{\mathcal{S}}(\boldsymbol{x}) \quad \boldsymbol{\alpha}(\boldsymbol{x}) \,)$ is representable as a 6×7 matrix in an appropriate basis. Now in a two phase composite the matrix $\boldsymbol{M}_1 - \boldsymbol{M}_2$, like any matrix which has more columns than rows, is necessarily singular, i.e. there necessarily exists a $\boldsymbol{v}$ such that $(\boldsymbol{M}_1 - \boldsymbol{M}_2)\boldsymbol{v} = 0$. Consequently the effective tensor $\boldsymbol{M}_*$ must satisfy $(\boldsymbol{M}_* - \boldsymbol{M}_2)\boldsymbol{v} = 0$. When the phases are isotropic this reduces to Levin's formula (5). When the phases are anisotropic (but each with constant orientation) it reduces to the formula of Rosen and Hashin [10]. Having obtained particular solutions for the stress and strain field with a non-zero value of θ it is an easy matter to determine the corresponding entropy field $\varsigma(\boldsymbol{x})$ and thereby obtain a formula for the effective constant of specific heat c_{*p} in terms of the effective elasticity tensor [10].

The same reasoning can be applied to obtain an exact expression for effective constant of specific heat and effective thermal expansion tensor in terms of the effective compliance tensor for polycrystaline materials constructed from a single crystal [11,12]. The compliance tensor $\boldsymbol{\mathcal{S}}_0$ and thermal expansion tensor $\boldsymbol{\alpha}_0$ of the single crystal must be such that $\boldsymbol{\mathcal{S}}_0 \boldsymbol{I}$ and $\boldsymbol{\alpha}_0$ are both uniaxial with a common axis of symmetry. This is ensured if the crystal has hexagonal, tetragonal or trigonal symmetry. The contact between crystals need not necessarily be ideal; slippage along grain boundaries is allowed [13]. When the constitutive relation involves more than one constant field in addition to θ, such as humidity causing expansion due to moisture absorption, then uniform field arguments yield exact relations even when $\boldsymbol{\mathcal{S}}_0\boldsymbol{I}$ and $\boldsymbol{\alpha}_0$ are biaxial, provided they share the same three principal axes [14].

There is also a direct mathematical correspondence between the equations of poroelasticity and those of thermoelasticity (the fluid pressure plays the role of the temperature). Consequently these exact microstructure independent relations extend to effective poroelastic moduli of two phase media [15,16].

Uniform field arguments are also important three dimensional elastic two phase media when the microstructure and stress and strain fields are independent of the x_3 coordinate. The constitutive relation can be expressed in the form

$$\begin{pmatrix} \boldsymbol{g}(\boldsymbol{x}) \\ \sigma_{33}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{L}(\boldsymbol{x}) & \boldsymbol{\alpha}(\boldsymbol{x}) \\ \boldsymbol{\alpha}(\boldsymbol{x})^T & C_{3333}(\boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ \epsilon_{33} \end{pmatrix}, \tag{10}$$

where

$$\boldsymbol{L} = \begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} & \sqrt{2}C_{1123} & -\sqrt{2}C_{1113} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} & \sqrt{2}C_{2223} & -\sqrt{2}C_{2213} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} & 2C_{2312} & -2C_{1312} \\ \sqrt{2}C_{1123} & \sqrt{2}C_{2223} & 2C_{2312} & 2C_{2323} & -2C_{2313} \\ -\sqrt{2}C_{1113} & -\sqrt{2}C_{2213} & -2C_{1312} & -2C_{2313} & 2C_{1313} \end{pmatrix}, \tag{11}$$

and

$$\boldsymbol{\alpha} = \begin{pmatrix} C_{1133} \\ C_{2233} \\ \sqrt{2}C_{3312} \\ \sqrt{2}C_{3323} \\ -\sqrt{2}C_{3313} \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \\ E_1' \\ E_2' \end{pmatrix}, \quad \boldsymbol{h} = \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \sqrt{2}\epsilon_{12} \\ D_1' \\ D_2' \end{pmatrix}, \tag{12}$$

in which $E_1'(x_1, x_2)$, $E_2'(x_1, x_2)$, $D_1'(x_1, x_2)$, and $D_2'(x_1, x_2)$ are components of the vector fields

$$\boldsymbol{E}' = \begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2}\sigma_{23} \\ -\sqrt{2}\sigma_{13} \end{pmatrix}, \quad \boldsymbol{D}' = \begin{pmatrix} D_1' \\ D_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2}\epsilon_{23} \\ -\sqrt{2}\epsilon_{13} \end{pmatrix}. \tag{13}$$

Here the $C_{ijk\ell}$ are the cartesian components of the elasticity tensor field $\boldsymbol{\mathcal{C}}(x_1, x_2)$. Let us also introduce two-dimensional stress and strain fields

$$\boldsymbol{\sigma}' = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \boldsymbol{\epsilon}' = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{pmatrix}. \tag{14}$$

Now the three dimensional displacement field is necessarily of the form $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{v}(x_1, x_2) + x_3\boldsymbol{w}$ where $\boldsymbol{w}$ is constant, and this together with the constraints on the three dimensional stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ implies that

$$\nabla \times \boldsymbol{E}' = 0, \quad \nabla \cdot \boldsymbol{D}' = 0, \quad \nabla \cdot \boldsymbol{\sigma}' = 0, \quad \boldsymbol{\epsilon} = [\nabla \boldsymbol{u}' + (\nabla \boldsymbol{u}')^T]/2, \tag{15}$$

and that ϵ_{33} [like θ in the thermoelastic problem (7)] is constant, where $\boldsymbol{u}' = (v_1, v_2)$ is a two dimensional displacement field. Thus if we interpret $\boldsymbol{E}'$ and $\boldsymbol{D}'$ as two-dimensional electric and electric displacement fields, then the equation $\boldsymbol{g} = \boldsymbol{L}\boldsymbol{h}$ can be regarded as a two-dimensional piezoelectric equation incorporating a positive definite symmetric tensor $\boldsymbol{L}(\boldsymbol{x})$, which will have an associated effective tensor $\boldsymbol{L}_*$. Since the field $\sigma_{33}(x_1, x_2)$ is not subject to any differential constraints we can drop it from the equation (10). By applying the uniform field argument we thereby obtain expressions for the components of the three-dimensional effective elasticity tensor $\boldsymbol{\mathcal{C}}_*$ in terms of the components of the two-dimensional effective piezoelectric tensor $\boldsymbol{L}_*$, assuming it is a two-phase medium. If the medium has more than two phases then, by setting $\epsilon_{33} = 0$, 15 of the 21 components of $\boldsymbol{\mathcal{C}}_*$ can be determined from the elements of $\boldsymbol{L}_*$.

When the elasticity tensors $\boldsymbol{\mathcal{C}}_1$ and $\boldsymbol{\mathcal{C}}_2$ of the two phases are both invariant under the reflection transformation $x_3 \to -x_3$ then the two-dimensional piezelectric problem decouples into a planar elastic problem and a two-dimensional dielectric problem (the antiplane elastic problem). In particular if both phases are elastically isotropic and the composite is transversely isotropic, then the relation between $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{L}_*$ reduces to Hill's formulae [17] for $\boldsymbol{\mathcal{C}}_*$ in terms of the effective bulk and shear moduli of the planar elastic problem and the effective axial shear modulus of the antiplane elastic problem.

3. TRANSLATING BY A NULL–LAGRANGIAN

The translation discussed below originates in the work of Lurie and Cherkaev [18] on the plate equation. Its existence accounts [19] for certain invariance properties of the stress field discovered by Dundurs [20]. More general stress invariance properties, under a linear rather than a constant shift of the compliance tensor, have recently been discovered by Dundurs and Markenscoff [21]. The following analysis is largely based on the papers of Cherkaev, Lurie and Milton [22] and Thorpe and Jasiuk [19].

In a two-dimensional, simply-connected, possibly inhomogeneous, elastic body with no body forces present the components of the stress field $\boldsymbol{\sigma}$ can be expressed in terms of a potential ϕ, known as the Airy stress function, through the equations $\sigma_{11} = \phi_{,22}$, $\sigma_{12} = -\phi_{,12}$ and $\sigma_{22} = \phi_{,11}$. Here as elsewhere we use a comma in a subscript to denote differentiation with respect to the indices that follow the comma: thus, for example, $\phi_{,22} = \partial^2\phi/\partial x_2^2$. The relation between the stress and Airy stress function can be expressed in the equivalent form

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{R}}\nabla\nabla\phi \quad \text{where} \quad \nabla\nabla\phi = \begin{pmatrix} \phi_{,11} & \phi_{,12} \\ \phi_{,12} & \phi_{,22} \end{pmatrix}, \tag{16}$$

and $\boldsymbol{\mathcal{R}}$ is the fourth order tensor with cartesian elements

$$\mathcal{R}_{ijk\ell} = \delta_{ij}\delta_{k\ell} - (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2, \tag{17}$$

whose action in two-dimensions is to rotate a matrix by 90°. Now the key point is to recognize that $\nabla\nabla\phi$ satisfies the same differential constraints as a strain field: it derives from the "displacement field" $\nabla\phi$. In other words, the stress field rotated at each point by 90° produces a strain field. We use this observation to rewrite any solution of the two-dimensional elasticity equations

$$\boldsymbol{\epsilon} = \boldsymbol{\mathcal{S}}\boldsymbol{\sigma}, \quad \boldsymbol{\epsilon} = [\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]/2, \quad \nabla\cdot\boldsymbol{\sigma} = 0, \tag{18}$$

in the equivalent form

$$\boldsymbol{\epsilon}' = \boldsymbol{\mathcal{S}}'\boldsymbol{\sigma}, \quad \boldsymbol{\epsilon}' = [\nabla \boldsymbol{u}' + (\nabla \boldsymbol{u}')^T]/2, \quad \nabla \cdot \boldsymbol{\sigma} = 0, \tag{19}$$

where

$$\boldsymbol{\mathcal{S}}' = \boldsymbol{\mathcal{S}} + t\boldsymbol{\mathcal{R}}, \qquad \boldsymbol{u}' = \boldsymbol{u} + \nabla\phi. \tag{20}$$

Evidently if the strain and stress fields $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ solve the elasticity equations in a medium with compliance tensor $\boldsymbol{\mathcal{S}}$ then the strain and stress fields $\boldsymbol{\epsilon}'$ and $\boldsymbol{\sigma}$ solve the elasticity equations in a translated medium with compliance tensor $\boldsymbol{\mathcal{S}}'$. The basic Euler-Lagrange equations for the Airy stress function are the same in both media: hence the name null-Lagrangian. General characterizations of null-Lagrangians, or quasicontinuous functionals have been given by Ball, Currie and Olver [23] and by Murat [24].

When the medium under consideration is a composite, we have from (19) that

$$\langle \boldsymbol{\epsilon}' \rangle = \langle \boldsymbol{\mathcal{S}}\boldsymbol{\sigma} \rangle + t\boldsymbol{\mathcal{R}}\langle \boldsymbol{\sigma} \rangle = (\boldsymbol{\mathcal{S}}_* + t\boldsymbol{\mathcal{R}})\langle \boldsymbol{\sigma} \rangle. \tag{21}$$

Since it is this linear relation which defines the effective tensor $\boldsymbol{\mathcal{S}}'_*$ of the translated medium we deduce that

$$\boldsymbol{\mathcal{S}}'_* = \boldsymbol{\mathcal{S}}_* + t\boldsymbol{\mathcal{R}}. \tag{22}$$

Thus the effective tensor undergoes precisely the same translation as the local tensor.

For example, consider a locally isotropic planar elastic material which is macroscopically elastically isotropic. The local compliance tensor $\boldsymbol{\mathcal{S}}(\boldsymbol{x})$ and effective compliance tensor $\boldsymbol{\mathcal{S}}_*$ have elements

$$\begin{aligned} \mathcal{S}_{ijk\ell}(\boldsymbol{x}) &= (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2E(\boldsymbol{x}) - [\delta_{ij}\delta_{k\ell} - (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2]\nu(\boldsymbol{x})/E(\boldsymbol{x}), \\ \mathcal{S}^*_{ijk\ell}(\boldsymbol{x}) &= (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2E_* - [\delta_{ij}\delta_{k\ell} - (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2]\nu_*/E_*, \end{aligned} \tag{23}$$

where $E(\boldsymbol{x})$ and E_* are the local and effective in plane Young's modulus and $\nu(\boldsymbol{x})$ and ν_* are the local and effective in plane Poisson's ratio. It follows from (17) and (20) that under translation these moduli transform to

$$E'(\boldsymbol{x}) = E(\boldsymbol{x}), \qquad \nu'(\boldsymbol{x}) = \nu(\boldsymbol{x}) - tE(\boldsymbol{x}), \tag{24}$$

i.e. the Young's modulus remains unchanged, while the ratio of the Poisson's ratio to Young's modulus is shifted uniformly by $-t$. Under this translation the result (22) implies that the effective Young's modulus E_* and Poisson's ratio ν_* transform in a similar fashion,

$$E'_* = E_*, \qquad \nu'_* = \nu_* - tE_*. \tag{25}$$

A nice application of this result is to a metal plate with constant moduli E and ν that has a statistically isotropic distribution of holes punched into it. Under the translation (24) the holes remain holes (since the holes effectively correspond to a material with zero Young's modulus) while the Young's modulus E of the metal is unchanged, and its Poisson's ratio is shifted from ν to $\nu - tE$. By dimensional analysis it is apparent that the ratio, E_*/E can only depend on ν and on the geometry. But (25) implies this ratio remains invariant as t and hence ν varies. We conclude that E_*/E only depends on the geometry, and is not influenced by ν. This result was observed numerically by Day, Snyder, Garboczi and Thorpe [25] and subsequently proved by Cherkaev, Lurie and Milton [22]. The extent to which it holds in three dimensions was explored by Christensen [26].

Moreover when there are so many holes that the plate is about to fall apart, then E_* is close to zero, and (25) implies that ν_* is also independent of ν in this limit [25,19]. This is a striking result: no matter what the geometry of the configuration happens to be (so long as it is just about to fall apart) the effective Poisson's ratio takes a universal value which is independent of both the Young's modulus and the Poisson's ratio of the plate.

Translations are also useful for deriving microstructure independent results in the context of three-dimensional elasticity. The following is an extension of a two-dimensional argument [18,22,19] used to rederive Hill's result [17] for the effective bulk modulus of a locally isotropic planar elastic medium with constant shear modulus. [22,19].

First consider the rather extreme example of a locally isotropic three-dimensional medium with a compliance tensor $\boldsymbol{\mathcal{S}}(\boldsymbol{x})$ with cartesian elements

$$\mathcal{S}_{ijk\ell}(\boldsymbol{x}) = \delta_{ij}\delta_{k\ell}/9\kappa(\boldsymbol{x}). \tag{26}$$

This material has infinite shear modulus and finite bulk modulus $\kappa(\boldsymbol{x})$, i.e. at each point its Poisson's ratio is -1. (Although seemingly unphysical, materials with Poisson's ratio arbitrarily close to -1 can in fact be constructed: see [27,28] and references therein.) The deformation of the material is conformal since any change of angles corresponds to shear. In three dimensions the only conformal mappings are inversion in a sphere and uniform dilation, and since we are looking for periodic solutions the first can be ruled out. Therefore the strain $\boldsymbol{\epsilon}(\boldsymbol{x})$ must equal $\alpha\boldsymbol{I}$ where α is constant. The three dimensional stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ being symmetric and divergence-free derives from a 3×3 symmetric matrix valued potential $\boldsymbol{\phi}(\boldsymbol{x})$:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \nabla \times (\nabla \times \boldsymbol{\phi}(\boldsymbol{x}))^T, \tag{27}$$

Substituting this in the constitutive equation $\boldsymbol{\mathcal{S}}\boldsymbol{\sigma} = \boldsymbol{\epsilon} = \alpha\boldsymbol{I}$ gives an equation for the matrix potential $\boldsymbol{\phi}$:

$$\mathrm{Tr}[\nabla \times (\nabla \times \boldsymbol{\phi}(\boldsymbol{x}))^T] = 9\kappa(\boldsymbol{x})\alpha. \tag{28}$$

Since this is a single equation it seems plausible to look for solutions of the form

$$\boldsymbol{\phi}(\boldsymbol{x}) = \phi(\boldsymbol{x})\boldsymbol{I} \quad \text{with} \quad \phi(\boldsymbol{x}) = \phi_0(\boldsymbol{x}) + (\boldsymbol{x}\cdot\boldsymbol{x})f, \tag{29}$$

in which the scalar function $\phi_0(\boldsymbol{x})$ is periodic and f is a constant determining the average value of the stress. The associated stress field is

$$\boldsymbol{\sigma} = \boldsymbol{I}\Delta\phi - \nabla\nabla\phi, \tag{30}$$

which when substituted in the constitutive law gives the equation

$$2\Delta\phi_0 = 9\kappa(\boldsymbol{x})\alpha - 6f, \tag{31}$$

for the periodic potential ϕ_0. This has a solution if and only if the average value of the right hand side is zero, which thereby determines the value of f and the associated average value of the stress:

$$f = 3\langle\kappa(\boldsymbol{x})\rangle\alpha/2, \quad \langle\boldsymbol{\sigma}\rangle = 2f\boldsymbol{I} = 3\langle\kappa(\boldsymbol{x})\rangle\alpha\boldsymbol{I} = 3\langle\kappa(\boldsymbol{x})\rangle\langle\boldsymbol{\epsilon}\rangle. \tag{32}$$

Clearly the effective bulk modulus of this composite is microstructure independent and equal to

$$\kappa_* = \langle\kappa(\boldsymbol{x})\rangle. \tag{33}$$

Now let $\boldsymbol{\mathcal{T}}$ denote the fourth order tensor with cartesian elements

$$\mathcal{T}_{ijk\ell} = \delta_{ij}\delta_{k\ell}/2 - (\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})/2. \tag{34}$$

This tensor has the important property that for stresses of the form (30) the field $\boldsymbol{\mathcal{T}}\boldsymbol{\sigma} = \nabla\nabla\phi$ satisfies the same differential constraints as a strain: it derives from the "displacement field" $\nabla\phi$. The translated medium $\boldsymbol{\mathcal{S}} + t\boldsymbol{\mathcal{T}}$ now will have three dimensional inverse bulk and shear moduli

$$1/\kappa'(\boldsymbol{x}) = 1/\kappa(\boldsymbol{x}) + 3t/2, \quad 1/\mu' = -2t. \tag{35}$$

By applying the same sort of analysis as before it follows that inverse effective bulk and shear moduli of the translated medium equal

$$1/\kappa'_* = 1/\kappa_* + 3t/2 = 1/\langle\kappa(\boldsymbol{x})\rangle + 3t/2, \quad 1/\mu'_* = -2t. \tag{36}$$

By combining these formulae we see that a locally isotropic elastic medium with constant shear modulus μ' and bulk modulus $\kappa'(\boldsymbol{x})$ has effective shear and bulk moduli μ'_* and κ'_* given by

$$\mu'_* = \mu, \quad \frac{1}{4/\kappa'_* + 3/\mu'} = \left\langle\frac{1}{4/\kappa'(\boldsymbol{x}) + 3/\mu'}\right\rangle \tag{37}$$

which is Hill's result [17].

4. DUALITY FOR ANTIPLANE ELASTICITY

The duality relations for antiplane elasticity have their origins in the work of Keller [29] and Dykhne [30]. Mendelson [31], upon whose work the following treatment is based, extended their analysis to arbitrary inhomogeneous anisotropic planar media.

Let $u(x_1, x_2)$ be the vertical displacement in a state of anti-plane shear and let $\varphi(x_1, x_2)$ be the shear stress potential. The equations of antiplane elasticity take the form

$$\begin{pmatrix}\sigma_{31}\\ \sigma_{32}\end{pmatrix} = \boldsymbol{m}(\boldsymbol{x})\begin{pmatrix}2\epsilon_{31}\\ 2\epsilon_{32}\end{pmatrix}, \quad \begin{pmatrix}\sigma_{31}\\ \sigma_{32}\end{pmatrix} = \begin{pmatrix}\varphi_{,2}\\ -\varphi_{,1}\end{pmatrix}, \quad \begin{pmatrix}2\epsilon_{31}\\ 2\epsilon_{32}\end{pmatrix} = \begin{pmatrix}u_{,1}\\ u_{,2}\end{pmatrix}, \tag{38}$$

where $\boldsymbol{m}(x_1, x_2)$ is a symmetric 2×2 anti-plane shear elasticity matrix. Now let us introduce a new vertical displacement $u'(x_1, x_2) = \varphi(x_1, x_2)$ and a new shear stress potential

$\varphi'(x_1, x_2) = -u(x_1, x_2)$. The associated stress and strain field components satisfy the relations

$$\begin{pmatrix} \sigma'_{31} \\ \sigma'_{32} \end{pmatrix} = \begin{pmatrix} \varphi'_{,2} \\ -\varphi'_{,1} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} 2\epsilon_{31} \\ 2\epsilon_{32} \end{pmatrix}, \quad \begin{pmatrix} 2\epsilon'_{31} \\ 2\epsilon'_{32} \end{pmatrix} = \begin{pmatrix} u'_{,1} \\ u'_{,2} \end{pmatrix} = \boldsymbol{R}_\perp \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \end{pmatrix}, \tag{39}$$

where

$$\boldsymbol{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{40}$$

is the matrix for a 90° rotation. In two dimensions a curl free vector field when rotated pointwise by 90° produces a divergence free vector field and vice-versa. This key fact explains why the new stress and strain fields given by (39) satisfy the required differential constraints. These fields are linked through the constitutive relation

$$\begin{pmatrix} \sigma'_{31} \\ \sigma'_{32} \end{pmatrix} = \boldsymbol{m}' \begin{pmatrix} 2\epsilon'_{31} \\ 2\epsilon'_{32} \end{pmatrix}, \tag{41}$$

in which

$$\boldsymbol{m}'(\boldsymbol{x}) = [\boldsymbol{R}_\perp^T \boldsymbol{m}(\boldsymbol{x}) \boldsymbol{R}_\perp]^{-1} = \boldsymbol{m}(\boldsymbol{x})/\det[\boldsymbol{m}(\boldsymbol{x})]. \tag{42}$$

In other words these potentials solve the antiplane shear problem in a dual medium with anti-plane shear elasticity matrix $\boldsymbol{m}'(\boldsymbol{x})$.

By taking averages of the fields we deduce that the dual medium has effective shear matrix

$$\boldsymbol{m}'_* = [\boldsymbol{R}_\perp^T \boldsymbol{m}_* \boldsymbol{R}_\perp]^{-1} = \boldsymbol{m}_*/\det[\boldsymbol{m}_*], \tag{43}$$

where $\boldsymbol{m}_*$ is the effective shear matrix of the original medium. Thus duality relations link the effective tensors of two different media. If the medium is a composite of two isotropic phases, then $\boldsymbol{m}(\boldsymbol{x})$ and $\boldsymbol{m}'(\boldsymbol{x})$ take the form

$$\boldsymbol{m}(\boldsymbol{x}) = \chi(\boldsymbol{x})\mu_1 \boldsymbol{I} + (1 - \chi(\boldsymbol{x}))\mu_2 \boldsymbol{I}, \quad \boldsymbol{m}'(\boldsymbol{x}) = [\chi(\boldsymbol{x})\mu_2 \boldsymbol{I} + (1 - \chi(\boldsymbol{x}))\mu_1 \boldsymbol{I}]/\mu_1\mu_2, \tag{44}$$

where μ_1 and μ_2 are the shear moduli of the two phases and $\chi(\boldsymbol{x})$ is the characteristic function representing the microstructure of phase 1 (taking the value 1 when $\boldsymbol{x}$ is in phase 1 and zero otherwise.) So, the phase interchanged medium is obtained from the dual medium by multiplying $\boldsymbol{m}'(\boldsymbol{x})$ by the factor $\mu_1\mu_2$ and therefore its effective tensor is obtained by multiplying $\boldsymbol{m}'_*$ by the same factor $\mu_1\mu_2$. If we consider the effective tensor $\boldsymbol{m}_*$ as a function $\boldsymbol{m}_*(\mu_1, \mu_2)$ of the two-phases then (43) implies

$$\boldsymbol{m}_*(\mu_2, \mu_1) = \mu_1\mu_2 \boldsymbol{m}_*(\mu_1, \mu_2)/\det[\boldsymbol{m}_*(\mu_1, \mu_2)]. \tag{45}$$

It may happen that the geometry is phase interchange invariant like a checkerboard. Then $\boldsymbol{m}_*(\mu_2, \mu_1) = \boldsymbol{m}_*(\mu_1, \mu_2)$ and it follows from the above equation that $\det[\boldsymbol{m}_*(\mu_1, \mu_2)] = \mu_1\mu_2$. In particular if the effective shear matrix is isotropic then we have $\boldsymbol{m}_* = \mu_* \boldsymbol{I}$ where $\mu_* = \sqrt{\mu_1\mu_2}$ [30]. By this procedure we have obtained an exact expression for the effective antiplane shear modulus μ_* of the composite without solving for the fields directly.

5. DUALITY FOR PLANAR ELASTICITY

The duality relations for incompressible planar elastic media discussed here are due to Berdichevski [32]. The extension of the duality relations to compressible planar elastic media with a constant bulk modulus and to certain other anisotropic planar media is due to Helsing, Milton and Movchan [33].

Consider a planar elastic medium which is incompressible at each point. Since $\nabla\cdot\boldsymbol{u}=0$ there exists a potential $\psi(\boldsymbol{x})$ such that $u_1=\psi_{,2}$ and $u_2=-\psi_{,1}$. Let us introduce the matrices

$$\boldsymbol{a}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \boldsymbol{a}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \boldsymbol{a}_3=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix},\tag{46}$$

as a basis on the space of 2×2 symmetric matrices. The two-dimensional stress and strain fields $\boldsymbol{\sigma}(\boldsymbol{x})$ and $\boldsymbol{\epsilon}(\boldsymbol{x})$ can be expanded in this basis,

$$\boldsymbol{\sigma}(\boldsymbol{x})=\sigma_1(\boldsymbol{x})\boldsymbol{a}_1+\sigma_2(\boldsymbol{x})\boldsymbol{a}_2+\sigma_3(\boldsymbol{x})\boldsymbol{a}_3,\quad \boldsymbol{\epsilon}(\boldsymbol{x})=\epsilon_2(\boldsymbol{x})\boldsymbol{a}_2+\epsilon_3(\boldsymbol{x})\boldsymbol{a}_3,\tag{47}$$

where the coefficients satisfy the equations

$$\begin{pmatrix}\epsilon_2\\\epsilon_3\end{pmatrix}=\boldsymbol{S}(\boldsymbol{x})\begin{pmatrix}\sigma_2\\\sigma_3\end{pmatrix},\quad \begin{pmatrix}\epsilon_2\\\epsilon_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}2\psi_{,12}\\\psi_{,22}-\psi_{,11}\end{pmatrix},\quad \begin{pmatrix}\sigma_2\\\sigma_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_{,22}-\phi_{,11}\\-2\phi_{,12}\end{pmatrix},\tag{48}$$

and $\sigma_1=(\phi_{,11}+\phi_{,22})/\sqrt{2}$. Here the 2×2 matrix $\boldsymbol{S}(\boldsymbol{x})$ represents the non-singular part of the compliance tensor in this basis and $\phi(\boldsymbol{x})$ is the Airy stress function. We now introduce dual potentials $\phi'(x_1,x_2)=\psi(x_1,x_2)$ and $\psi'(x_1,x_2)=-\phi(x_1,x_2)$ and the associated stress and strain field components

$$\begin{pmatrix}\epsilon'_2\\\epsilon'_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}2\psi'_{,12}\\\psi'_{,22}-\psi'_{,11}\end{pmatrix}=\boldsymbol{R}_\perp\begin{pmatrix}\sigma_2\\\sigma_3\end{pmatrix},\quad \begin{pmatrix}\sigma'_2\\\sigma'_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi'_{,22}-\phi'_{,11}\\-2\phi'_{,12}\end{pmatrix}=\boldsymbol{R}_\perp\begin{pmatrix}\epsilon_2\\\epsilon_3\end{pmatrix},\tag{49}$$

and $\sigma'_1=(\phi'_{,11}+\phi'_{,22})/\sqrt{2}$. These field components satisfy the constitutive relation

$$\begin{pmatrix}\epsilon'_2\\\epsilon'_3\end{pmatrix}=\boldsymbol{S}'(\boldsymbol{x})\begin{pmatrix}\sigma'_2\\\sigma'_3\end{pmatrix}\quad\text{where}\quad \boldsymbol{S}'(\boldsymbol{x})=[\boldsymbol{R}_\perp^T\boldsymbol{S}(\boldsymbol{x})\boldsymbol{R}_\perp]^{-1}=\boldsymbol{S}(\boldsymbol{x})/\det[\boldsymbol{S}(\boldsymbol{x})].\tag{50}$$

In other words the dual potentials solve the planar elasticity equations in an incompressible medium with $\boldsymbol{S}'(\boldsymbol{x})$ being the non-singular part of the compliance tensor in the basis (46). By taking averages of the fields we deduce that the non-singular part of the effective compliance tensor for the dual medium is

$$\boldsymbol{S}'_*=[\boldsymbol{R}_\perp^T\boldsymbol{S}_*\boldsymbol{R}_\perp]^{-1}=\boldsymbol{S}_*/\det[\boldsymbol{S}_*],\tag{51}$$

in which $\boldsymbol{S}_*$ is the non-singular part of the compliance tensor of the original medium in the basis (46).

As an example, consider an isotropic incompressible planar elastic composite of two isotropic phases. Then the matrices $\boldsymbol{S}(\boldsymbol{x})$ and $\boldsymbol{S}_*$ take the form

$$\boldsymbol{S}(\boldsymbol{x})=\chi(\boldsymbol{x})\boldsymbol{I}/(2\mu_1)+(1-\chi(\boldsymbol{x}))\boldsymbol{I}/(2\mu_2),\quad \boldsymbol{S}_*=\boldsymbol{I}/(2\mu_*),\tag{52}$$

where μ_1, μ_2 and μ_* are the shear moduli of the phases and composite, while $\chi(\boldsymbol{x})$ is the characteristic function representing the microstructure of phase 1 (taking the value 1 when $\boldsymbol{x}$ is in phase 1 and zero otherwise.) Then, by direct analogy with the relation (45) for antiplane shear, we see that the effective shear modulus μ_* as a function $\mu_*(\mu_1, \mu_2)$ of the shear moduli of the phases satisfies the phase interchange relation $\mu_*(\mu_1, \mu_2)\mu_*(\mu_2, \mu_1) = \mu_1\mu_2$. By translation we can extend this result to two phase planar elastic composites that have compressible phases sharing a common bulk modulus κ. Then the phase interchange relation takes the form

$$E_*(E_1, E_2)E_*(E_2, E_1) = E_1E_2, \tag{53}$$

where $E_*(E_1, E_2)$ is the effective in plane Young's modulus expressed as a function of the in-plane Young's moduli E_1 and E_2 of the two phases. In particular if the composite is phase interchange invariant, like a two-dimensional checkerboard, then (53) implies its effective in plane Young's modulus is $\sqrt{E_1E_2}$. If the bulk modulus is not the same in both phases then Gibiansky and Torquato [34] have shown that the effective elastic moduli of the composite and phase interchanged material are linked by inequalities which reduce to the relation (53) when the bulk moduli are equal.

A related example is that of an isotropic two-dimensional polycrystal of incompressible crystals. The individual crystals, being incompressible, necessarily have square symmetry and are characterized by two shear moduli $\mu^{(1)}$ and $\mu^{(2)}$. The duality result (51) implies that the effective shear modulus μ_* of the polycrystal is given by the formula $\mu_* = \sqrt{\mu^{(1)}\mu^{(2)}}$ of Lurie and Cherkaev [18]. Using translations they generalized this result to two-dimensional polycrystals, comprised of compressible grains with square symmetry and found that the effective shear modulus μ_* of the polycrystal is given by

$$\mu_* = \frac{\kappa}{-1 + \sqrt{(\kappa + \mu^{(2)})(\kappa + \mu^{(1)})/(\mu^{(1)}\mu^{(2)})}}, \tag{54}$$

where κ, $\mu^{(1)}$ and $\mu^{(2)}$ are the planar bulk and two shear moduli of the crystal.

More generally, planar elastic duality transformations can be applied whenever there exists a matrix $\boldsymbol{v}$ and constant t such that $(\boldsymbol{\mathcal{S}}(\boldsymbol{x}) - t\boldsymbol{\mathcal{R}})\boldsymbol{v} = 0$ for all $\boldsymbol{x}$ [33]. High accuracy numerical results for the effective compliance tensor of periodic media comprised of two orthotropic phases confirm the predictions of the theory.

6. LINKING ANTIPLANE AND PLANAR ELASTICITY PROBLEMS

Given that duality relations hold for both antiplane and planar elasticity, one might wonder if these problems are are linked in some way. Such a link would be a surprise because antiplane problems involve a second order shear matrix, whereas planar elastic problems involve a fourth order elasticity tensor. A formal similarity between incompressible elasticity and antiplane elasticity is known [35] but this does not provide a correspondence between the fields solving the planar and antiplane problems. Here we establish a direct correspondence. The ensuing analysis is based on the papers of Milton and Movchan [36,37] and Helsing, Milton and Movchan [33].

The constitutive relation in a simply connected, planar, locally orthotropic medium, with the axes of orthotropy aligned with the coordinate axes takes the form

$$\begin{pmatrix} u_{1,1} \\ u_{2,2} \\ (u_{1,2}+u_{2,1})/\sqrt{2} \end{pmatrix} = \boldsymbol{S}\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{21} \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_4 & 0 \\ 0 & 0 & s_6 \end{pmatrix}, \tag{55}$$

and the equilibrium constraint $\nabla \cdot \sigma = 0$ implies there exist stress potentials $\phi_1(\boldsymbol{x})$ and $\phi_2(\boldsymbol{x})$ such that

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \phi_{1,2} & \phi_{2,2} \\ -\phi_{1,1} & -\phi_{2,1} \end{pmatrix}. \tag{56}$$

Let us substitute these expressions back into the constitutive law and into the relation $\sigma_{12} = \sigma_{21}$, implied by symmetry of the stress field. Manipulating the resulting four equations so the terms involving derivatives with respect to x_2 appear on the left while terms involving derivatives with respect to x_1 appear on the right gives an equivalent form of the elasticity equations

$$\boldsymbol{\eta}_{,2} = \boldsymbol{N}\boldsymbol{\eta}_{,1} \tag{57}$$

introduced by Ingebrigtsen and Tonning [38], where

$$\boldsymbol{\eta} = \begin{pmatrix} u_1 \\ u_2 \\ \phi_1 \\ \phi_2 \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} 0 & -1 & -s_6 & 0 \\ s_2/s_1 & 0 & 0 & s_2^2/s_1 - s_4 \\ 1/s_1 & 0 & 0 & s_2/s_1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{58}$$

The matrix $\boldsymbol{N}(\boldsymbol{x})$ is known as the fundamental elasticity matrix. The associated effective fundamental elasticity matrix $\boldsymbol{N}_*$ governs the relation between the average fields,

$$\langle \boldsymbol{\eta}_{,2} \rangle = \boldsymbol{N}_* \langle \boldsymbol{\eta}_{,1} \rangle, \tag{59}$$

and is related to the effective compliance matrix $\boldsymbol{S}_*$ in the same way that the fundamental elasticity matrix $\boldsymbol{N}(\boldsymbol{x})$ is related to the local compliance matrix $\boldsymbol{S}(\boldsymbol{x})$.

Now notice that the equations can be rewritten in the equivalent form

$$\boldsymbol{\eta}'_{,2} = \boldsymbol{N}'\boldsymbol{\eta}'_{,1}, \quad \langle \boldsymbol{\eta}'_{,2} \rangle = \boldsymbol{N}'_* \langle \boldsymbol{\eta}'_{,1} \rangle, \tag{60}$$

where

$$\boldsymbol{\eta}'(\boldsymbol{x}) = \boldsymbol{K}\boldsymbol{\eta}(\boldsymbol{x}), \quad \boldsymbol{N}'(\boldsymbol{x}) = \boldsymbol{K}^{-1}\boldsymbol{N}(\boldsymbol{x})\boldsymbol{K}, \quad \boldsymbol{N}'_* = \boldsymbol{K}^{-1}\boldsymbol{N}_*\boldsymbol{K}, \tag{61}$$

and $\boldsymbol{K}$ is an arbitrary constant, non-singular 4×4 matrix. In other words, when the fundamental matrix field $\boldsymbol{N}(\boldsymbol{x})$ undergoes a constant similarity transformation then the effective fundamental matrix $\boldsymbol{N}_*$ undergoes the same similarity transformation. The translation of the compliance tensor discussed in section 3 corresponds a particular similarity transformation of the fundamental matrix, as do the duality transformations for antiplanar and planar elasticity. (For antiplane elasticity the associated fundamental matrix is a 2×2 matrix). Other duality transformations of the fundamental matrix form of the equations have been analysed by Nemat-Nasser and Ni [39].

Similar mappings between equivalent sets of equations, obtained by taking linear combinations of potentials and fluxes separately, have been applied to coupled field problems by Straley [1] and Milgrom and Shtrikman [2] among others. For media with isotropic phases they use these mappings to transform to a diagonal form of the equations where no couplings are present, and thereby obtain exact relations between the effective thermoelectric moduli in a two-phase medium. By mixing the potentials and flux potentials one obtains a more general class of equivalence transformations for two-dimensional coupled field problems (see Milton [40], Benveniste [5] and references therein). Under these the tensor entering the constitutive law undergoes a fractional linear transformation. Such transformations are equivalent to the similarity transformations of the fundamental matrices considered here. Working with the fundamental matrices has the advantage that the transformation takes a simpler form and is therefore easier to analyze.

For simplicity, let us suppose the moduli are such that for all $\boldsymbol{x}$

$$\Delta(\boldsymbol{x}) = (s_2(\boldsymbol{x}) + s_6(\boldsymbol{x}))^2 - s_1(\boldsymbol{x})s_4(\boldsymbol{x}) > 0. \tag{62}$$

Then the eigenvalues of the $\boldsymbol{N}(\boldsymbol{x})$ at each point $\boldsymbol{x}$ are

$$\lambda_1 = -\lambda_2 = -i\alpha_1, \quad \lambda_3 = -\lambda_4 = -i\alpha_2, \tag{63}$$

where $\alpha_1(\boldsymbol{x})$ and $\alpha_2(\boldsymbol{x})$ are the two real positive roots of the polynomial

$$s_1(\boldsymbol{x})\alpha^4 - 2(s_2(\boldsymbol{x}) + s_6(\boldsymbol{x}))\alpha^2 + s_4(\boldsymbol{x}) = 0. \tag{64}$$

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} -p_1 \\ i\alpha_1 p_2 \\ i\alpha_1 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} -p_1 \\ -i\alpha_1 p_2 \\ -i\alpha_1 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_3^{(j)} = \begin{pmatrix} -p_2 \\ i\alpha_2 p_1 \\ i\alpha_2 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_4^{(j)} = \begin{pmatrix} -p_2 \\ -i\alpha_2 p_1 \\ -i\alpha_2 \\ 1 \end{pmatrix}, \tag{65}$$

in which

$$p_1(\boldsymbol{x}) = -s_6(\boldsymbol{x}) + \sqrt{\Delta(\boldsymbol{x})}/2, \qquad p_2(\boldsymbol{x}) = -s_6(\boldsymbol{x}) - \sqrt{\Delta(\boldsymbol{x})}/2. \tag{66}$$

Now suppose p_1 and p_2 do not depend on $\boldsymbol{x}$. (This holds if and only if s_6 and Δ are both independent of $\boldsymbol{x}$.) Then $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will span a two-dimensional space that does not depend on $\boldsymbol{x}$, and $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ will span a two-dimensional space that does not depend on $\boldsymbol{x}$. Thus with an appropriate choice of $\boldsymbol{K}$ the matrix $\boldsymbol{N}'(\boldsymbol{x})$ will be block diagonal. Specifically, the choice

$$\boldsymbol{K} = \begin{pmatrix} -p_1 & 0 & 0 & -p_2 \\ 0 & p_2 & p_1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{gives} \quad \boldsymbol{N}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \alpha_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2^2 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{67}$$

As a consequence the equation $\boldsymbol{\eta}'_{,2} = \boldsymbol{N}'\boldsymbol{\eta}'_{,1}$ decouples into a pair of equations that can be expressed in the form

$$\begin{pmatrix} \eta'_{2,2} \\ -\eta'_{2,1} \end{pmatrix} = \boldsymbol{m}_1 \begin{pmatrix} \eta'_{1,1} \\ \eta'_{1,2} \end{pmatrix}, \quad \begin{pmatrix} \eta'_{3,2} \\ -\eta'_{3,1} \end{pmatrix} = \boldsymbol{m}_2 \begin{pmatrix} \eta'_{4,1} \\ \eta'_{4,2} \end{pmatrix}, \tag{68}$$

where $\boldsymbol{m}_1(\boldsymbol{x})$ and $\boldsymbol{m}_2(\boldsymbol{x})$ are the 2×2 matrix valued fields

$$\boldsymbol{m}_1 = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{m}_2 = \begin{pmatrix} \alpha_2^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{69}$$

These can be regarded as equations of antiplane elasticity in two different inhomogeneous anisotropic media with $\boldsymbol{m}_1(\boldsymbol{x})$ and $\boldsymbol{m}_2(\boldsymbol{x})$ being the antiplane shear matrix fields of these media. In other words, when s_6 is constant and Δ is constant and positive, the original planar elasticity equations can be reduced to a pair of uncoupled antiplane elasticity equations. The uniform field argument implies that when s_6 is constant the effective compliance matrix $\boldsymbol{S}_*$ is necessarily orthotropic with its axes aligned with the co-ordinate axes having $s_{*6} = s_6$. From the effective antiplane shear matrices

$$\boldsymbol{m}_{*1} = \begin{pmatrix} \alpha_{*1}^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{m}_{*2} = \begin{pmatrix} \alpha_{*2}^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{70}$$

associated with $\boldsymbol{m}_1(\boldsymbol{x})$ and $\boldsymbol{m}_2(\boldsymbol{x})$ we can compute the remaining elements s_{*1} s_{*2} and s_{*4} of the effective compliance matrix $\boldsymbol{S}_*$ associated with $\boldsymbol{S}(\boldsymbol{x})$ by solving the three equations

$$(s_{*2} + s_{*6})^2 - s_{*1}s_{*4} = \Delta, \quad s_{*1}\alpha_{*j}^4 - 2(s_{*2} + s_{*6})\alpha_{*j}^2 + s_{*4} = 0, \quad j = 1, 2. \tag{71}$$

This correspondence between the moduli of the effective antiplane shear matrices and the moduli of the effective compliance tensor has been verified numerically [33]. When s_6 is constant and Δ is constant and negative there is still a correspondence with antiplane elasticity: the original planar elasticity equations can then be reduced to a single viscoelastic antiplane problem, with a complex shear matrix field $\boldsymbol{m}(\boldsymbol{x})$.

7. A QUESTION OF PERCOLATION

The following is joint work with Yury Grabovsky. For more details, and for references to the relevant results from partial differential equation theory, see [41].

When we think of percolation it is usually in the context of current flow, where the current may be electrical, thermal or fluid flow. Perhaps one is considering a composite of two isotropic phases where phase 1 is permeable to current while phase 2 is impermeable to it. Or perhaps one is considering a polycrystal where each individual crystalline grain only allows the current to flow in certain directions within that crystal. At a percolation transition the rank of the effective conductivity or permeability tensor changes. For isotropic three-dimensional composites of two isotropic phases. the transition is from a tensor of rank 0, when phase 2 blocks all current flow, to a tensor of rank 3, when paths of phase 1 form a connected labyrinth of infinite extent. In the context of elasticity we can study the analogous percolation question: given that the local compliance tensor (or local elasticity tensor) is singular in some parts or in all of the material, is the rank of the effective compliance tensor (or effective elasticity tensor) dependent on the microstructure?

Curiously, the rank of the effective compliance tensor is independent of the microstructure in a planar elastic material where the local compliance tensor is rank 1, of the form

$$\boldsymbol{\mathcal{S}}(\boldsymbol{x}) = \boldsymbol{s}(\boldsymbol{x}) \otimes \boldsymbol{s}(\boldsymbol{x}), \tag{72}$$

where the 2×2 matrix valued field $\boldsymbol{s}(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x}$. Specifically, the effective elasticity tensor takes exactly the same rank 1 form,

$$\boldsymbol{\mathcal{S}}_* = \boldsymbol{s}_* \otimes \boldsymbol{s}_*, \tag{73}$$

where $\boldsymbol{s}_*$ is a 2×2 positive definite matrix. The assumption of positive definiteness of $\boldsymbol{s}(\boldsymbol{x})$ is necessary: it follows from the work Bhattacharya and Kohn [42] (see sections 5.2 and 5.3) that the existance of "percolating" stress or strain fields can be microstructure dependent when $\boldsymbol{s}(\boldsymbol{x})$ is not positive definite.

This result has a surprising corollary. If we take a two-dimensional planar elastic polycrystal constructed from a crystal with a positive definite compliance tensor $\boldsymbol{\mathcal{S}}_0$ such that the translated tensor $\boldsymbol{\mathcal{S}}_0' = \boldsymbol{\mathcal{S}}_0 - t\boldsymbol{\mathcal{R}}$ is rank 1 of the form $s_0' \otimes s_0'$ for some value of t, then necessarily the effective compliance tensor $\boldsymbol{\mathcal{S}}_*$ of the polycrystal must be such that the translated effective tensor $\boldsymbol{\mathcal{S}}_*' = \boldsymbol{\mathcal{S}}_* - t\boldsymbol{\mathcal{R}}$ is rank 1 of the form $s_*' \otimes s_*'$. (The positive definiteness of the tensor $\boldsymbol{\mathcal{S}}_0$ ensures the positive definiteness of the matrix s_0'.) In particular, if the polycrystal is elastically isotropic then its two-dimensional shear modulus is microstructure independent and equal to $1/(2t)$. The bulk modulus, by contrast, is microstructure dependent. This result was first derived [43] from the optimal bounds on the bulk and shear moduli of two-dimensional planar elastic polycrystals constructed from an orthotropic crystal.

A related result is that if a planar elasticity tensor field $\boldsymbol{\mathcal{C}}(\boldsymbol{x})$ is rank 2 with a positive definite matrix in its null-space for all $\mathbf{x}$, then the associated effective elasticity tensor $\boldsymbol{\mathcal{C}}_*$ is rank 2 with a positive definite matrix in its null-space.

The proof of (73) is technical and rests upon certain results from elliptic partial differential equation theory. Only those readers interested in getting a rough idea of the steps involved should read the remainder of this section as it is rather condensed.

If we apply an average stress field such that the resulting strain field in the material is non-zero then this "percolating" strain field $\boldsymbol{\epsilon}(\boldsymbol{x})$ must be of the form

$$\boldsymbol{\epsilon}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\boldsymbol{s}(\boldsymbol{x}), \tag{74}$$

for some scalar field $\alpha(\boldsymbol{x})$. The infinitesimal strain compatibility condition that $\nabla \cdot (\nabla \cdot \boldsymbol{\mathcal{R}}\boldsymbol{\epsilon}(\boldsymbol{x})) = 0$ (which ensures that $\boldsymbol{\epsilon}(\boldsymbol{x})$ derives from some displacement field) requires that $\alpha(\boldsymbol{x})$ satisfies the second order elliptic partial differential equation,

$$\nabla \cdot (\nabla \cdot (\alpha(\boldsymbol{x})\hat{\boldsymbol{s}}(\boldsymbol{x})) = 0 \quad \text{where} \quad \hat{\boldsymbol{s}}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}}\boldsymbol{s}(\boldsymbol{x}). \tag{75}$$

If we impose the normalization constraint that $\langle \alpha \rangle = 1$, then it is known from partial differential equation theory that a solution to the above equation for $\alpha(\boldsymbol{x})$ exists and is unique. [For example, when $\boldsymbol{s}(\boldsymbol{x}) = \beta(\boldsymbol{x})\boldsymbol{I}$ where $\beta(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$, as in (26) but in two dimensions, the solution is $\alpha(\boldsymbol{x}) = \langle 1/\beta \rangle / \beta(\boldsymbol{x})$]. From the solution for $\alpha(\boldsymbol{x})$ we determine the average strain

$$\langle \boldsymbol{\epsilon}(\boldsymbol{x}) \rangle = \langle \alpha(\boldsymbol{x})\boldsymbol{s}(\boldsymbol{x}) \rangle = \alpha_* \boldsymbol{s}_*(\boldsymbol{x}), \tag{76}$$

where α_* is some constant. Thus up to a proportionality constant, we can determine the matrix $\boldsymbol{s}_*(\boldsymbol{x})$ from the solution for $\alpha(\boldsymbol{x})$. The uniqueness of the solution for $\alpha(\boldsymbol{x})$ is what guarantees that $\boldsymbol{\mathcal{S}}_*$ is at most rank 1, of the form (73). It is known that the field $\alpha(\boldsymbol{x})$ is positive everywhere and consequently $\boldsymbol{s}_*$ is either a positive definite or negative definite matrix, depending on the sign of α_*. Since we are free to change the signs of $\boldsymbol{s}_*$ and α_* we can take $\boldsymbol{s}_*$ to be positive definite.

The associated stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ must be such that

$$\mathrm{Tr}(\boldsymbol{s}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})) = \alpha(\boldsymbol{x}). \tag{77}$$

By substituting the relation (16) into this we obtain another second order elliptic partial differential equation,

$$\mathrm{Tr}(\hat{\boldsymbol{s}}(\boldsymbol{x})\nabla\nabla\phi) = \alpha(\boldsymbol{x}), \tag{78}$$

this time for the Airy stress function $\phi(\boldsymbol{x})$, which can be taken to have the form

$$\phi(\boldsymbol{x}) = \phi_0(\boldsymbol{x}) + \boldsymbol{x}\cdot\boldsymbol{F}\boldsymbol{x}, \tag{79}$$

where $\phi_0(\boldsymbol{x})$ is periodic and the constant matrix $\boldsymbol{F}$ is determined by the average value of the stress field: $\langle\boldsymbol{\sigma}\rangle = 2\boldsymbol{\mathcal{R}}\boldsymbol{F}$. Thus the periodic function $\phi_0(\boldsymbol{x})$ satisfies

$$\mathrm{Tr}(\hat{\boldsymbol{s}}(\boldsymbol{x})\nabla\nabla\phi_0) = \alpha(\boldsymbol{x}) - 2\mathrm{Tr}(\hat{\boldsymbol{s}}(\boldsymbol{x})\boldsymbol{F}). \tag{80}$$

From elliptic partial differential equation theory it is known that (80) has a unique solution for $\phi_0(\boldsymbol{x})$ if and only if the right hand side is orthogonal to $\alpha(\boldsymbol{x})$, i.e. if and only if

$$0 = \langle\alpha^2 - 2\mathrm{Tr}(\alpha\hat{\boldsymbol{s}}\boldsymbol{F})\rangle = \langle\alpha^2\rangle - \alpha_*\mathrm{Tr}(\boldsymbol{s}_*\langle\boldsymbol{\sigma}\rangle)\rangle, \tag{81}$$

where we have used (76). [When $s(\boldsymbol{x}) = \beta(\boldsymbol{x})\boldsymbol{I}$ and $\alpha(\boldsymbol{x}) = \langle 1/\beta\rangle/\beta(\boldsymbol{x})$ equation (80) becomes $\Delta\phi_0 = \langle 1/\beta\rangle/\beta^2 - 2\mathrm{Tr}(\boldsymbol{F})$ and because $\Delta\phi_0$ has zero average value so must $\langle 1/\beta\rangle/\beta^2 - 2\mathrm{Tr}(\boldsymbol{F})$ which accounts, in this case, for the condition (81).]

From (76) and the effective constitutive law we have $\mathrm{Tr}(\boldsymbol{s}_*\langle\boldsymbol{\sigma}\rangle) = \alpha_*$ which with (81) gives $\alpha_* = \langle\alpha^2\rangle^{1/2}$. Thus, we can determine the effective compliance tensor completely by solving (75) for $\alpha(\boldsymbol{x})$ and using (76) and the identity $\alpha_* = \langle\alpha^2\rangle^{1/2}$ to determine $\boldsymbol{s}_*$.

Incidentally, percolation type questions often arise in the context of finding optimal microgeometries that attain bounds on effective moduli. For example, the task of finding three-dimensional polycrystalline microstructures that have the lowest possible effective conductivity is equivalent [44,45] to the task of finding non-trivial periodic rotation fields $\boldsymbol{R}(\boldsymbol{x})$ (satisfying $\boldsymbol{R}(\boldsymbol{x})^T\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{I}$) such that the equation

$$\nabla\boldsymbol{u}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x})^T\boldsymbol{A}\boldsymbol{R}(\boldsymbol{x}) \tag{82}$$

has a solution for the vector potential $\boldsymbol{u}(\boldsymbol{x})$ for some choice of scalar field $\alpha(\boldsymbol{x})$, where $\boldsymbol{A}$ is a given positive definite 3×3 diagonal matrix. Notice the similarity of (74) and (82).

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