Green's Function of the Biharmonic Operator is not Positive Definite

John “Peter” Whitney
ES 240 Fall 2007

January 11, 2008

Green's Function

For the following inhomogeneous linear differential equation,

$$Lu(x) = f(x),$$

The Green's function is defined as

$$LG(x,y) = \delta(x-y).$$

If the Green's function is known, the solution can be formed by the integral

$$u(x) = \int G(x,y) f(y) dy.$$

For electrostatics, the Green's function of the Laplacian represents the electric field due to a unit point charge. For the biharmonic equation, we can interpret it as the deflection field of a thin plate due to a point load.

The Green's function of the Laplacian is positive definite. Consider

$$\nabla^2 u = f.$$

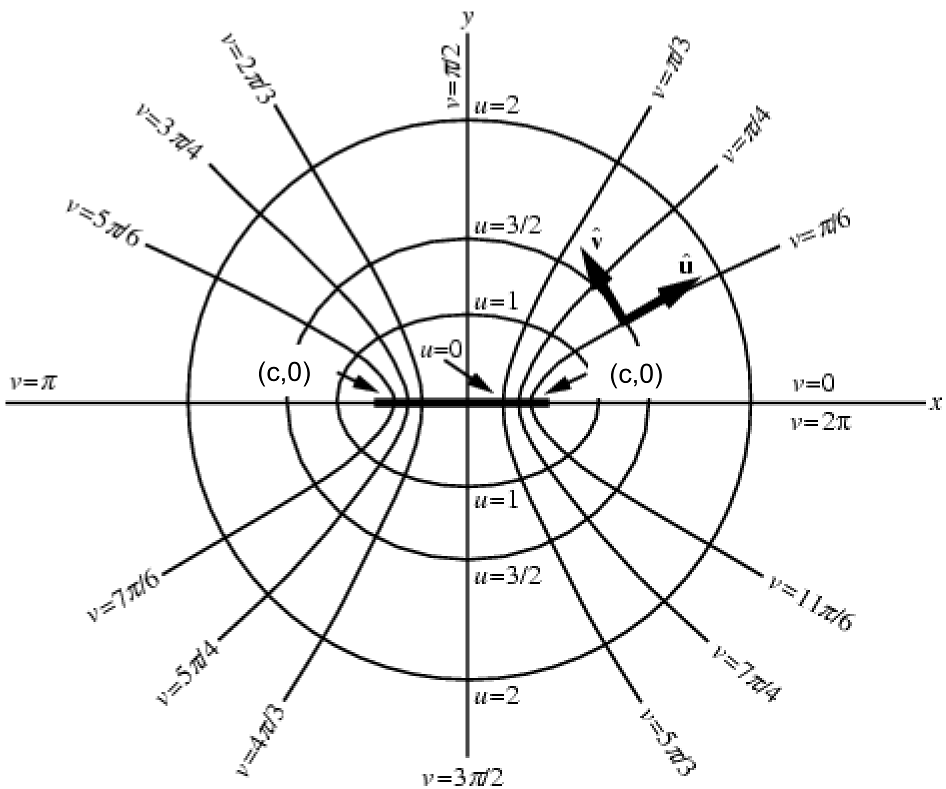
This equation models the deflection of a thin elastic membrane. Since the Laplacian is positive definite, a point load results in all deflected points moving in the direction of the applied force. Hadamard (1908) proposed that this might also be true of the biharmonic equation (which models a thin rigid plate instead of an elastic membrane).

But this is not true!

Hadamard's conjecture was disproven by counter example in 1949 by Duffin (semi-infinite strip), and in 1951 by Garabedian, for a finite, but sufficiently eccentric ellipse.

We will show this behavior by constructing the analytical solution for a point-loaded elliptical plate. We will also make mention of a curious feature of the eigenfunctions of the analogous eigenvalue problem, which is related to the non-positive definiteness of the biharmonic operator.

Elliptical Coordinates



$$x = c \cosh u \cos v$$

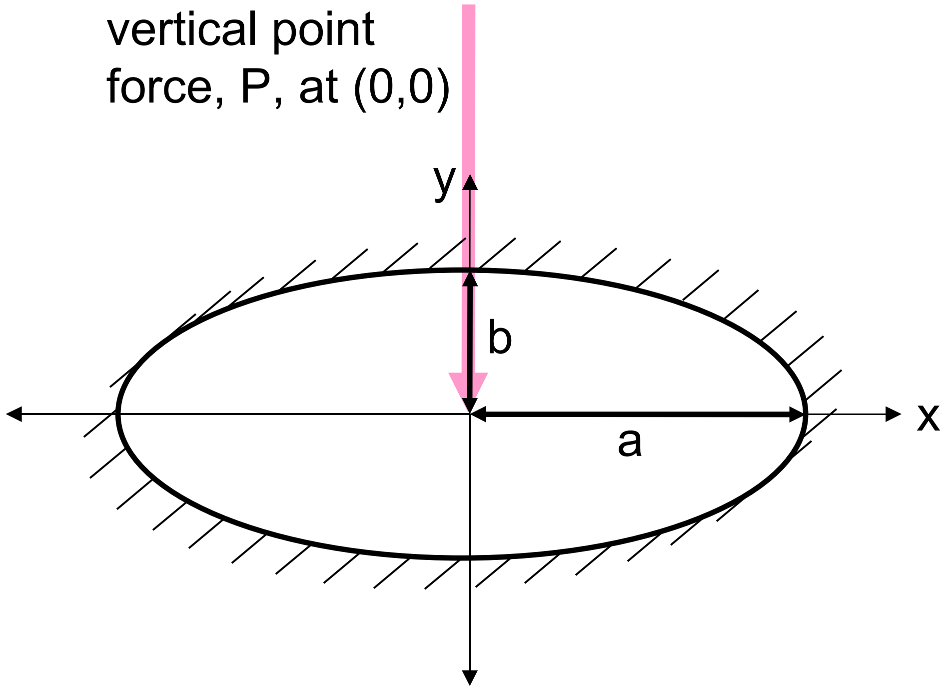
$$y = c \sinh u \sin v$$

boundary ellipse equation:

$$u_1 = \sinh^{-1}\left(\frac{1}{c}\right)$$

picture from *http://mathworld.wolfram.com/EllipticCylindricalCoordinates.html*

Clamped Elliptical Plate, Point Load



Clamped boundary:

$$w = 0$$

$$\frac{\partial w}{\partial u} = 0$$

$$c = \sqrt{\left(\frac{a}{b}\right)^2 - 1}$$

scaled deflection:

$$\overline{w} = \frac{wD}{P}$$

Thin Plate Equations

$$\nabla^2 \nabla^2 w = \frac{p}{K}$$

$$K = \frac{Eh^2}{12(1-\nu^2)}$$

$$w = w_p + w_h$$

particular solution (Cartesian) $w_p = \frac{P}{16\pi K}\left(x^2 + y^2\right)\log\left(x^2 + y^2\right)$

particular solution (elliptical) $w_p = \frac{Pc^2}{16\pi K}\left[\alpha_0(u) + \sum_{n=1}^{\infty} \alpha_n(u)\cos(n\nu)\right]$

(The equations for the alphas are quite complex, and are omitted here. See reference given on next slide for details.)

Homogeneous solution is composed of a sum of hyperbolic cosines in u, and cosines in nu.

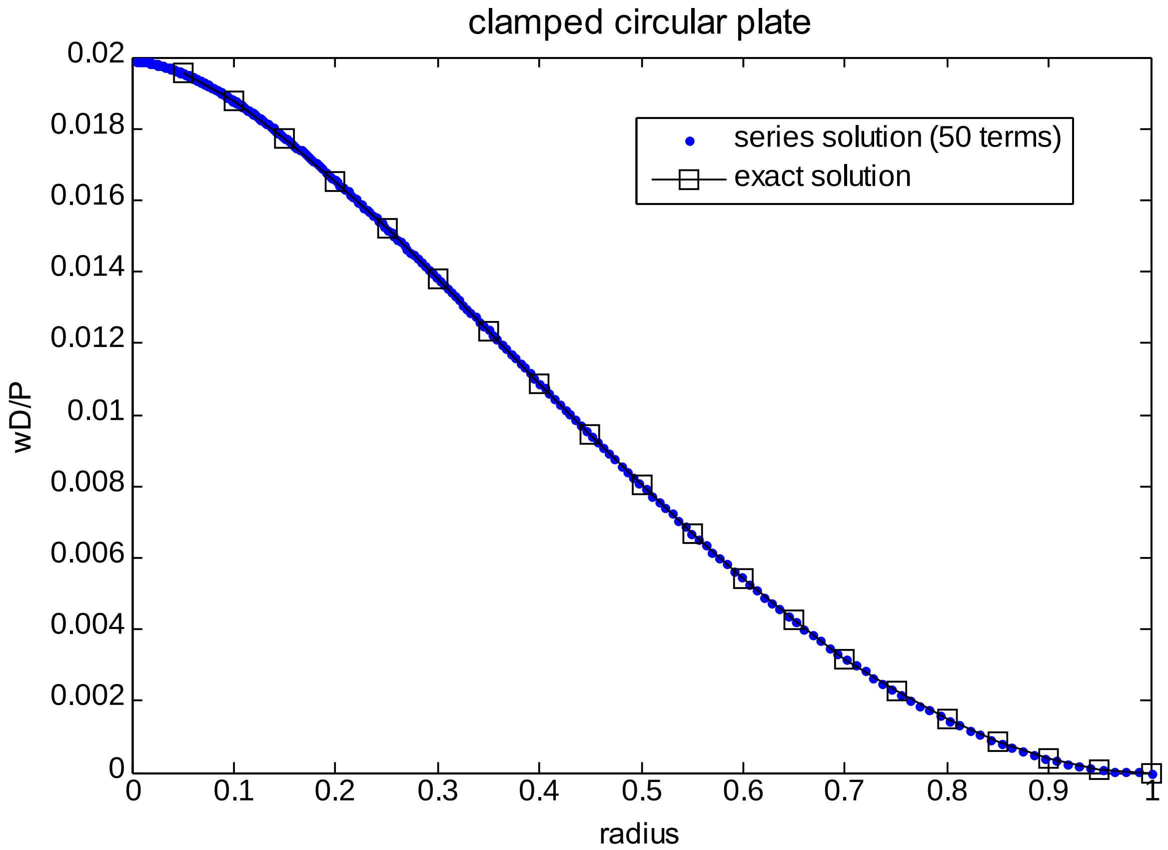
“Missing” odd terms are required if point load is not at origin (not considered here).

$$w_h = K_0 + E_0 \cosh(2u) + \sum_{n=2}^{\infty}\left[A_n \cosh(nu) + E_{n-2}\cosh((n-2)u) + E_n \cosh((n+2)u)\right]\cos(n\nu)$$

Constants are found by adding homogeneous solution to particular solution, and ensuring the boundary conditions are satisfied (details omitted).

Solution based on Saito (1959) “Bending of Elliptic Plates under a Concentrated Load” JSME, Vol. 2, No. 6 (Warning, this paper contains errors in the coefficients for the homogeneous solution).

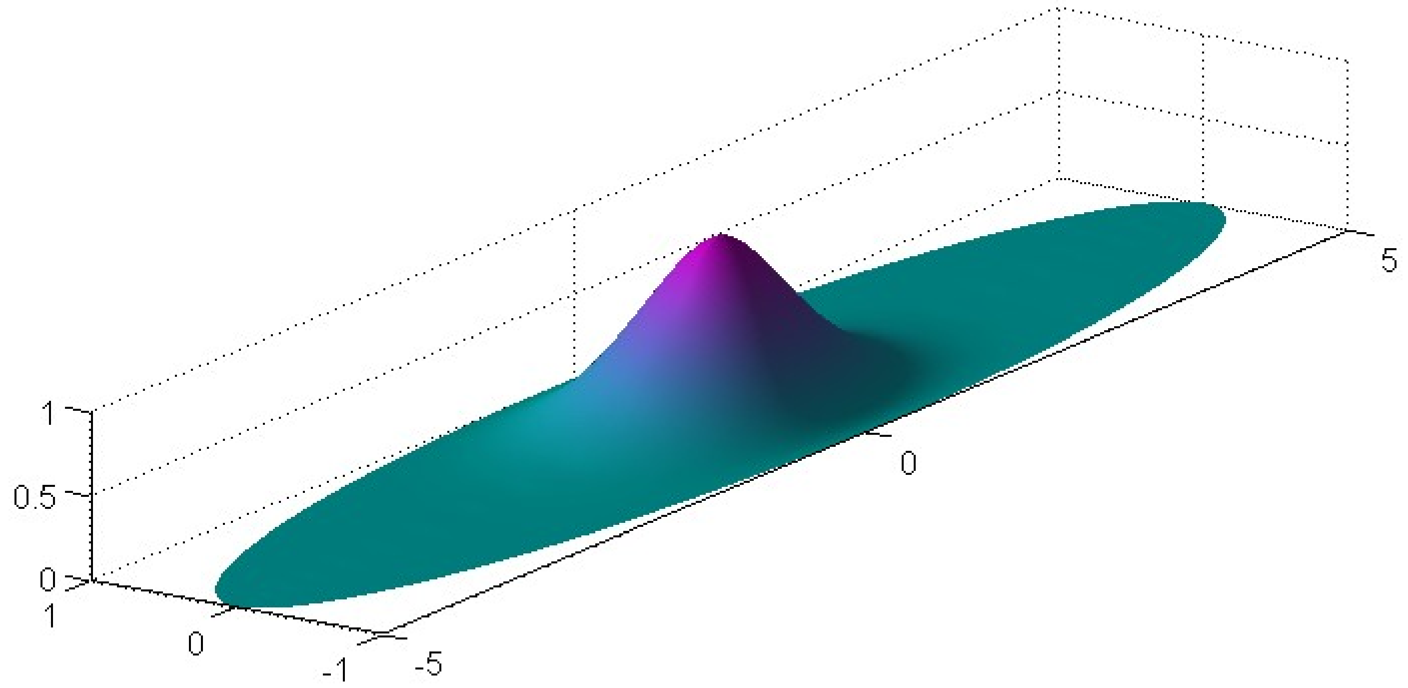
Verification in the Limiting Case: Circular Plate



$$w(r) = \frac{Pa^2}{16\pi K}\left[1 - \frac{r^2}{a^2} + 2\frac{r^2}{a^2}\log\left(\frac{r}{a}\right)\right]$$

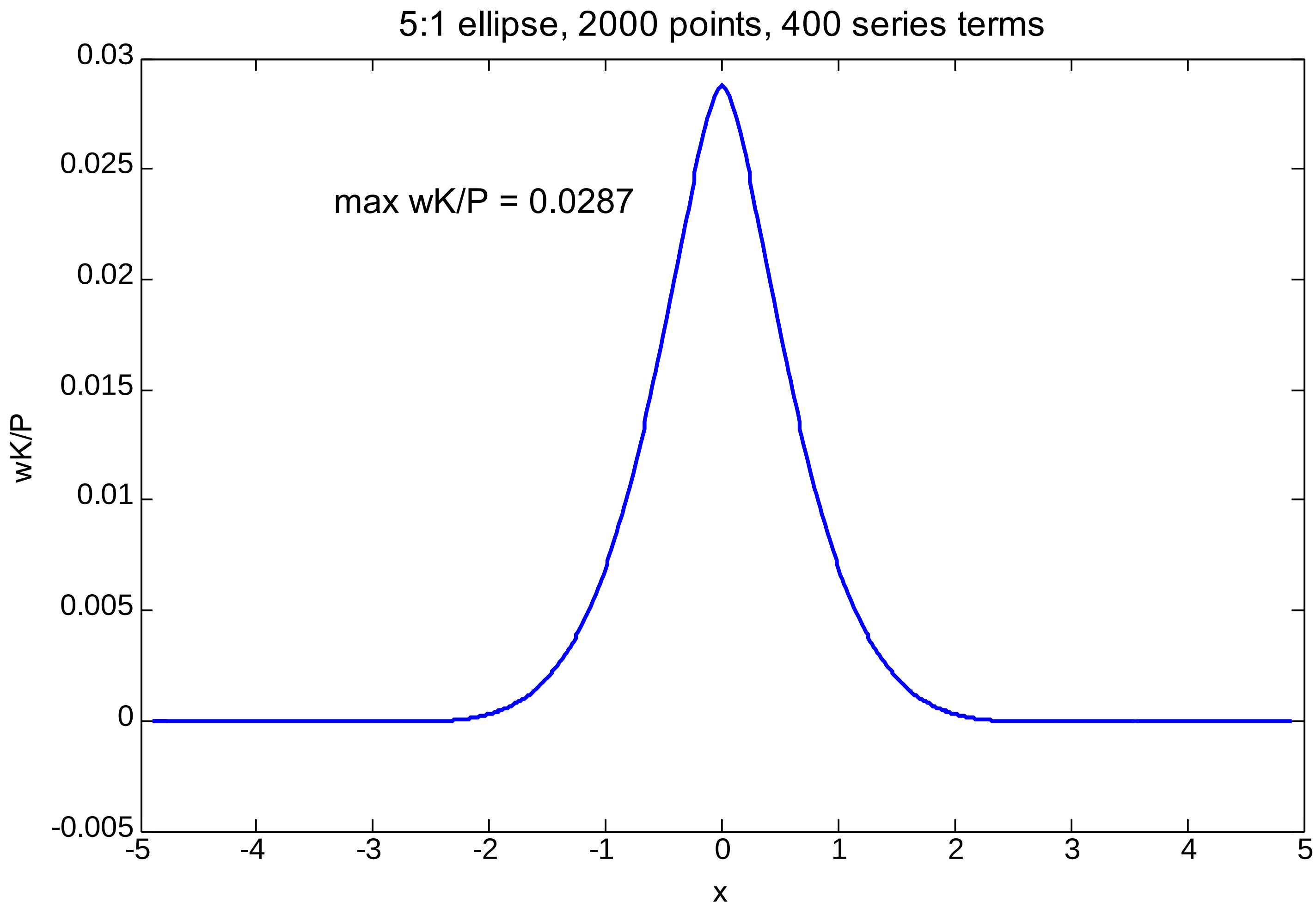
$$\frac{w(0)D}{P} = \frac{1}{16\pi} \approx 0.0199$$

5:1 Ellipse, Central Point Load



(deflection normalized by maximum deflection)

Y=0 Cross-section



Y=0 Cross-section

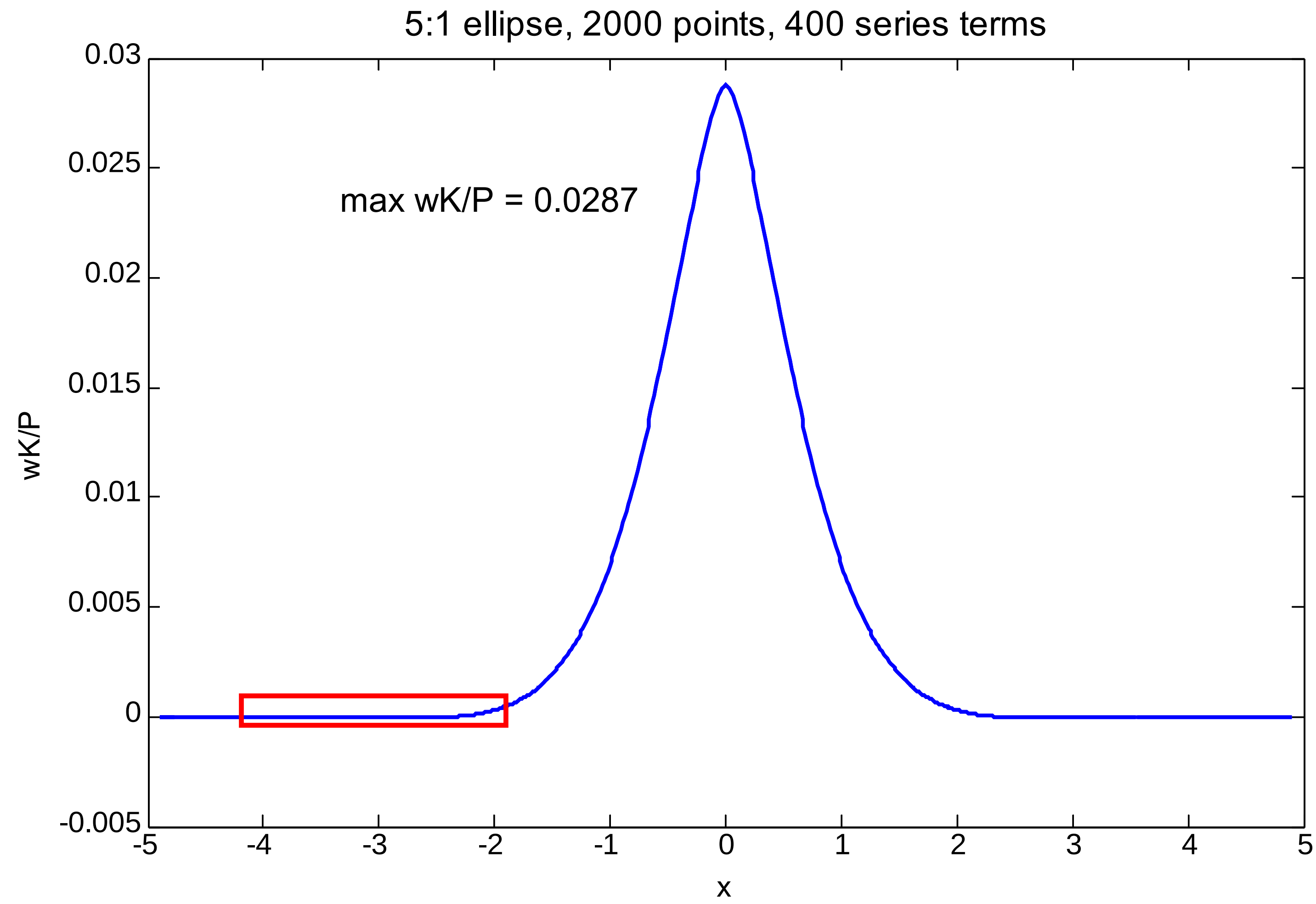
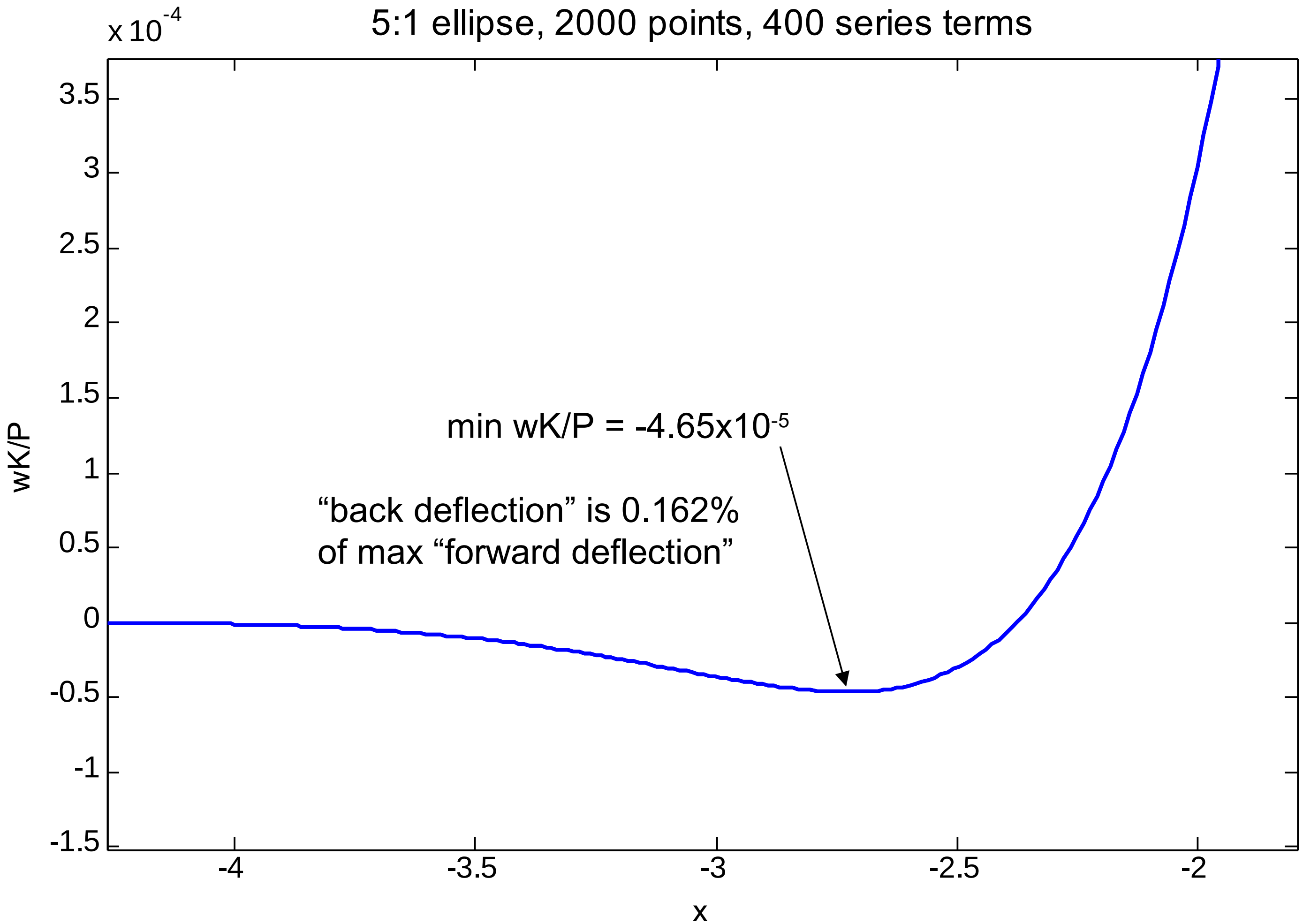
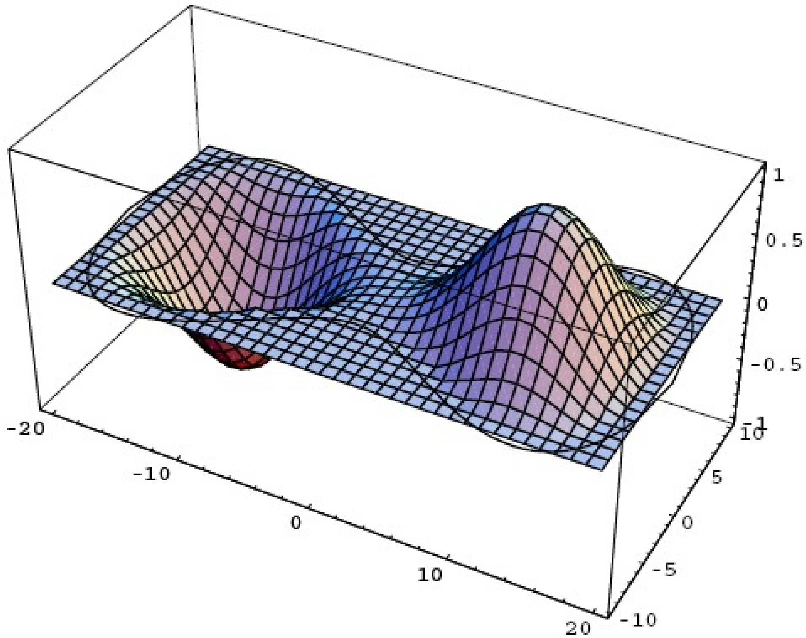


Plate Pushes Back!

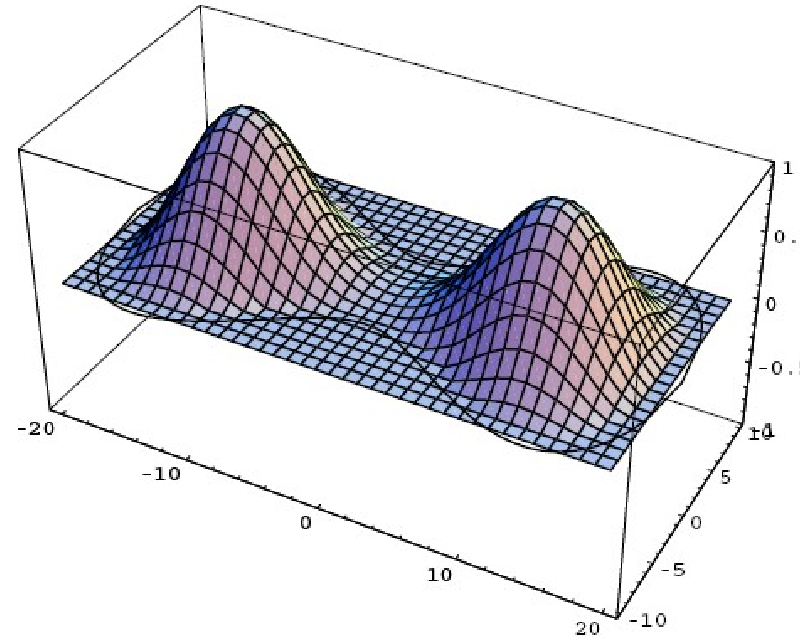


Related Phenomenon: Eigenmode Inversion

$$\nabla^2 \nabla^2 \phi = \lambda \phi$$



N = 1 mode!



N = 2 mode

plots from Sweers (2001)

Questions?