LINEAR EQUATION

Chickens and Rabbits

A farm has chickens and rabbits. The farmer counts 26 heads and 82 feet. How many chickens and rabbits are in the farm?

Trial and error. Before learning algebra, you solve this problem by trial and error. Each chicken has one head and two feet. Each rabbit has one head and four feet. You guess some number of chickens and some number of rabbits, and then see if the guess gives 26 heads and 82 feet.

For example, you may guess 10 chickens and 16 rabbits. This guess gives that

$$10+16=26 \text{ heads},$$

$$10\times 2+16\times 4=84 \text{ feet}.$$

This guess gives the right number of heads, but too many feet.

You may make another guess. Since a rabbit has more feet than a chicken, you might reduce the number of rabbits. For example, you guess 11 chickens and 15 rabbits. This guess gives that

$$11+15=26 \text{ heads},$$

$$11\times 2+15\times 4=82 \text{ feet}.$$

Bingo! You guess right.

Algebraic method. After learning algebra, you solve this problem by following a procedure, in two steps.

Step 1: Set up equations. Name every unknown by a symbol, and use the symbols to set up equations. Let the number of chickens be x, and the number of rabbits be y. Each unknown is also called a *variable*. The total number of heads is 26, and the total number of feet is 82. You translate the two sentences to two equations:

$$x+y=26,$$

$$2x+4y=82.$$

These equations place constraints on the variables x and y. This step requires that you know some facts of the real world: each chicken has one head and two feet, and each rabbit has one head and four feet.

Step 2: solve equations. Solve these equations to determine the variables x and y. There are many ways to solve the equations. For example, write the first equation as

$$y=26-x.$$

Use this expression to substitute y in the second equation:

$$2x+4\left(26-x\right)=82.$$

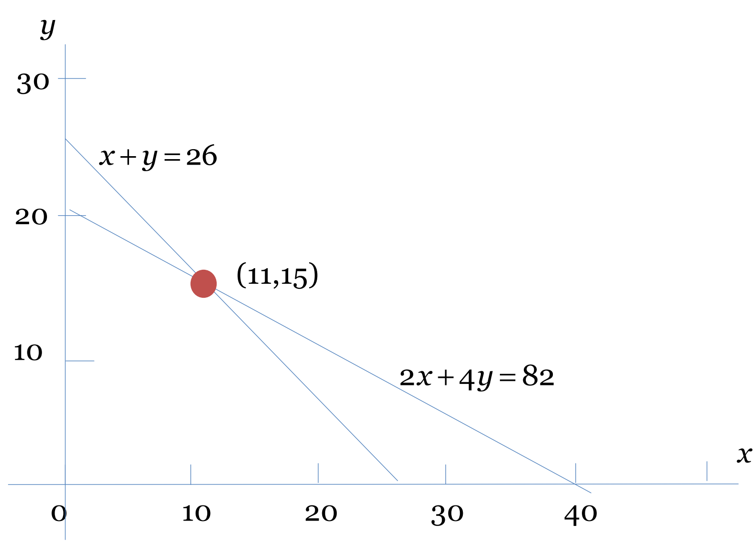
This equation has a single unknown, x. Solving for x, you obtain that $x=11$. Inserting $x=11$ into the equation for heads $x+y=26$, you obtain that $y=15$.

The algebraic method reduces the solution of this "real-world problem" to a two-step procedure: set up equations and solve them. The procedure kills the joy of guessing, but will lead to an algorithm for a computer to solve many equations of many variables.

Graphical representation of two variables. This problem has two variables. The number "two" has a special significance ever since humans learned to draw pictures in a plane. The plane is two-dimensional. Even though we live in a three-dimensional space, we still have not invented a good way to draw pictures in the three-dimensional space. Thus, all graphical representations are in a plane.

In a plane, draw a horizontal axis to represent the number of chickens, x, and draw a vertical axis to represent the number of rabbits, y. A point in the plane represents a pair of numbers, (x,y). A straight line in the plane represents a linear equation of the two variables, x and y.

Now look at the two equations in the problem of chickens and hamsters. The equation $x+y=26$ represents a straight line that intersects the x-axis at 26 and intersects the y-axis as 26. The equation $2x+4y=82$ represents a straight line that intersects the x-axis at 41 and intersects the y-axis as 20.5. The two straight lines intersect at one point, $(11,15)$. This point in the plane gives the solution to the problem: 11 chickens and 15 rabbits.



Rabbits and Hamsters

Rabbits and hamsters, first try. Next consider a farm having rabbits and hamsters. The farmer counts 26 heads and 82 feet in total. How many rabbits and hamsters are in the farm? Let the number of rabbits be y, and the number of hamsters be z. The two variables satisfy two equations:

$$y+z=26,$$

$$4y+4z=82.$$

Keep the first equation unchanged and divide the second equation by four, and the two equations become that

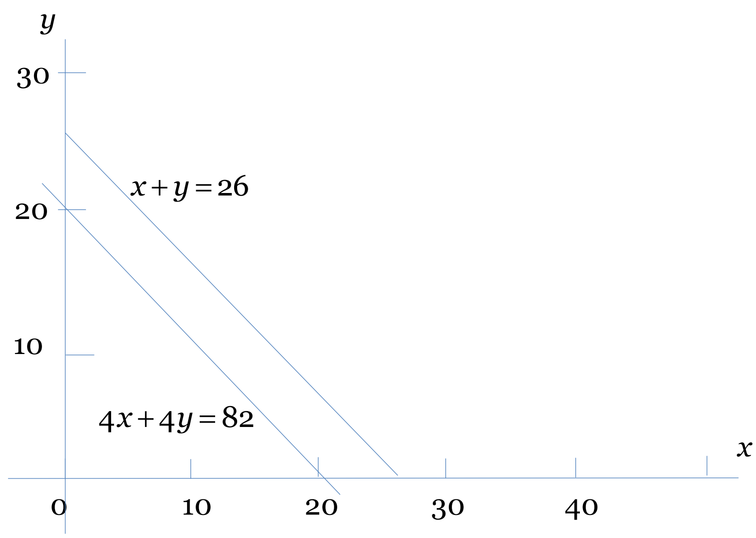
$$y+z=26,$$

$$y+z=20.5.$$

The two equations are incompatible. Consequently, this system of equations has no solution.

The cause of no solution is obvious from the given data. Each rabbit or hamster has one head and four feet. Thus, the total number of feet must be four times the total number of heads. However, the total number of feet 82 is not four times the total number heads 26. The farmer must have counted heads or feet wrong.

In this example, the two equations represent two parallel lines in the plane (x,y). The two parallel lines do not intersect, and give no solution to the system of equations.



Rabbits and hamsters, another try. What happens if the number of feet is four times the number of heads, say 26 and 104? In this case, the equations are

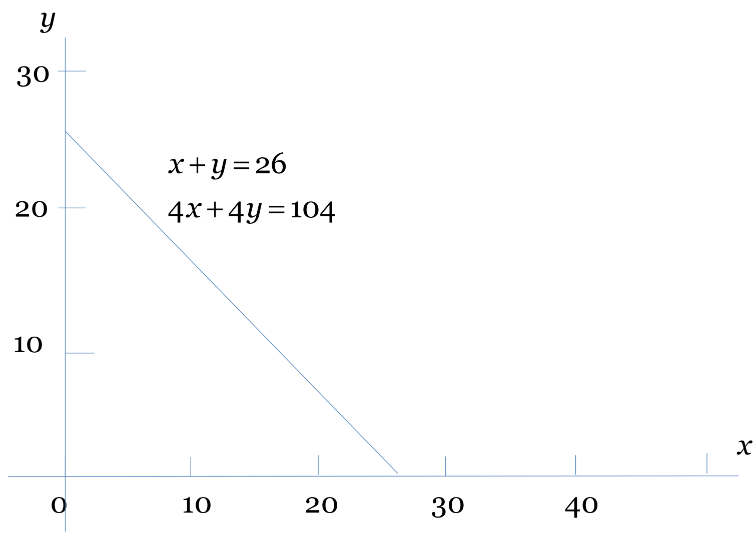
$$y+z=26,$$

$$4y+4z=104.$$

The two equations are equivalent to

$$y+z=26,$$
$$y+z=26.$$

This system of equations has many solutions. In this example, the two equations represent one single line in the plane (x,y). Every point on the line gives a solution. (Of course, in this example, we look for solutions such that the numbers of rabbits and hamsters are both nonnegative integers.)



The graphical solution is helpful when the number of variables is two. Two linear equations of two variables correspond to two straight lines in a plane. The two lines either intersect at a single point, or be parallel, or coincide. The three situations correspond to a system of equations of one solution, or no solution, or many solutions.

Chickens, Rabbits, and Hamsters

Chickens, rabbits, and hamsters. Now consider a farm having chickens, rabbits, and hamsters. Still there are 26 heads and 82 feet in total. How many chickens, rabbits, and hamsters are there?

Let the number of chickens be x, the number of rabbits be y, and the number of hamsters be z. The three variables satisfy two equations:

$$x+y+z=26,$$
$$2x+4y+4z=82.$$

The two equations are equivalent to

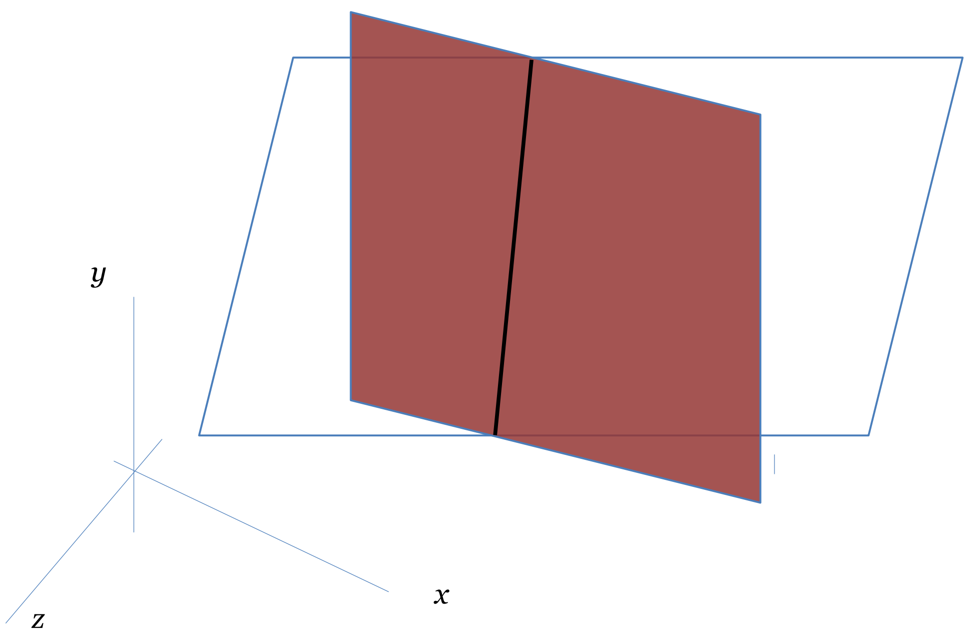
$$x=11,$$
$$y+z=15.$$

The two equations lead to many solutions.

Graphical representation of three variables. The above problem has three variables. In a three-dimensional space, use three axes to represent the number of chickens x, the number of rabbits y, and the number of hamsters z. A point in the space represents a triple of numbers (x,y,z). A plane in the space represents a linear equation of the three variables x, y and z.

We draw a fake three-dimensional space on a two-dimensional paper (or screen). The three axes represent the numbers of chickens, rabbits, and hamsters. The equation of heads, $x+y+z=26$, corresponds to a plane that intersects the three axes at 26, 26, 26. The equation of feet, $2x+4y+4z=82$, corresponds to a plane that intersects the three axes at 41, 20.5, 20.5.

It takes forever to draw a picture on screen. The drawing is inaccurate, and is of course fake no matter how much you try. But even this fake picture makes an essential point clear: the two planes intersect at a straight line, corresponding to many solutions to the system of equations.



The three variables live in a three-dimensional space. Yet we can only draw a picture in a two-dimensional plane. This imperfect method still helps us visualize the structure of the solutions. For example, two planes may either intersect at a line, or be parallel, or coincide.

A system of three equations of three variables corresponds to three planes in a three-dimensional space. Many situations may happen. Here are some examples:

- The three planes intersect at a single point. In this case, the system of equations has a unique solution.

- The three planes are parallel to one another. In this case, the system of equations has no solution.
- The three planes intersect at a line. In this case, the system of equations has many solutions.

The graphical representation becomes impossible when we deal with more than three variables.

Summary and Generalization

Solution set of three types. A system of linear algebraic equations may have either no solution, or one solution, or many solutions. The collection of all solutions is called the *solution set* of the system of equations.

In the example of chickens and rabbits, the system of equations is

$$x + y = 26,$$

$$2x+4y=82.$$

The system has a single solution:

$$x = 11$$

$$y = 15$$

Thus, the solution set of this system has a single element.

In the first example of rabbits and hamsters, the system of equations is

$$y + z = 26,$$

$$4y + 4z = 82.$$

The system has no solution, so that the solution set is the empty set.

In the second example of rabbits and hamsters, the system of equations is

$$y + z = 26,$$

$$4y + 4z = 104.$$

The system has many solutions. We write the solution set in the form

$$y = 26 - z\ .$$

Any value of z gives a solution. We call y the basic variable, and z the free variable.

In the example of chickens, rabbits and hamsters, the system of equations is

$$x + y + z = 26,$$

$$2x + 4y + 4z = 82.$$

The system has many solutions. We write the solution set in the form

$$x = 11,$$

$$y = 15 - z.$$

Any value of z gives a solution. Here x and y are the basic variables, and z is the free variable.

Observations. The above examples illustrate some general observations:

- Solve a real-world problem by a two-step procedure. Step 1: set up equations. Step 2: solve equations.
- A system of linear algebraic equations has m equations and n variables. In general, $m \neq n$.
- A system of linear algebraic equations may have no solution, one solution, or many solutions. All solutions to a system of equations form a set, called the solution set of the system of equations.
- Graphical representation is sometimes helpful, but not always.

Row Reduction Algorithm

The two-step procedure is still too creative for computers. The first step, setting up equations for a real-world problem, is beyond the capability of computers today, and is performed by humans. By contrast, the second step, solving the equations, is almost exclusively done by computers.

For the time being, we skip the step of setting up the equations, and assume that we already have the equations. We focus on solving the equations. A computer solves any system of linear algebraic equations using a procedure, known as the *row reduction algorithm*.

Example. The row reduction algorithm is readily understood by an example. Consider a system of equations:

$$x - 2y + z = 0$$

$$2x - 2y - 6z = 8$$

$$-4x + 5y + 9z = -9$$

The system has three equations, as well as three variables, x, y, z. Values of the variables that satisfy the three equations is called a *solution*. We wish to find all solutions—that is, the solution set of the system of equations.

The fundamental method to solve a system of linear algebraic equations is the method of eliminating variables. In this method, we multiply equations by nonzero numbers, and add equations. The aim is to eliminate variables, until we find a solution. We now streamline the method of elimination, so that even a computer can follow it.

We list the *coefficients* of the system of equations by a table:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & -6 \\ -4 & 5 & 9 \end{bmatrix}.$$

This table is called the *matrix of coefficients*.

We list the *constant terms* of the system of equations by a column:

$$\begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}.$$

We collect all the numbers in a table:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & 9 \end{bmatrix}.$$

This table is called the *augmented matrix* of the system of equations. Each row of the augmented matrix corresponds to one equation. The first three columns collect the coefficients, and the last column collects the constant terms.

Now we watch the row reduction algorithm in action:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \begin{array}{l} \\ R_2 / 2 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} R_1 - R_3 \\ R_2 + 4R_3 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ \\ \end{array}$$

Translate the table back to a system of equations, and we obtain the solution:

$$\begin{aligned} x &= 29 \\ y &= 16 \\ z &= 3 \end{aligned}$$

We can verify this solution by inserting into the original equations.

Row operations. Now we abstract the ideas. Note row operations of three types:

1. Swap the positions of two rows, $R_i \rightleftarrows R_j$.
2. Multiply a row by a nonzero number, $R_i \to cR_i$.
3. Replace a row by the addition of itself and another row, $R_i \to R_i + cR_j$.

Reduced row echelon form (rref). The object of the row reduction algorithm is to use row operations to change the augmented matrix of a system of linear equations to a simple form, from which we can see the solution set of the system of equations. We now define this "simple form".

A matrix is in reduced row echelon form (rref) if it has the following properties:

1. In each nonzero row, the leftmost nonzero entry is 1. This entry is called a *pivot,* its position a *pivot position*, and its column a *pivot column*.
2. In each pivot column, all other entries are zero.
3. The pivot position in one row is to the right of the pivot position in the row above it.
4. As a consequence of 3, rows with all entries being zero are at the bottom of the matrix.

Here are several matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row reduction algorithm. Given any matrix, the row reduction algorithm changes the matrix to its reduced row echelon form. The algorithm goes as follows.

1. Begin with the leftmost pivot column. If the entry at the pivot position in this column is zero, swap rows to bring a nonzero entry to the pivot position.
2. Use row operations to eliminate entries in the column below the pivot.
3. Move right to the next pivot column. Repeat 1-2 for all pivot columns.
4. Begin with the rightmost pivot. Use row operations to eliminate entries in the column above the pivot.
5. Move left to the next pivot column. Repeat 5 for all pivot columns

Example. In the above example, we do not need to use the row operation of type 1. Let us look at an example that requires row operations of all three types.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 2 & 1 & 2 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 2 & 1 & 2 & 8 & 4 \end{bmatrix} \quad R_1 \rightleftarrows R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 4 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 / 2$$

Observations. Here are some general observations of row operations, row reduction algorithm, and reduced row echelon form.

1) Row operations do not change the solution set of the system of linear equations.
2) If matrix B is obtained by row operations from matrix A, we say that B is *row-equivalent* to A, and write $A \sim B$.
3) Elementary row operations are reversible. That is, if $A \sim B$, then $B \sim A$.
4) Each matrix A is row-equivalent to one and only one reduced row echelon form, denoted $\mathrm{rref}(A)$.

Solution Set of Three Types

The collection of all solutions to a system of equations is called the solution set. Given a system of linear equations, we can form its augmented matrix, and then use row operations to change the augmented matrix to the reduced row echelon form. The reduced row echelon form lets us write the solution set. Three types of solution sets are possible.

The solution set is the empty set. If the rightmost column is a pivot column, the system has no solution.

The solution set has a single element. If the rightmost column is not a pivot column, and the number of pivots is the same as the number of variables, the system has a unique solution.

The solution set has infinitely many elements. If the rightmost column is not a pivot column, and the number of pivots is less than the number of variables, the system has infinite many solutions.

We next look at the three types of solution sets in turn.

A system having no solution. A system of linear equations has no solution if and only if the rightmost column of the augmented matrix is a pivot column.

For example, look at the following augmented matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is already in the reduced row echelon form, and has three pivot columns. For the rightmost column to be a pivot column, its pivot position must be in the bottom nonzero row (the third row in the example). In this row, the rightmost entry must be 1, and all other entries must be zero. This row corresponds to equation

$$0 = 1.$$

Thus, the system of equations has no solution.

A system having a unique solution. A system of linear equations has a unique solution if and only if the number of pivots is the same as the number of variables.

For example, look at the following augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

This augmented matrix corresponds to a system of equations:

$$x = 5$$
$$y = 2$$
$$z = 3$$

A system having infinitely many solutions. A system of linear equations has an infinite many solutions if and only if the number of pivots is less than the number of variables.

Look at a system of equation:

$$x + 2y + 3z = 1,$$
$$4x + 5y + 6z = 2,$$
$$7x + 8y + 9z = 3.$$

This system of equations corresponds to the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix}.$$

The numbers are very special, so you can easily remember this example.

We next follow the row reduction algorithm to change this matrix to its reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 7 & 8 & 9 & 3 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & -6 & -12 & -4 \end{bmatrix} \quad R_3 \to R_3 - 7R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2/3 \\ 0 & -6 & -12 & -4 \end{bmatrix} \quad R_2 \to R_2 / (-3)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 6R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + 2R_2$$

From the reduced row echelon form, we see that the matrix of coefficients has two pivot columns. The system of equations has three variables.

The augmented matrix in the reduced row echelon form means three equations:

$$\begin{aligned} & x - z = -1/3 \\ & y + 2z = 2/3 \\ & 0 = 0 \end{aligned}$$

The two variables x and y correspond to two pivot columns, and are called basic variables. The variable z does not correspond to a pivot column, and is called a free variable. The free variable can take any value. The system of equations has infinitely many solutions. We write the solution set in a parametric form:

$$x = -\frac{1}{3} + z$$

$$y = \frac{2}{3} - 2z$$

The free variable z can take any value. Each value of z gives a solution to the system of equations.

Next look at the following augmented matrix of a system of equations:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in the reduced echelon form. In this example, the augmented matrix has four rows and six columns, corresponding to a system of four equations of five variables. The system has three pivot columns: column 1, column 2, and column 4. The three pivot columns correspond to three basic variables: x_1, x_2, x_4. The matrix of coefficients has two nonpivot columns: column 2 and column 5. The two nonpivot columns correspond to two free variables: x_3, x_5.

Translate the augmented matrix back to a system of equations:

$$x_1 + 3x_3 + x_5 = 2$$

$$x_2 + 2x_3 + 3x_5 = 4$$

$$x_4 + 5x_5 = 6$$

Move the free variables to the right side:

$$x_1 = 2 - 3x_3 - x_5$$

$$x_2 = 4 - 2x_3 - 3x_5$$

$$x_4 = 6 - 5x_5$$

These expressions express the solution set of the system of equations, with the basic variables on the left side, and the free variables on the right side. Each choice of the free variables x_3, x_5 gives a solution to the system of equations.

From a system of equations to its solution set. Given a system of linear algebraic equations, how do you find its complete solution set?

1. Translate the system of linear algebraic equation into its augmented matrix.
2. Convert the augmented matrix to its reduced row echelon form.
3. In the reduced row echelon form, identify pivot columns, which correspond to basic variables, and identify nonpivot columns, which correspond to free variables.
4. Translate the reduced row echelon form back to a system of equations.
5. Move the free variables to the right side of the equations.

The resulting expressions give the complete solution set of the system of equations. Each choice of values of the free variables gives a solution.

Same solution set, different choices of free variables. Given a system of equations, we have some flexibility in choosing free variables. For example, inspecting the solution set of the last problem, we can chose x_4, rather than x_5, as free variable. Rewrite the last equation as

$$x_5 = \frac{6}{5} - \frac{1}{5}x_4$$

Then insert this equation into the first two equations. We obtain a new expression of the solution set:

$$x_1 = 2 - 3x_3 - \left(\frac{6}{5} - \frac{1}{5}x_4\right)$$

$$x_2 = 4 - 2x_3 - 3\left(\frac{6}{5} - \frac{1}{5}x_4\right)$$

$$x_5 = \frac{6}{5} - \frac{1}{5}x_4$$

Of course, the system of equations has one solution set. The two choices of free variables give alternative expressions of the same solution set. You can find other choices of free variables.

Rank and Nullity

So far we have focused on row operations. Given a matrix, its reduced row echelon form is unique.

Given a system of equations, we can list its variables in any order. A change of the order of the variables corresponds to a swap of columns of the matrix of coefficients. The change in the order will not change the solution set, but will change the augmented matrix, and will change the row reduced echelon form. Consequently, the positions of the pivots may change. But a change in the order of the variables will *not* change the number of pivots.

The number of pivots of a matrix A is called the *rank* of the matrix, written $\text{rank}A$. The rank of the matrix is invariant when we swap columns. The rank of the matrix is, of course, also invariant when we swap rows.

The number of nonpivot columns of a matrix A is called the *nullity* of the matrix, written $\text{nullity}A$. Of course,

$$\text{rank}A + \text{nullity}A = n.$$

Example. To compute the rank of a given matrix A, we use row operations to convert A to its reduced row echelon form, and then count the number of pivots. Look at the reduced row echelon form of a matrix A:

$$\operatorname{rref}(A)=\begin{bmatrix} 1 & 0 & 3 & 0 & 7 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is the number of pivot columns, $\operatorname{rank}A = 4$. The nullity of the matrix is the number of nonpivot columns, $\operatorname{nullity}A = 2$.

The Invention of the Row Reduction Algorithm

The row reduction algorithm was invented in antiquity. The earliest record of this algorithm is found in a textbook, *The Nine Chapters on the Mathematical Art* (九章算术), written in Chinese, over 2200 years ago, by unknown author(s). Ancient Chinese did not know Arabic numbers. The algorithm represented numbers by rods placed in an array of cells. The arrangement looks the same as the matrix used today. The algorithm might have been taught to illiterates.

Incidentally, in the 1950s, George Forsythe, the founding chair of the computer science department at Stanford University, started to call the algorithm *Gaussian elimination*, in honor of the German mathematician Gauss (1777-1855). Both names, row reduction and Gaussian elimination, are in common use today.

Division of Labor

Divide labor between humans and computers. We live in a historical period when computers are coming between us. Some tasks are easier for humans, and others easier for computers. We divide labor between humans and computers:

1. Pose a real-world problem, and set up a system of linear algebraic equations (humans).
2. Solve the equations and determine the solution set (computers).
3. Use the solution set of the equations to act on the real-world problem (humans)

The division of labor solves problems economically.

Automation, loss of jobs, and new division of labor. Of course, what is easy to do changes when circumstances change. As new tools and applications emerge, it behooves us to renegotiate a more economical division of labor. The history of computers offers excellent lessons on such divisions and renegotiations. The division of labor is better appreciated historically and economically.

The division of labor between humans and computers is recent. Before electronic computers were developed in 1950s, people had to solve equations.

The people in the profession of solving equations are called computers. Since then, electronic computers have completely replaced human computers.

Because the division of labor will evolve in your lifetime, it is important that you appreciate the entire process, rather than just particular tasks. In some limited ways, computers have already begun to replace humans in tasks 1 and 3. These replacements will continue in your lifetime. You may become a person who creates a new division of labor, or a person who loses job.

The choice may be yours.

References

J.F. Grcar, Mathematicians of Gaussian elimination. Notices of the American Mathematical Society 58, 782-792 (2011).