LINEAR FORM

Previous notes have described numbers, vectors, and scalars. These objects lead to a new type of objects: linear forms.

Linear Form

Map. First recall the basics of a map. A map f associates very element x in a set V to an element s in a set S. We write the map in several ways:

$$s = f(x),$$

or

$$x \mapsto f(x),$$

or

$$f : V \to S.$$

We call the set V the domain of the map, and the set S the range of the map. In specifying a map, we must identify a domain, a range, and a rule to associate elements in the domain to elements in the range.

Definition. Let V be a vector space and S be a scalar set, both over a number field F. A *linear form* from V to S is a map $f : V \to S$ with the property

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for any x and y in V and any α and β in F. The vector space V is the domain of the linear form, and the scalar set S is the range of the linear form. A linear form is also known as a linear functional, a one-form, or a covector.

Remark. A linear form maps a vector space to a scalar set. To specify a linear form, we must identify a number field F, a vector space V, a scalar set S, as well as a rule to associate every vector x in V to a scalar s in S, $s = f(x)$.

Because a scalar set is a one-dimensional vector space, a linear form is a special case of a linear map from one vector space to another vector space.

Example. Each element in a set V is a piece of alloy made of some amount of gold and some amount of silver. We call this set goldsilver. We model the set goldsilver as a two-dimensional vector space over the field of real numbers. Summing two pieces means putting them together. Scaling a piece by a real number α means a piece α times of amount.

The set C of various amounts of money is a scalar set over the field of real numbers. We will call this set the set of cost.

The linear map $f : V \to C$ is a linear form. The linear form tells us the cost of every piece in the set of goldsilver. The linear form f can be represented by two real numbers: the unit price for gold, and the unit price for silver.

Example. Let V be the vector space of goldsilver, and M be the scalar set of various masses. The linear map $g: V \to M$ is a linear form. This linear form tells us the mass of every piece in the set of goldsilver. The linear form g can be represented by two real numbers: the mass per gold atom, and the mass per silver atom.

Example. Let set U be the vector space of appleorange. Each element in the set is a pile containing some quantities of apples and oranges. The addition of two piles means putting the two piles together. Multiplication a pile by a number α means a new pile α times the old pile. Let C be the scalar set of cost. Each element in this scalar set is some amount of money. The linear form $p: U \to C$ tells us the cost of every pile of appleorange. The linear form p can be represented by two real numbers: the unit price for apple, and the unit price for orange.

Remark. We do not confuse goldsilver with appleorange. Nor do we confuse cost with mass. In defining a linear form, we identify both a vector space and a scalar set. The maps p, f, and g are distinct linear maps.

Components of a Linear Form

Definition. Let V be an n-dimensional vector space over a field F, and $e_1,...,e_n$ be a basis of V. Let S be a scalar set over the field F, and u be a unit of S. Let $f: V \to S$ be a linear map. Observe that e_i is a vector in V, and $f(e_i)$ is a scalar in S. Consequently, $f(e_i)$ scales with the unit u of S. Write

$$f(e_i) = f_i u .$$

This expression defines the numbers $f_1,...,f_n$ in F as the components of the linear form a relative to the basis $e_1,...,e_n$ of V and the unit u of S.

A linear form maps the components of a vector to the magnitude of a scalar. A vector x in V is a linear combination of the base vectors of V:

$$x = x^1 e_1 + ... + x^n e_n .$$

The numbers $x^1,...,x^n$ in F are the components of the vector x in V relative to the basis $e_1,...,e_n$. We index the base vectors using a subscript, and index the components using a superscript. The merit of this choice will become evident later.

A scalar s in S scales with the unit u of S:

$$s = s_M u .$$

The number s_M in F is the magnitude of the scalar s in S relative to the unit u.

Because $f(x)$ is a linear map, we write

$$f(x) = f\left(x^1 e_1 + ... + x^n e_n\right)$$
$$= x^1 f(e_1) + ... + x^n f(e_n)$$
$$= \left(f_1 x^1 + ... + f_n x^n\right) u$$

Comparing this expression with $s = a(x) = s_m u$, we obtain that

$$s_M = f_1 x^1 + ... + f_n x^n .$$

Thus, upon choosing a basis for V and a unit for S, we can represent the linear map using the components $f_1,...,f_n$, and the above expression calculates the magnitude of the scalar s in S associated with the vector x in V.

Example. For the vector space of goldsilver, each vector is a piece that contains some amount of gold and some amount of silver. The cost of each piece is a scalar. Suppose we determine the cost of each piece in a simple way:

$$\begin{aligned}&(\text{cost of a piece})\\&= (\text{unit price of gold})(\text{amount of gold})\\&+ (\text{unit price of silver})(\text{amount of silver})\end{aligned}$$

The linear form has two components: the unit price of gold, and the unit price of silver. The linear form maps each vector (piece) to a scalar (cost).

Example. Let V be the vector space of goldsilver. Each element in this vector space contains some amounts of gold and silver. This vector space is two-dimensional. Consider a basis of the vector space: e_1 contains 1 gold atom and no silver atom, and e_2 contains no gold atom and 1 silver atom. Let M be the scalar set of masses. We use kilogram as the unit of mass. The linear form $g: V \to M$ has two components: the mass per gold atom $g_1 = 3.27 \times 10^{-25}$, and the mass per silver atom $g_2 = 1.79 \times 10^{-25}$. These two numbers are physical properties of gold and silver.

A piece of goldsilver is an element x in V. The components of the vector are the number of gold atoms x^1 in the piece, and the number of silver atoms x^2 in the piece. Write $x = x^1 e_1 + x^2 e_2$. The linear form maps the piece to its mass, $m = g(x)$. We have taken kilogram as the unit of the scalar set of mass M, and write each element in M as $m = (m_M)(\text{kilogram})$, with m_M being the magnitude of the mass. We can write the linear form $m = g(x)$ in terms of the magnitude of the scalar, the components of the vector, and the components of the linear form:

$$m_M = g_1 x^1 + g_2 x^2 .$$

We translate this mathematics to English:

(mass of a piece of goldsilver)

= (mass of a gold atom)(number of gold atoms)

+ (mass of a silver atom)(number of silver atoms)

This linear form simply states a fact in chemistry: mass is additive when we put atoms together.

Remarks. In the expression $s = f(x)$, *x* is any vector in *V*, but *f* is a fixed linear map. Once we choose a basis $e_1,...,e_n$ for *V* and a unit *u* for *S*, the list of components of the linear form is unique.

We can understand this remark as follows. Assume the linear form has two distinct sets of components: $f_1,...,f_n$ and $f_1',...,f_n'$. Thus, we can write the linear form *f* in two ways:

$$s_M = f_1 x^1 + ... + f_n x^n$$

$$s_M = f_1' x^1 + ... + f_n' x^n$$

Subtracting the two equations, we obtain that

$$0 = (f_1 - f_1')x^1 + ... + (f_n - f_n')x^n .$$

This equation holds any *n*-tuple $x^1,...,x^n$. Consequently, $f_1 = f_1',...,f_n = f_n'$.

Alternative Representations of a Linear Form

Summation convention. To save time, we often adopt a summation convention. Let *V* be an *n*-dimensional vector space, and let $e_1,...,e_n$ be a basis. A vector *x* in *V* is a linear combination of the base vectors:

$$x = x^i e_i .$$

The repeated index implies a sum from 1 to *n*. The repeated index appears twice, once as a superscript, and once as a subscript.

Recall that $f(x)$ is a linear map and $f(e_i) = f_i u$. We write

$$f(x) = f(x^i e_i) = x^i f(e_i) = (f_i x^i) u .$$

Comparing this expression with $s = f(x) = s_m u$, we obtain that

$$s_M = f_i x^i .$$

Once again, the repeated index implies a sum from 1 to *n*.

Column and row. A yet another alternative representation uses a column and a row. As an example, consider a linear from *a* that maps a two-dimensional vector space V to a scalar set S. Upon choosing a basis for V and a unit for S, we list the components of a vector by a column,

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix},$$

and list the components of the linear form by a row,

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix}.$$

The product of the row and the column gives the magnitude of the scalar:

$$s_M = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}.$$

Dual Space

Let V be a vector space and S be a scalar set, both over a field F. If there exist many linear maps from V to S, we can study their collective properties.

Definition. Let f and g be two linear maps from V to S. Define a linear combination of the two linear maps, $\alpha f + \beta g$, such that

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$$

for every element x in V and any α and β in F.

Given a vector space V and a scalar set S over a field F, the collection of all linear maps from V to S, denoted by $L(V,S)$, is a vector space over the number field F. This space is called the *dual space* of V with respect to the scalar set S.

Example. The collection of all lists of prices is the dual space of the vector space of goldsilver with respect to the set of cost. Adding two lists of prices means forming a new list of prices. This operation is useful, for example, when we fix a list of prices for the metals sold in one country, and then fix another list of prices for the metals sold in one country. Multiplying a list of prices by a real number α means changing the price of every metal by a factor of α.

Counterexample. Let V be the vector space of goldsilver, and M be the scalar set of various masses. The linear map $g: V \to M$ is a linear form. This linear form tells us the mass of every piece in the set of goldsilver. The linear form m can be represented by two real numbers: the mass per gold atom, and the mass per silver atom. In this case, the mass per gold atom and the mass per silver atom are both fixed, so only a single linear map $g: V \to M$ exists. Thus, we cannot form a dual space of goldsilver with respect to mass.

Dimension of dual space. For an n-dimensional vector space V, its dual space with respect to any scalar set is an n-dimensional vector space.

Example. The set X of directed segments is a vector space over the field of real numbers. The set E of various amounts of energy is a scalar set over the field of real numbers. The linear map $F: X \to E$ is the force. The set of forces is the dual space of the vector space of directed segments with respect to energy.

Example. The set X of directed segments is a vector space over the field of real numbers. The set U of various levels of electric potential is a scalar set over the field of real numbers. The linear map $E: X \to U$ is the electric field. The set of electric fields is the dual space of the vector space of directed segments with respect to electric potential.

Remark. Given a vector space, its dual space is not unique, but depends on the scalar set.

Dual Basis

Given an n-dimensional vector space V and a scalar set S, both over a number field F, the dual space $L(V,S)$ is also an n-dimensional vector space. Any list of n linearly independent linear forms constitutes a basis of the dual space $L(V,S)$. We now consider a special basis of the dual space.

Definition. Let $e_1,...,e_n$ be a basis of V, and u be a unit of S. A basis $e^1,...,e^n$ of the dual space $L(V,S)$ is called a *dual basis* if

$$e^\gamma(e_i) = \begin{cases} 0 & \text{if } i \neq v \\ u & \text{if } i = v \end{cases}$$

The dual basis has a useful property as follows.

Recall that the components of a vector x in V are defined by $x = x^i e_i$, and the components of a linear form f are defined by $f(e_i) = f_i u$. Write

$$f(x) = f(x^i e_i) = x^i f(e_i) = x^i f_i u = x^i f_\gamma e^\gamma(e_i) = f_\gamma e^\gamma(x).$$

Thus, $f_1,...,f_n$ are also the components of the element f in the dual space $L(V,S)$ relative to the dual basis $e^1,...,e^n$.

Remark. To serve as a basis for the space $L(V,S)$, the linear forms $e^1,...,e^n$ defined above need be linearly independent.

Remark. The notion ties a basis $e_1,\ldots,e_n$ for the vector space V to a basis $e^1,\ldots,e^n$ for the dual space $L(V,S)$. The notation is economic, but has a cost. Given a vector space V, if we consider dual spaces relative to different scalar sets, $L(V,S)$ and $L(V,T)$, each dual space should have a distinct dual basis.

Change of Basis and Change of Unit

For a given linear form $f:V\to S$, the components of the linear form transform when we change the basis of the vector space V and the unit of the scalar set S.

Transformation of the components of a vector associated with a change of basis. Let $e_1,\ldots,e_n$ be an old basis of the vector space V, and $\tilde{e}_1,\ldots,\tilde{e}_n$ be a new basis of the same vector space. Each base vector in the new basis is a linear combination of the base vectors in the old basis. Write

$$\tilde{e}_i = p_i^j e_j .$$

The numbers p_j^i in F relate the new basis to the old basis.

A vector x in V is a linear combination of either basis:

$$x = x^i e_i = \tilde{x}^i \tilde{e}_i ,$$

where the numbers $x^1,\ldots,x^n$ in F are the components of x relative to the basis $e_1,\ldots,e_n$, and the numbers $\tilde{x}^1,\ldots,\tilde{x}^n$ in F are the components of x relative to the basis $\tilde{e}_1,\ldots,\tilde{e}_n$. The two sets of components are related as

$$x^j = p_i^j \tilde{x}^i .$$

Thus, the components of the vector in V transform in the opposite way as the basis. We say that the vector is *contravariant*.

Transformation of the magnitude of a scalar associated with a change of unit. Let u be a unit of the scalar set S, and $\tilde{u}$ be another unit of the same scalar set. The two units are related as

$$\tilde{u} = ru ,$$

where the number r in F converts the two units.

A scalar s in S scales with either unit:

$$s = s_M u = \tilde{s}_M u ,$$

where the number s_M in F is the magnitude of s relative to the unit u, and the number $\tilde{s}_M$ in F is the magnitude of s relative to the unit $\tilde{u}$. The two magnitudes are related as

$$s_M = r\tilde{s}_M .$$

Thus, the magnitude of the scalar in S transforms in the opposite way as the unit. We say that the scalar is *contravariant*.

Transformation of the components of a linear form. Associated with a change of the basis of V and a change with the unit of S, the components of the linear form also transform. Because $\tilde{e}_i$ is a vector in V, the linear form $f\left(\tilde{e}_i\right)$ maps the vector $\tilde{e}_i$ to an element in S. Write

$$f\left(\tilde{e}_i\right)=\tilde{f}_i\tilde{u} .$$

The numbers $\tilde{f}_1,...,\tilde{f}_n$ are the components of the linear form a relative to the new basis $\tilde{e}_1,...,\tilde{e}_n$ and the unit $\tilde{u}$.

We write

$$f\left(\tilde{e}_i\right)=f\left(p_i^j e_j\right)=p_i^j f\left(e_j\right)=p_i^j f_j u .$$

A combination of the above two expressions gives

$$\tilde{f}_i=p_i^j f_j / r .$$

The components of the linear form transform in the same way as the basis of V, but in the opposite way as the unit of S. We say that the linear form is *covariant* with respect to the basis of the vector space, but *contravariant* with respect to the unit of the scalar set.

Transformation of dual basis. Once we change the basis $e_1,...,e_n$ of V, we also change the dual basis $e^1,...,e^n$ of $L\left(V,S\right)$ accordingly. Let the new basis $\tilde{e}_1,...,\tilde{e}_n$ be a new basis of V, which relate to the old basis as

$$\tilde{e}_i=p_i^j e_j .$$

$$\tilde{e}^\xi\left(\tilde{e}_j\right)=\delta_j^\xi u$$

$$e^\gamma\left(\tilde{e}_j\right)=e^\gamma\left(p_j^i e_i\right)=p_j^i e^\gamma\left(e_i\right)=p_j^i u\delta_i^\gamma$$

The new dual basis relate to the old dual basis by

$$e^\gamma=p_\xi^\gamma\tilde{e}^\xi .$$

The dual basis is contravariant.

Indexing with superscripts and subscripts. Now the merit of the indexing with superscripts and subscripts becomes evident. By a convention, we index a basis of the vector space with subscripts. Associated with a change of basis, the components of an object will transform. If the components of the object transform in the same way as the basis, we index the components of the object using subscripts. If the components of an object transform in an opposite way as the basis, we index the components using superscripts.

A Bad, but Commonly Adopted, Definition of Linear Form

Textbooks of linear algebra define a linear form *f* as a linear map from a vector space *V* to a number field *F*,

$$f : V \to F .$$

This definition confuses the role of a number field and that of a scalar set. The definition is bad for several reasons.

First, a scalar set in a one-dimensional vector space, defined by a vector space and a number field. The field gives the magnitudes of the scalars in the scalar set. The scalar set is a different mathematical object from the field.

Second, a field has extra properties that do not exist in a scalar set. The multiplication of two elements in a field gives another element in the same field. By contrast, the multiplication of two elements in a scalar field is undefined. For example, various amounts of mass form a scalar set over the field of real numbers. The multiplication of two masses does not give another mass, and the operation of multiplication is undefined in the scalar set of masses.

Third, given a vector space *V* over a field *F*, we may have several scalar sets over the field *F*. We may wish to define a linear form for each scalar set. Let *S* and *T* be two scalar sets over the field *F*. Define two scalar sets *a* and *b* by

$$f : V \to S ,$$

$$g : V \to T .$$

The two maps are distinct: they have the same domain *V*, but different ranges *S* and *T*.

We always identify a domain and a range whenever we define a map. For a linear map, the domain and the range must be both vector spaces. A linear form is just a special case of a linear map. In defining a linear form, we will always identify both a domain (a vector space) and a range (a scalar set).