Compatibility Conditions for the Left Cauchy Green Tensor Field in 3-D

Amit Acharya

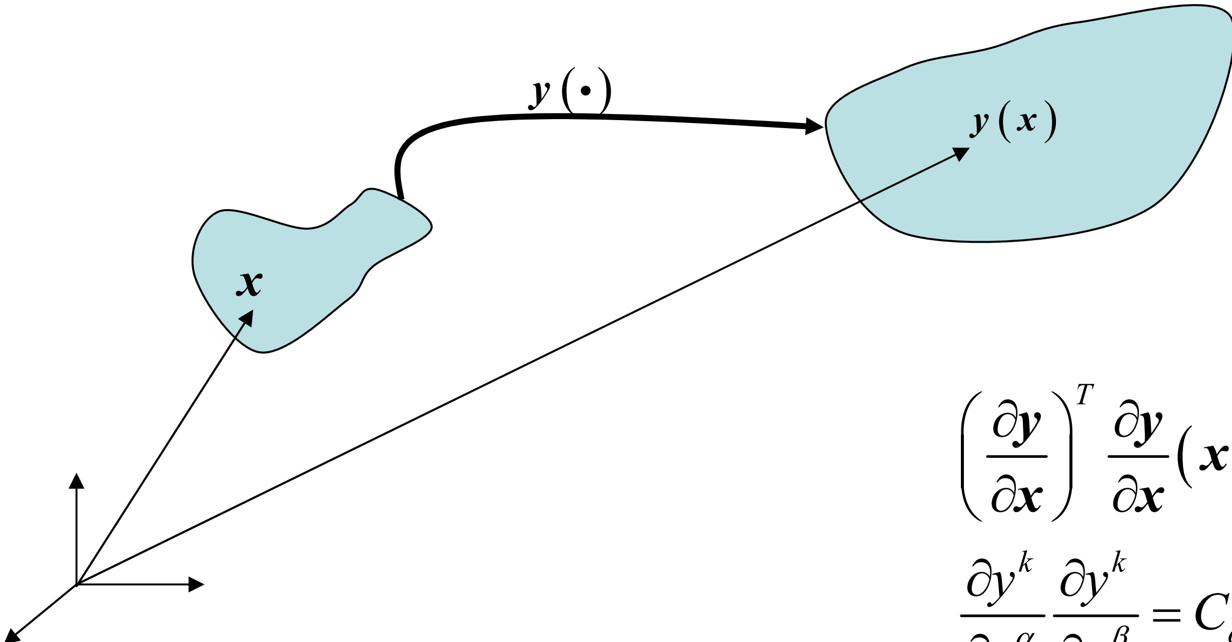
OxPDE Seminar Series

Oxford, Mathematical Institute, Nov. 12, 2008

Carnegie Mellon

Civil & Environmental ENGINEERING

Continuum Mechanics Problem for Left/Right Cauchy Green Compatibility (LCG/RCG)



Prescribed $\boldsymbol{C}(\cdot), \boldsymbol{B}(\cdot)$

as functions of $\boldsymbol{x}$

Find $\boldsymbol{y}(\cdot)$

$$\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right)^T \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x}) \in P_{sym}$$

RCG

$$\frac{\partial y^k}{\partial x^\alpha}\frac{\partial y^k}{\partial x^\beta} = C_{\alpha\beta}$$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right)^T (\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}) \in P_{sym}$$

LCG

$$\frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\alpha} = B^{ij}$$

Are the RCG and LCG compatibility problems really different?

RCG

$$\frac{\partial y}{\partial x} = F \Leftrightarrow curl\, F = 0$$

$$F^T F = \text{ specified}(x) \in Psym$$

LCG

$$\frac{\partial y}{\partial x} = F \Leftrightarrow curl\, F = 0$$

$$FF^T = B$$

$$\Leftrightarrow F^{-T} F^{-1} = B^{-1}$$

$$\Im(F)^T \Im(F) = \text{ specified}(x) \in P_{sym}$$

Using RCG method one has

$$curl\, \Im(F(x)) = 0$$

$$\varepsilon_{ijk} \frac{\partial F_{an}}{\partial x^j} F_{nk}^{-1} = 0 \text{ but need } \varepsilon_{ijk} \frac{\partial F_{ak}}{\partial x^j} = 0$$

Motivation

- Continuum Mechanics
  - Interesting geometry question for classical kinematical measure
    - Cauchy stress for Frame-indifferent, isotropic elastic material is a function only of B
  - Sharp contrast in uniqueness from the more studied RCG case
  - Open in 3-D

- Mathematics
  - Interesting questions involving
  - Geometry
  - Nonlinear PDE
  - Algebra
    - Even in the C^{∞} local existence case

Riemannian Geometry in charts

Given two coordinate patches for a Riemannian manifold, with points denoted generically by

$$x \Leftrightarrow x^{\alpha} \Leftrightarrow \left(x^1, x^2, x^3\right)$$

$$y \Leftrightarrow y^{\alpha} \Leftrightarrow \left(y^1, y^2, y^3\right)$$

and

$$y = y(x) \qquad \det\left(\frac{\partial y}{\partial x}\right) \neq 0$$

$\exists \quad 3 \times 3$ sym, + def matrix fields

$$_{(x)}C, \quad _{(y)}C, \quad _{(x)}B := {}_{(x)}C^{-1}, \quad _{(y)}B := {}_{(y)}C^{-1}$$

satisfying

$$_{(x)}C_{\alpha\beta} = \frac{\partial y^k}{\partial x^{\alpha}} \left[{}_{(y)}C\right]_{km} \frac{\partial y^m}{\partial x^{\beta}}$$

Matrix inverse

$$_{(y)}B^{ij} = \frac{\partial y^i}{\partial x^{\alpha}} \left[{}_{(x)}B\right]^{\alpha\beta} \frac{\partial y^j}{\partial x^{\beta}}$$

$$\left[\partial y / \partial x\right]_{k\alpha} = \frac{\partial y^k}{\partial x^{\alpha}}$$

Notational agreement:
Evaluate anything like

$_{(x)}(\cdot)$ at x

and

$_{(y)}(\cdot)$ at $y(x)$

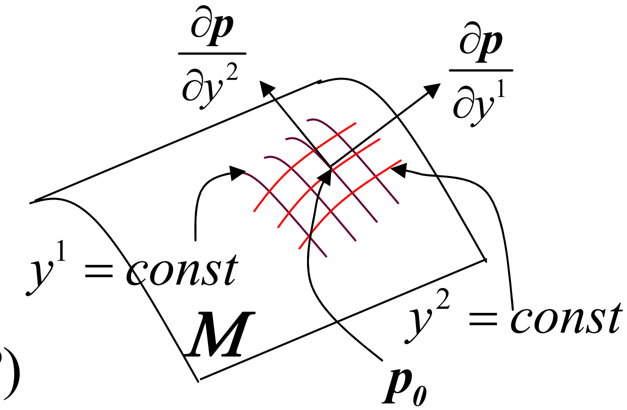
The question of equivalence of quadratic forms

Given two local P_{sym} matrix fields on manifold $\boldsymbol{M}$,

$\dim(\boldsymbol{M}) = N$, they are equivalent if one can find

two local charts in $\mathbb{R}^N$ related $1-1$ with

$\det\left(\dfrac{\partial y}{\partial x}\right) \neq 0$ satisfying transformation rules. (WHY???)



On tangent space T_{p_0} of $\boldsymbol{p_0} \in \boldsymbol{M}$ spanned by

$\left(\dfrac{\partial \boldsymbol{p}}{\partial y^1}, \dfrac{\partial \boldsymbol{p}}{\partial y^2}\right)$ or $\left(\dfrac{\partial \boldsymbol{p}}{\partial x^1}, \dfrac{\partial \boldsymbol{p}}{\partial x^2}\right)$ any vector $\boldsymbol{a} \in T_{p_0}$

$$\boldsymbol{a} = {}_{(x)}a^\alpha \frac{\partial \boldsymbol{p}}{\partial x^\alpha} = {}_{(x)}a^\alpha \frac{\partial \boldsymbol{p}}{\partial y^i}\frac{\partial y^i}{\partial x^\alpha} = {}_{(y)}a^i \frac{\partial \boldsymbol{p}}{\partial y^i} \quad \Rightarrow \quad {}_{(y)}a^i = {}_{(x)}a^\alpha \frac{\partial y^i}{\partial x^\alpha}$$

Now, let there be a quadratic form for each chart s.t. for $\boldsymbol{a}, \boldsymbol{b}$

${}_{(x)}a^\alpha \, {}_{(x)}C_{\alpha\beta} \, {}_{(x)}b^\beta =:$ physical scalar indep. of chart

But, no chart is special; $\therefore {}_{(x)}a^\alpha \, {}_{(x)}C_{\alpha\beta} \, {}_{(x)}b^\beta = {}_{(y)}a^i \, {}_{(y)}C_{ij} \, {}_{(y)}b^j$

$$ {}_{(x)}a^\alpha \, {}_{(x)}b^\beta \left[\frac{\partial y^i}{\partial x^\alpha} \, {}_{(y)}C_{ij} \frac{\partial y^j}{\partial x^\beta} - {}_{(x)}C_{\alpha\beta}\right] = 0 $$

$\forall \; {}_{(x)}a^\alpha, {}_{(x)}b^\beta$

Mapping compatibility question to Riemannian Geometry

$$ {}_{(x)}C_{\alpha\beta} = \frac{\partial y^k}{\partial x^\alpha} \left[{}_{(y)}C \right]_{km} \frac{\partial y^m}{\partial x^\beta} $$

$$ {}_{(y)}B^{ij} = \frac{\partial y^i}{\partial x^\alpha} \left[{}_{(x)}B \right]^{\alpha\beta} \frac{\partial y^j}{\partial x^\beta} $$

choose

$$ {}_{(y)}C \equiv I $$

RCG compatibility

$$ \frac{\partial y^k}{\partial x^\alpha} \frac{\partial y^k}{\partial x^\beta} = C_{\alpha\beta} $$

given C, B as fns. of x
find $y(x)$

$$ {}_{(x)}B \equiv I $$

LCG compatibility

$$ \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\alpha} = B^{ij} $$

Overdetermined Problems

Machinery of Riemannian Geometry thanks to Christoffel

Recall notation, for any patch z $\qquad {}_{(z)}B = {}_{(z)}C^{-1}$

$$ {}_{(z)}\Gamma^{i}_{rs} := \frac{{}_{(z)}B^{ip}}{2}\left[\frac{\partial\,{}_{(z)}C_{rp}}{\partial z^{s}} + \frac{\partial\,{}_{(z)}C_{sp}}{\partial z^{r}} - \frac{\partial\,{}_{(z)}C_{rs}}{\partial z^{p}}\right] $$

Necessary condition for existence of $y(x)$ satisfying metric transformation rules is

$$ \frac{\partial y^{i}}{\partial x^{\alpha}\partial x^{\beta}} = {}_{(x)}\Gamma^{\rho}_{\alpha\beta}\frac{\partial y^{i}}{\partial x^{\rho}} - {}_{(y)}\Gamma^{i}_{rs}\frac{\partial y^{r}}{\partial x^{\alpha}}\frac{\partial y^{s}}{\partial x^{\beta}} $$

Roughly:

For RCG $\qquad {}_{(y)}C \equiv I \quad \Rightarrow \quad {}_{(y)}\Gamma^{i}_{rs} \equiv 0 \qquad$ Linear problem for $\partial y/\partial x$

For LCG $\qquad {}_{(x)}B \equiv I \quad \Rightarrow \quad {}_{(x)}\Gamma^{\rho}_{\alpha\beta} \equiv 0 \qquad$ Quasilinear problem for $\partial y/\partial x$!!!!

RCG compatibility

- Riemann
- Christoffel
- Brothers Cosserat
- ……..
- ……
- Shield
- Deturck and Yang
- Interesting associated facts, especially uniqueness question
  - if two deformations have same RCG field, then they differ by rigid deformation
    - Reshetnyak (according to Ball and James)
      - **Inadequacy of single-well energy for prediction of microstructure with compatible elastic deformation (Ball & James)**
    - Friesecke, Muller, James
  - An invertible tensor field F may have nonvanishing curl even if its RCG field ($F^T F$) is compatible

Complete Integrability of Pfaff PDE (T.Y. Thomas, 1934)

Theorem: Consider PDE

$$\frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha\left(w(x), x\right) \qquad i = 1 \text{ to } R \; ; \; \alpha = 1 \text{ to } n$$

$\psi^i_\alpha \in C^1(\Omega)$, Ω open connected subset of $\mathbb{R}^R \times \mathbb{R}^n$

Suppose the integrability condition

$$\frac{\partial \psi^i_\alpha}{\partial w^j}\psi^j_\beta + \frac{\partial \psi^i_\alpha}{\partial x^\beta} = \frac{\partial \psi^i_\beta}{\partial w^j}\psi^j_\alpha + \frac{\partial \psi^i_\beta}{\partial x^\alpha}$$

holds in Ω. (motivated by equality of second partial derivs.)

Then for arbitrary $(w_0, x_0) \in \Omega, \exists$ a unique local solution around x_0 satisfying $w(x_0) = w_0$. Therefore, solution allows R arbitrary constants to be specified.

RCG compatibility

Motivated by necessary condition, consider

$$_{(x)}B$$

$$\frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha$$

$$\frac{\partial u^i_\alpha}{\partial x^\beta} = {}_{(x)}\Gamma^\gamma_{\alpha\beta}\, u^i_\gamma = u^i_\gamma \frac{C^{\gamma\mu}}{2}\left[\frac{\partial C_{\alpha\mu}}{\partial x^\beta} + \frac{\partial C_{\beta\mu}}{\partial x^\alpha} - \frac{\partial C_{\alpha\beta}}{\partial x^\mu}\right]$$

Integrability Condition (nice and 'separably' factored in x-dependent terms and u-dependent terms)

$$u^i_\mu\left[\frac{\partial\, {}_{(x)}\Gamma^\mu_{\alpha\beta}}{\partial x^\rho} - \frac{\partial\, {}_{(x)}\Gamma^\mu_{\alpha\rho}}{\partial x^\beta} + {}_{(x)}\Gamma^\mu_{\gamma\rho}\, {}_{(x)}\Gamma^\gamma_{\alpha\beta} - {}_{(x)}\Gamma^\mu_{\gamma\beta}\, {}_{(x)}\Gamma^\gamma_{\alpha\rho}\right] = 0$$

$\therefore$ require Riemann-Christoffel curvature tensor to vanish. Guarantees existence of u with arbitrarily specifiable value u_0 at one point, and because of symmetry of Γ in lower indices, of y.

Remains to be shown that $u^i_\alpha u^i_\beta = C_{\alpha\beta}$

RCG Compatibility

Assign $u(x_0)$ such that $u^T u(x_0) = C(x_0)$.

Continuity $\Rightarrow$ u invertible locally around x_0.

Define $v = u^{-1}$; noting $\delta_{ij} = v_i^{\alpha} C_{\alpha\beta} v_j^{\beta}(x_0)$

$$\frac{\partial}{\partial x^{\mu}}\left(v_i^{\alpha} C_{\alpha\beta} v_j^{\beta}\right) = v_j^{\beta} v_i^{\rho}\left[\frac{\partial C_{\rho\beta}}{\partial x^{\mu}} - C_{\alpha\beta}\,{}_{(x)}\Gamma^{\alpha}_{\rho\mu} - C_{\rho\alpha}\,{}_{(x)}\Gamma^{\alpha}_{\beta\mu}\right] \quad = \quad 0 \text{ !!!!}$$

covariant derivative of covariant metric tensor

Ricci: metric tensors covariantly constant

(merely smoothness, and defn. of Γ !!!)

$$\therefore \delta_{ij} = v_i^{\alpha} C_{\alpha\beta} v_j^{\beta} \quad \Rightarrow \quad u^T u = C = \left(\frac{\partial y}{\partial x}\right)^T \frac{\partial y}{\partial x} \quad \text{locally } \square$$

Left Cauchy Green Compatibility

- 2-D
  - **Blume (1989)**
    - Formulation based on Polar Decomposition (for both 2/3-D)
      - Find a rotation tensor field
    - Compatibility condition for 2-d problem is derived
    - 'Explicit' characterization of the condition
    - Uniqueness is analyzed
  - **Duda & Martins (1995)**
    - Plane case
    - Polar Decomposition
    - Analysis of possible cases; construction of the rotation field
    - Insightful and detailed analysis of the uniqueness question
    - Demonstration of nonuniqueness through constructive examples systematically using Thomas

- 3-D - open
  - **Acharya (1999)**
    - geometric formulation
    - provides condition for local existence in 3-d
    - Much can be done in 'explicit' characterization of the existence condition

LCG Compatibility

Consider necessary condition for existence of y^i such that

$$\frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\alpha}(x) = {}_{(y)}B^{ij}(x) \text{ holds. (and } {}_{(x)}B^{\alpha\beta}(x) = \delta^{\alpha\beta} \text{ constant)}$$

Around arbitrary x_0, the map y is then locally invertible and so

$$\frac{\partial B_{rp}}{\partial y^s} = \frac{\partial B_{rp}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^s} \quad \text{where } \left[{}_{(y)}B^{-1}\right]_{rp} =: B_{rp} = {}_{(y)}C_{rp}$$

Recall $\quad {}_{(z)}B = {}_{(z)}C^{-1} \qquad {}_{(z)}\Gamma^i_{rs} := \dfrac{{}_{(z)}B^{ip}}{2}\left[\dfrac{\partial\, {}_{(z)}C_{rp}}{\partial z^s} + \dfrac{\partial\, {}_{(z)}C_{sp}}{\partial z^r} - \dfrac{\partial\, {}_{(z)}C_{rs}}{\partial z^p}\right]$

$$\frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = {}_{(x)}\Gamma^\rho_{\alpha\beta}\frac{\partial y^i}{\partial x^\rho} - {}_{(y)}\Gamma^i_{rs}\frac{\partial y^r}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta}$$

$$\therefore {}_{(x)}\Gamma^\rho_{\alpha\beta} \equiv 0, \quad \text{and} \quad \frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = -\frac{B^{im}}{2}\left[\frac{\partial B_{rm}}{\partial x^\beta}\frac{\partial y^r}{\partial x^\alpha} + \frac{\partial B_{sm}}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta} - \frac{\partial B_{rs}}{\partial x^\rho}\frac{\partial x^\rho}{\partial y^m}\frac{\partial y^r}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta}\right]$$

Left Cauchy Green Compatibility
Governing PDE System

Original – nonlinear, first order system
Formulated as – Quasilinear, Pfaffian system

$$u^i_\rho u^j_\rho \quad = \quad B^{ij}$$

$$\left[{}_{(y)}B\right]^{ij}(x) =: \left[B\right]^{ij}(x)$$

$$\left[{}_{(y)}B^{-1}\right]_{rp} =: B_{rp} = {}_{(y)}C_{rp}$$

$$\frac{\partial y^i}{\partial x^\alpha} \quad = \quad u^i_\alpha$$

Matrix inverse function

$$\frac{\partial u^i_\alpha}{\partial x^\beta} \quad = \quad -\frac{B^{im}}{2}\left[\frac{\partial B_{rm}}{\partial x^\beta}u^r_\alpha + \frac{\partial B_{sm}}{\partial x^\alpha}u^s_\beta - \frac{\partial B_{rs}}{\partial x^\rho}\mathfrak{I}^\rho_m(u)u^r_\alpha u^s_\beta\right]$$

Setup

Seek functions

$$w^i(x), \quad i = 1, 2, \ldots, R$$

that satisfy

$$\frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha\left(w(x), x\right) \qquad \alpha = 1, 2, \ldots, n$$

(for definiteness think of domain of ψ^i_α to be open connected set of $\mathbb{R}^R \times \mathbb{R}^n$)

Refer to domain of ψ^i_α as (z, x)

Associate $w^i \to u^i_\alpha$; so $R = 3 \times 3 = 9$; $n = 3$.

$$\frac{\partial u^i_\alpha}{\partial x^\beta} = A^i_{\alpha\beta}\left(u, \; {}_{(y)}B(x), \; \frac{\partial \; {}_{(y)}C}{\partial x}(x)\right) \quad ; \quad \frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha$$

Sufficient condition for local existence: the completely-integrable situation

Hypothesis: Suppose

$$F^{(1)}(u,x) := \left(\frac{\partial A^i_{\alpha\beta}}{\partial u^k_\mu} A^k_{\mu\gamma} + \frac{\partial A^i_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial A^i_{\alpha\gamma}}{\partial u^k_\mu} A^k_{\mu\beta} - \frac{\partial A^i_{\alpha\gamma}}{\partial x^\mu} \right)(u,x) \quad \equiv \quad 0 \quad \text{locally in} \quad (u,x)$$

(seek symmetry in β,γ for each i,α $\Rightarrow$ 27 nonlinear algebraic equations)

If so, Thomas guarantees solution to u and therefore y with arbitrary data at one x_0.

So - specify conditions on B field for when identity can be satisfied:

Unlike RCG case, $F^{(1)}$ is

- cumbersome (downright scary!) • nonlinear in u • does not readily

SEPARABLY factorize into at least 1 solely $x-$dependent term

- need separability (seems to me) for identity with control only on $x-$dependent terms
- need (algebraic-geometric?) theorem for when this can happen given field ${}_{(y)}B(x)$ and algebraic structure of array A.

Sufficient condition for local existence: the completely-integrable situation

Initial data to match $uu^T(x_0) = {}_{(y)}B(x_0) \in P_{ym}$ can be constructed. Then local diffeomorphism y around x_0 satisfying $\partial y/\partial x = u$ exists (and u is invertible)

Now define
$$ {}_{(y)}\Gamma^i_{rs}(y') := \left(\frac{{}_{(y)}B^{ip} \circ x}{2} \left[\frac{\partial\, {}_{(y)}C_{rp} \circ x}{\partial y^s} + \frac{\partial\, {}_{(y)}C_{sp} \circ x}{\partial y^r} - \frac{\partial\, {}_{(y)}C_{rs} \circ x}{\partial y^p} \right] \right)(y') $$

(Notation: $y^{-1} =: x$) Then,

$$ \frac{\partial u^i_\alpha}{\partial x^\beta}\left(x(y')\right) = -{}_{(y)}\Gamma^i_{rs}(y')\left[u^r_\alpha u^s_\beta\right]\left(x(y')\right) $$

Define $v = u^{-1}$, consider $\frac{\partial}{\partial y^m}\left[\left(B^{ij} v^\alpha_i v^\beta_j\right) \circ x\right]$

$$ \frac{\partial}{\partial y^m}\left[B^{ij} v^\alpha_i v^\beta_j\right] = \left[\frac{\partial_{(y)}B^{ij}}{\partial y^m} + {}_{(y)}B^{kj}\, {}_{(y)}\Gamma^i_{km} + {}_{(y)}B^{ik}\, {}_{(y)}\Gamma^j_{km} \right] v^\alpha_i v^\beta_j = 0 $$

Ricci: covariant deriv. of contravariant Metric tensor = 0

Since u invertible,

$$ \frac{\partial[\ ]}{\partial x} u^{-1} = 0 \Rightarrow \frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial x}\right)^T = B $$

□

Sufficient condition for NOT completely integrable case

Let $F^{(1)}(z,x) \neq 0$ identically. ($F^{(1)}$ defines complete integrability condition)

Define $F_\alpha^{(j+1)}(z,x) \; := \; \left(\sum_{i=1}^{R} \frac{\partial F^{(j)}}{\partial z^i} \psi_\alpha^i + \frac{\partial F^{(j)}}{\partial x^\alpha} \right)(z,x)$

and consider two integers N, R with

$1 \le N \le R \quad ; \quad 1 \le M \le R.$

Assume that

- there exist M equations in the sets $F^{(1)} = 0$ through $F^{(N)} = 0$ denoted by

$\tilde{G}_\lambda = 0, \;\; \lambda = 1$ to M,

and M of the variables z^i (from the list $z^i, i = 1$ to R) denoted by

$\overline{z}^i, \quad i = 1$ to M identified through a known one-to-one map $\kappa : \{1, 2, \ldots, M\} \to \{1, 2, \ldots R\}$

by $\overline{z}^i \; := \; z^{\kappa(i)}$

which satisfy $\det\left[\frac{\partial \tilde{G}_\lambda}{\partial \overline{z}^i}(z,x) \right] \; \neq \; 0$ locally in (z,x) space.

Not Completely Integrable Case, contd.

Denote remaining $R - M =: P$ variables z as $\hat{z}^i, i = 1$ to P, defined by

$$\mu : \{1, 2, \ldots, P\} \to \{1, 2, \ldots, R\} \qquad \hat{z}^i := z^{\mu(i)}.$$

- Assume that around a point $(\hat{z}_0, x_0) := (\hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n)$, the solution

$$\bar{z}^i = \varphi^i(\hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n), \quad i = 1, 2, \ldots, M, \text{ of } \tilde{G}_\lambda = 0, \quad \lambda = 1, 2, \ldots, M$$

satisfies all the equations of the sets

$$F^{(1)} = 0 \qquad \text{through} \qquad F^{(N+1)} = 0$$

identically in a neighborhood of $(\hat{z}_0, x_0)$ in $(\hat{z}, x)$ space.

Then $\exists$ a function $w(x)$ satisfying

$$\frac{\partial w^i}{\partial x^\alpha}(x) = \psi_\alpha^i(w(x), x) \qquad \alpha = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, R$$

determined by P constants.

Proof: Not completely Integrable case

Let $\Pi\left(\bar{z},\hat{z},x\right)\mapsto\left(z,x\right)$. Define,

$$G_\lambda\left(\bar{z},\hat{z},x\right)=\tilde{G}_\lambda\left(\Pi\left(\bar{z},\hat{z},x\right)\right),\quad \lambda=1,2,\ldots,M$$

$$\bar{\psi}^i_\alpha\left(\bar{z},\hat{z},x\right)=\psi^{\kappa(i)}_\alpha\left(\Pi\left(\bar{z},\hat{z},x\right)\right),\quad i=1,2,\ldots,M,\quad \alpha=1,2,\ldots,n$$

$$\hat{\psi}^j_\alpha\left(\bar{z},\hat{z},x\right)=\psi^{\mu(j)}_\alpha\left(\Pi\left(\bar{z},\hat{z},x\right)\right),\quad j=1,2,\ldots,P,\quad \alpha=1,2,\ldots,n$$

Since $F^{(N+1)}\left(\varphi\left(\hat{z},x\right),\hat{z},x\right)\equiv 0$ in $\left(\hat{z},x\right)$

$$\left[\frac{\partial G_\lambda}{\partial \bar{z}^i}\bar{\psi}^i_\alpha+\frac{\partial G_\lambda}{\partial \hat{z}^j}\hat{\psi}^j_\alpha+\frac{\partial G_\lambda}{\partial x^\alpha}\right]\left(\varphi\left(\hat{z},x\right),\hat{z},x\right)=0 \qquad (I)$$

For arbitrarily fixed path $f\left(t\right)$ with $f\left(0\right)=x^*$ $\exists$ solution to ODE

$$\frac{dg^j}{dt}\left(t\right)=\left[\hat{\psi}^j_\alpha\circ\left(\varphi\circ\left(g,f\right),g,f\right)\frac{df^\alpha}{dt}\right]\left(t\right)\quad ;\quad g^j\left(0\right)=\hat{z}^*\ \text{(arbitrary)},$$

and g satisfies $G_\lambda\circ\left(\varphi\circ\left(g,f\right),g,f\right)\left(t\right)=0$ for t in some local interval around 0.

Carnegie Mellon

Civil & Environmental ENGINEERING

Proof: Not Completely Integrable Case

Differentiating w.r.t. t, combining with (I), and using arbitrariness of path f, $\hat{z}^*$, and x^* yields,

$$\bar{\psi}^i_\alpha\left(\varphi(\hat{z},x),\hat{z},x\right) - \frac{\partial \varphi^i}{\partial \hat{z}^j}(\hat{z},x)\,\hat{\psi}^j_\alpha\left(\varphi(\hat{z},x),\hat{z},x\right) - \frac{\partial \varphi^i}{\partial x^\alpha}(\hat{z},x) \equiv 0 \quad (II)$$

locally in $(\hat{z},x)$ space.

$F^{(1)}\left(\varphi(\hat{z},x),\hat{z},x\right)\equiv 0$ combined with (II) implies that the conditions of complete integrability for

$$\frac{\partial g^j}{\partial x^\alpha}(x) = \hat{\psi}^j_\alpha\left(\varphi\left(g(x),x\right),g(x),x\right) \qquad (III)$$

are satisfied. Thus $\exists$ solution to (III) with P arbitrary constants.

Then it can be shown that

$$h^i(x) := \varphi^i\left(g(x),x\right) \quad i=1,2,\ldots,M$$

$$g^j(x) \quad j=1,2,\ldots,P$$

satisfy $\frac{\partial h^i}{\partial x^\alpha}(x) = \psi^{\kappa(i)} \circ (h,g,Id)(x) \quad \frac{\partial g^j}{\partial x^\alpha}(x) = \psi^{\mu(j)} \circ (h,g,Id)(x)$. □

If $u_0 u_0^T = B(x_0)$ cannot be accommodated by P constants, stick in $uu^T(x) = B(x)$ in $F^{(1)}$. Then use standard idea with Ricci to show B compatibility.

Comments

- Proof adapted from Eisenhart, 1927, Veblen and Thomas, 1926
  - They claim necessity as well; I couldn't
- Guess:
  - Related to Cartan's Method of Equivalence
  - Modern treatment – R B Gardner CBMS-NSF-SIAM