Compatibility Conditions for the Left Cauchy Green Tensor Field in 3-D

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OxPDE Seminar Series
Oxford, Mathematical Institute, Nov. 12, 2008
Continuum Mechanics Problem for Left/Right Cauchy Green Compatibility (LCG/RCG)

Prescribed \( C(\cdot) \), \( B(\cdot) \) as functions of \( x \)

Find \( y(\cdot) \)

\[
\frac{\partial y}{\partial x}
\begin{bmatrix}
\frac{\partial y}{\partial x}^T
\end{bmatrix}
(x) = C(x) \in P_{sym} \quad \text{RCG}
\]

\[
\frac{\partial y^k}{\partial x^\alpha} \frac{\partial y^k}{\partial x^\beta} = C_{\alpha\beta}
\]

\[
\frac{\partial y}{\partial x}
\begin{bmatrix}
\frac{\partial y}{\partial x}^T
\end{bmatrix}
(x) = B(x) \in P_{sym} \quad \text{LCG}
\]

\[
\frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\alpha} = B^{ij}
\]
Are the RCG and LCG compatibility problems really different?

RCG

\[ \frac{\partial y}{\partial x} = F \iff \text{curl } F = 0 \]

\[ F^T F = \text{specified}(x) \in P_{\text{sym}} \]

LCG

\[ \frac{\partial y}{\partial x} = F \iff \text{curl } F = 0 \]

\[ FF^T = B \]

\[ \iff F^{-T} F^{-1} = B^{-1} \]

\[ \Im(F)^T \Im(F) = \text{specified}(x) \in P_{\text{sym}} \]

Using RCG method one has

\[ \text{curl } \Im(F(x)) = 0 \]

\[ \varepsilon_{ijk} \frac{\partial F}{\partial x^j} F_{nk}^{-1} = 0 \]

but need \[ \varepsilon_{ijk} \frac{\partial F}{\partial x^j} F_{nk}^{-1} = 0 \]
Motivation

- **Continuum Mechanics**
  - Interesting geometry question for classical kinematical measure
    - Cauchy stress for Frame-indifferent, isotropic elastic material is a function only of $B$
  - Sharp contrast in uniqueness from the more studied RCG case
  - Open in 3-D

- **Mathematics**
  - Interesting questions involving
    - Geometry
    - Nonlinear PDE
    - Algebra
      - Even in the $C^\infty$ local existence case
Riemannian Geometry in charts

Given two coordinate patches for a Riemannian manifold, with points denoted generically by

\[ x \leftrightarrow x^\alpha \leftrightarrow (x^1, x^2, x^3) \]
\[ y \leftrightarrow y^\alpha \leftrightarrow (y^1, y^2, y^3) \]

and

\[ y = y(x) \]
\[ \det \left( \frac{\partial y}{\partial x} \right) \neq 0 \]

\[ \exists 3 \times 3 \text{ sym, + def matrix fields} \]

\[ (x) C, \quad (y) C, \quad (x) B := (x) C^{-1}, \quad (y) B := (y) C^{-1} \]

satisfying

\[ (x) C_{\alpha\beta} = \frac{\partial y^k}{\partial x^\alpha} \left[ (y) C \right]_{km} \frac{\partial y^m}{\partial x^\beta} \]

\[ (y) B^{ij} = \frac{\partial y^i}{\partial x^\alpha} \left[ (x) B \right]^{\alpha\beta} \frac{\partial y^j}{\partial x^\beta} \]

Notational agreement:
Evaluate anything like

\[ (x) (\cdot) \text{ at } x \]
and

\[ (y) (\cdot) \text{ at } y(x) \]
The question of equivalence of quadratic forms

Given two local $P_{sym}$ matrix fields on manifold $M$, $\text{dim}(M) = N$, they are equivalent if one can find two local charts in $\mathbb{R}^N$ related $1-1$ with

$$\det \left( \frac{\partial y}{\partial x} \right) \neq 0$$

satisfying transformation rules. (WHY???)

On tangent space $T_{p_0}$ of $p_0 \in M$ spanned by

$$\left( \frac{\partial p}{\partial y^1}, \frac{\partial p}{\partial y^2} \right) \text{ or } \left( \frac{\partial p}{\partial x^1}, \frac{\partial p}{\partial x^2} \right)$$

any vector $a \in T_{p_0}$

$$a = (x) a^\alpha \frac{\partial p}{\partial x^\alpha} = (x) a^\alpha \frac{\partial p}{\partial y^i} \frac{\partial y^i}{\partial x^\alpha} = (y) a^i \frac{\partial p}{\partial y^i} \Rightarrow (y) a^i = (x) a^\alpha \frac{\partial y^i}{\partial x^\alpha}$$

Now, let there be a quadratic form for each chart s.t. for $a, b$

$$(x) a^\alpha C_{\alpha\beta} (x) b^\beta =: \text{physical scalar indep. of chart}$$

But, no chart is special; $\therefore (x) a^\alpha C_{\alpha\beta} (x) b^\beta = (y) a^i C_{ij} (y) b^j$
Mapping compatibility question to Riemannian Geometry

\[(x) C_{\alpha\beta} = \frac{\partial y^k}{\partial x^\alpha} \left[ (y) C \right]_{km} \frac{\partial y^m}{\partial x^\beta} \]

\[(y) B^{ij} = \frac{\partial y^i}{\partial x^\alpha} \left[ (x) B \right]^{\alpha\beta} \frac{\partial y^j}{\partial x^\beta} \]

choose

\[(y) C \equiv I \quad \text{RCG compatibility} \]

\[\frac{\partial y^k}{\partial x^\alpha} \frac{\partial y^k}{\partial x^\beta} = C_{\alpha\beta} \]

\[(x) B \equiv I \quad \text{LCG compatibility} \]

\[\frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\alpha} = B^{ij} \]

Overdetermined Problems
Machinery of Riemannian Geometry thanks to Christoffel

Recall notation, for any patch $Z$ where

$$(z) B = (z) C^{-1}$$


defined as

$$(z) \Gamma_{rs}^i := \frac{(z) B^{ip}}{2} \left[ \frac{\partial (z) C_{rp}}{\partial z^s} + \frac{\partial (z) C_{sp}}{\partial z^r} - \frac{\partial (z) C_{rs}}{\partial z^p} \right]$$

Necessary condition for existence of $y(x)$ satisfying metric transformation rules is

$$\frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = (x) \Gamma^\rho_{\alpha \beta} \frac{\partial y^i}{\partial x^\rho} - (y) \Gamma_{rs}^i \frac{\partial y^r}{\partial x^\alpha} \frac{\partial y^s}{\partial x^\beta}$$

Roughly:

For RCG

$$(y) C \equiv I \quad \Rightarrow \quad (y) \Gamma_{rs}^i \equiv 0$$

Linear problem for $\frac{\partial y}{\partial x}$

For LCG

$$(x) B \equiv I \quad \Rightarrow \quad (x) \Gamma^\rho_{\alpha \beta} \equiv 0$$

Quasilinear problem for $\frac{\partial y}{\partial x}$
RCG compatibility

- Riemann
- Christoffel
- Brothers Cosserat
- ........
- .......
- Shield
- Deturck and Yang

Interesting associated facts, especially uniqueness question
  - if two deformations have same RCG field, then they differ by rigid deformation

- Reshetnyak (according to Ball and James)
  - Inadequacy of single-well energy for prediction of microstructure with compatible elastic deformation (Ball & James)

- Friesecke, Muller, James

- An invertible tensor field $F$ may have nonvanishing curl even if its RCG field $(F^TF)$ is compatible
Complete Integrability of Pfaff PDE (T.Y. Thomas, 1934)

Theorem: Consider PDE

$$\frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha(w(x), x) \quad i = 1 \text{ to } R ; \alpha = 1 \text{ to } n$$

$$\psi^i_\alpha \in C^1(\Omega), \Omega \text{ open connected subset of } \mathbb{R}^R \times \mathbb{R}^n$$

Suppose the integrability condition

$$\frac{\partial \psi^i_\alpha}{\partial w^j} \psi^j_\beta + \frac{\partial \psi^i_\alpha}{\partial x^\beta} = \frac{\partial \psi^i_\beta}{\partial w^j} \psi^j_\alpha + \frac{\partial \psi^i_\beta}{\partial x^\alpha}$$

holds in $\Omega$. (motivated by equality of second partial derivs.)

Then for arbitrary $(w_0, x_0) \in \Omega$, $\exists$ a unique local solution around $x_0$ satisfying $w(x_0) = w_0$. Therefore, solution allows $R$ arbitrary constants to be specified.
RCG compatibility

Motivated by necessary condition, consider

\[
\frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha
\]

\[
\frac{\partial u^i_\alpha}{\partial x^\beta} = (x) \Gamma^\gamma_{\alpha\beta} u^i_\gamma = u^i_\gamma \frac{C^{\gamma\mu}}{2} \left[ \frac{\partial C_{\alpha\mu}}{\partial x^\beta} + \frac{\partial C_{\beta\mu}}{\partial x^\alpha} - \frac{\partial C_{\alpha\beta}}{\partial x^\mu} \right]
\]

Integrability Condition (nice and 'separably' factored in \(x\)-dependent terms and \(u\)-dependent terms)

\[
u^i_\mu\left[ \frac{\partial (x) \Gamma^\nu_{\alpha\beta}}{\partial x^\rho} - \frac{\partial (x) \Gamma^\nu_{\alpha\rho}}{\partial x^\beta} + (x) \Gamma^\nu_{\gamma\rho} (x) \Gamma^\gamma_{\alpha\beta} - (x) \Gamma^\nu_{\gamma\beta} (x) \Gamma^\gamma_{\alpha\rho} \right] = 0
\]

\[\therefore\] require Riemann-Christoffel curvature tensor to vanish. Guarantees existence of \(u\) with arbitrarily specifiable value \(u_0\) at one point, and because of symmetry of \(\Gamma\) in lower indices, of \(y\).

Remains to be shown that \(u^i_\alpha u^i_\beta = C_{\alpha\beta}\)
RCG Compatibility

Assign $u(x_0)$ such that $u^T u(x_0) = C(x_0)$. 

Continuity $\Rightarrow$ $u$ invertible locally around $x_0$.

Define $\nu = u^{-1}$; noting $\delta_{ij} = \nu_i^\alpha C_{\alpha\beta} \nu_j^\beta(x_0)$

$$
\frac{\partial}{\partial x^\mu} \left( \nu_i^\alpha C_{\alpha\beta} \nu_j^\beta \right) = \nu_j^\beta \nu_i^\rho \left[ \frac{\partial C_{\rho\beta}}{\partial x^\mu} - C_{\alpha\beta}(x) \Gamma_{\rho\mu}^\alpha - C_{\rho\alpha}(x) \Gamma_{\beta\mu}^\alpha \right] = 0 \text{ !!!} 
$$

covariant derivative of covariant metric tensor

Ricci: metric tensors covariantly constant

(merely smoothness, and defn. of $\Gamma$ !!!)

$$
\therefore \delta_{ij} = \nu_i^\alpha C_{\alpha\beta} \nu_j^\beta \Rightarrow u^T u = C = \left( \frac{\partial y}{\partial x} \right)^T \frac{\partial y}{\partial x} \text{ locally } \square
$$

Carnegie Mellon
Left Cauchy Green Compatibility

- **2-D**
  - **Blume (1989)**
    - Formulation based on Polar Decomposition (for both 2/3-D)
      - Find a rotation tensor field
    - Compatibility condition for 2-d problem is derived
    - ‘Explicit’ characterization of the condition
    - Uniqueness is analyzed
  - **Duda & Martins (1995)**
    - Plane case
    - Polar Decomposition
    - Analysis of possible cases; construction of the rotation field
    - Insightful and detailed analysis of the uniqueness question
    - Demonstration of nonuniqueness through constructive examples systematically using Thomas

- **3-D - open**
  - **Acharya (1999)**
    - geometric formulation
    - provides condition for local existence in 3-d
    - Much can be done in ‘explicit’ characterization of the existence condition
Consider necessary condition for existence of $y^i$ such that
\[
\frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\alpha}(x) = (y)B^{ij}(x) \quad \text{holds.} \quad \text{(and } (x)B^\alpha{}^\beta(x) = \delta^\alpha{}^\beta \text{ constant)}
\]

Around arbitrary $x_0$, the map $y$ is then locally invertible and so
\[
\frac{\partial B_{rp}}{\partial y^s} = \frac{\partial B_{rp}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^s} \quad \text{where } \left[ (y)B^{-1} \right]_{rp} : = B_{rp} = (y)C_{rp}
\]

Recall $(z)B = (z)C^{-1}$
\[
(z)\Gamma^i_{rs} := \frac{(z)B^{ip}}{2} \left[ \frac{\partial (z)C_{rp}}{\partial z^s} + \frac{\partial (z)C_{sp}}{\partial z^r} - \frac{\partial (z)C_{rs}}{\partial z^p} \right]
\]

\[
\frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = (x)\Gamma^\rho_{\alpha\beta} \frac{\partial y^i}{\partial x^\rho} - (y)\Gamma^i_{rs} \frac{\partial y^r}{\partial x^\alpha} \frac{\partial y^s}{\partial x^\beta}
\]

\[
\therefore (x)\Gamma^\rho_{\alpha\beta} \equiv 0, \quad \text{and} \quad \frac{\partial y^i}{\partial x^\alpha \partial x^\beta} = -\frac{B^{im}}{2} \left[ \frac{\partial B_{rm}}{\partial y^r} \frac{\partial y^r}{\partial x^\alpha} + \frac{\partial B_{sm}}{\partial y^s} \frac{\partial y^s}{\partial x^\alpha} - \frac{\partial B_{rs}}{\partial y^r} \frac{\partial y^r}{\partial x^\alpha} \frac{\partial y^s}{\partial x^\beta} - \frac{\partial B_{rs}}{\partial y^r} \frac{\partial y^r}{\partial x^\alpha} \frac{\partial y^s}{\partial x^\beta} \right]
\]
Left Cauchy Green Compatibility
Governing PDE System

Original – nonlinear, first order system
Formulated as – Quasilinear, Pfaffian system

\[ u^i_\rho u^j_\rho = B^{ij} \]

\[ \frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha \]

\[ \frac{\partial u^i_\alpha}{\partial x^\beta} = -\frac{B^{im}}{2} \left[ \frac{\partial B_{rm}}{\partial x^\beta} u^r_\alpha + \frac{\partial B_{sm}}{\partial x^\alpha} u^s_\beta - \frac{\partial B_{rs}}{\partial x^\rho} \sum_m (u) u^r_\alpha u^s_\beta \right] \]

Matrix inverse function

\[ \left[ (y) B \right]^{ij} (x) =: \left[ B \right]^{ij} (x) \]

\[ \left[ (y) B^{-1} \right]_{rp} =: B_{rp} = (y) C_{rp} \]
Setup

Seek functions

\[ w^i(x), \quad i = 1, 2, \ldots, R \]

that satisfy

\[ \frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha(w(x), x) \quad \alpha = 1, 2, \ldots, n \]

(for definiteness think of domain of \( \psi^i_\alpha \) to be open connected set of \( \mathbb{R}^R \times \mathbb{R}^n \))

Refer to domain of \( \psi^i_\alpha \) as \((z, x)\)

Associate \( w^i \rightarrow u^i_\alpha \); so \( R = 3 \times 3 = 9 \); \( n = 3 \).

\[ \frac{\partial u^i_\alpha}{\partial x^\beta} = A^i_{\alpha\beta} \left( u, (y)B(x), \frac{\partial (y)C}{\partial x}(x) \right) \]

\[ \frac{\partial y^i}{\partial x^\alpha} = u^i_\alpha \]
Sufficient condition for local existence: the completely-integrable situation

Hypothesis: Suppose

\[ F^{(1)}(u,x) := \left( \frac{\partial A^i_{\alpha \beta}}{\partial u^k_{\mu}} A^k_{\mu \gamma} + \frac{\partial A^i_{\alpha \beta}}{\partial x^\gamma} - \frac{\partial A^i_{\alpha \gamma}}{\partial u^k_{\mu}} A^k_{\mu \beta} - \frac{\partial A^i_{\alpha \gamma}}{\partial x^\mu} \right) (u,x) \equiv 0 \quad \text{locally in } (u,x) \]

(seek symmetry in \( \beta, \gamma \) for each \( i, \alpha \) \( \Rightarrow \) 27 nonlinear algebraic equations)

If so, Thomas guarantees solution to \( u \) and therefore \( y \) with arbitrary data at one \( x_0 \).

So - specify conditions on \( B \) field for when identity can be satisfied:

Unlike RCG case, \( F^{(1)} \) is

- cumbersome (downright scary!) • nonlinear in \( u \) • does not readily separably factorize into at least 1 solely \( x \)–dependent term

- need separability (seems to me) for identity with control only on \( x \)–dependent terms
- need (algebraic-geometric?) theorem for when this can happen

Given field \( (y)B(x) \) and algebraic structure of array \( A \).
Sufficient condition for local existence: the completely-integrable situation

Initial data to match $uu^T(x_0) = (y)B(x_0) \in P_{ym}$ can be constructed. Then local diffeomorphism $y$ around $x_0$ satisfying $\partial y / \partial x = u$ exists (and $u$ is invertible)

Now define $(y)\Gamma^{i}_{rs}(y') := \left( \frac{(y)B^{ip}_{rp} \circ x}{2} \left[ \frac{\partial (y)C_{rp} \circ x}{\partial y^s} + \frac{\partial (y)C_{sp} \circ x}{\partial y^r} - \frac{\partial (y)C_{rs} \circ x}{\partial y^p} \right] \right)(y')$

(Notation: $y^{-1} := x$) Then,

$$\frac{\partial u^i}{\partial x^\beta}(x(y')) = -(y)\Gamma^{i}_{rs}(y') [u^r_{\alpha} u^s_{\beta}](x(y'))$$

Define $v = u^{-1}$, consider$$\frac{\partial}{\partial y^m} \left[ \left( B^{ij}_v v^\alpha_i v^\beta_j \right) \circ x \right]$$

Since $u$ invertible,$$
\frac{\partial}{\partial x} u^{-1} = 0 \Rightarrow \frac{\partial y}{\partial x} \left( \frac{\partial y}{\partial x} \right)^T = B
$$

Ricci: covariant deriv. of contravariant Metric tensor = 0

\[ \begin{bmatrix}
\frac{\partial}{\partial y^m} \left[ B^{ij}_v v^\alpha_i v^\beta_j \right] = \begin{bmatrix}
\frac{\partial (y)B^{ij}}{\partial y^m} + (y)B^{kj}_{(y)} \Gamma^{i}_{km} + (y)B^{ik}_{(y)} \Gamma^{j}_{km}
\end{bmatrix} v^\alpha_i v^\beta_j = 0
\]
Sufficient condition for NOT completely integrable case

Let \( F^{(l)}(z, x) \neq 0 \) identically. \( (F^{(l)} \) defines complete integrability condition)

Define \( F^{(j+1)}_{\alpha}(z, x) := \left( \sum_{i=1}^{R} \frac{\partial F^{(j)}}{\partial z^i} \psi^i_{\alpha} + \frac{\partial F^{(j)}}{\partial x^\alpha} \right)(z, x) \)

and consider two integers \( N, R \) with
\[ 1 \leq N \leq R \quad ; \quad 1 \leq M \leq R. \]

Assume that

- there exist \( M \) equations in the sets \( F^{(1)} = 0 \) through \( F^{(N)} = 0 \) denoted by
  \( \tilde{G}_{\lambda} = 0, \ \lambda = 1 \) to \( M, \)

and \( M \) of the variables \( z^i \) (from the list \( z^i, i = 1 \) to \( R \)) denoted by
  \( \tilde{z}^i, \quad i = 1 \) to \( M \) identified through a known one-to-one map \( \kappa: \{1, 2, \ldots, M\} \rightarrow \{1, 2, \ldots R\} \)

by \( \tilde{z}^i := z^{\kappa(i)} \)

which satisfy \( \det \left[ \frac{\partial \tilde{G}_{\lambda}}{\partial \tilde{z}^i}(z, x) \right] \neq 0 \) locally in \((z, x)\) space.
Not Completely Integrable Case, contd.

Denote remaining \( R - M =: P \) variables \( z \) as \( \hat{z}^i, i = 1 \) to \( P \), defined by
\[
\mu : \{1, 2, \ldots, P\} \rightarrow \{1, 2, \ldots, R\} \quad \hat{z}^i := z^{\mu(i)}.
\]

Assume that around a point \( (\hat{z}_0, x_0) := (\hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n) \), the solution \( \bar{z}^i = \varphi^i \left( \hat{z}_0^1, \ldots, \hat{z}_0^P, x_0^1, \ldots, x_0^n \right), \quad i = 1, 2, \ldots, M \), of \( \tilde{G}_\lambda = 0, \quad \lambda = 1, 2, \ldots, M \) satisfies all the equations of the sets
\[
F^{(1)} = 0 \quad \text{through} \quad F^{(N+1)} = 0
\]
identically in a neighborhood of \( (\hat{z}_0, x_0) \) in \( (\hat{z}, x) \) space.

Then \( \exists \) a function \( w(x) \) satisfying
\[
\frac{\partial w^i}{\partial x^\alpha}(x) = \psi^i_\alpha \left( w(x), x \right) \quad \alpha = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, R
\]
determined by \( P \) constants.
Proof: Not completely Integrable case

Let $\Pi(\bar{z}, \hat{z}, x) \mapsto (z, x)$. Define,

$$G_{\lambda}(\bar{z}, \hat{z}, x) = \tilde{G}_{\lambda}(\Pi(\bar{z}, \hat{z}, x)), \quad \lambda = 1, 2, \ldots, M$$

$$\psi_{\alpha}^i(\bar{z}, \hat{z}, x) = \psi_{\alpha}^{k(i)}(\Pi(\bar{z}, \hat{z}, x)), \quad i = 1, 2, \ldots, M, \quad \alpha = 1, 2, \ldots, n$$

$$\psi_{\alpha}^j(\bar{z}, \hat{z}, x) = \psi_{\alpha}^{\mu(j)}(\Pi(\bar{z}, \hat{z}, x)), \quad j = 1, 2, \ldots, P, \quad \alpha = 1, 2, \ldots, n$$

Since $F^{(N+1)}(\varphi(\hat{z}, x), \hat{z}, x) \equiv 0$ in $(\hat{z}, x)$

$$\left[ \frac{\partial G_{\lambda}}{\partial \bar{z}^i} \psi_{\alpha}^i + \frac{\partial G_{\lambda}}{\partial \hat{z}^j} \psi_{\alpha}^j + \frac{\partial G_{\lambda}}{\partial x^\alpha} \right] \varphi(\hat{z}, x), \hat{z}, x) = 0 \quad (I)$$

For arbitrarily fixed path $f(t)$ with $f(0) = x^* \exists$ solution to ODE

$$\frac{dg^j}{dt}(t) = \left[ \psi_{\alpha}^j \circ (\varphi \circ (g, f), g, f) \frac{df^\alpha}{dt} \right](t) \quad ; \quad g^j(0) = \hat{z}^* \text{ (arbitrary)},$$

and $g$ satisfies $G_{\lambda} \circ (\varphi \circ (g, f), g, f)(t) = 0$ for $t$ in some local interval around 0.
Proof: Not Completely Integrable Case

Differentiating w.r.t. $t$, combining with $(I)$, and using arbitrariness of path $f$, $\hat{\alpha}^*$, and $x^*$ yields,

$$\overline{\psi}_\alpha^i \left( \varphi \left( \hat{\alpha}, x \right), \hat{\alpha}, x \right) - \frac{\partial \varphi^j}{\partial \hat{\alpha}^j} \left( \hat{\alpha}, x \right) \hat{\psi}_\alpha^j \left( \varphi \left( \hat{\alpha}, x \right), \hat{\alpha}, x \right) - \frac{\partial \varphi^i}{\partial x^\alpha} \left( \hat{\alpha}, x \right) \equiv 0 \ (II)$$

locally in $(\hat{\alpha}, x)$ space.

$F^{(1)}(\varphi(\hat{\alpha}, x), \hat{\alpha}, x) \equiv 0$ combined with $(II)$ implies that the conditions of complete integrability for

$$\frac{\partial g^j}{\partial x^\alpha}(x) = \hat{\psi}_\alpha^j \left( \varphi \left( g \left( x \right), x \right), g \left( x \right), x \right) \ (III)$$

are satisfied. Thus $\exists$ solution to $(III)$ with $P$ arbitrary constants.

Then it can be shown that

$$h^i(x) := \varphi^i \left( g \left( x \right), x \right) \ i = 1, 2, \ldots, M$$

$$g^j(x) \ j = 1, 2, \ldots, P$$

satisfy

$$\frac{\partial h^i}{\partial x^\alpha}(x) = \psi^j \circ \left( h, g, Id \right)(x) \quad \frac{\partial g^j}{\partial x^\alpha}(x) = \psi^\mu \circ \left( h, g, Id \right)(x). \ \square$$

If $u_0 u_0^T = B \left( x_0 \right)$ cannot be accommodated by $P$ constants, stick in $uu^T(x) = B(x)$ in $F^{(1)}$. Then use standard idea with Ricci to show $B$ compatibility.
Comments

- Proof adapted from Eisenhart, 1927, Veblen and Thomas, 1926
  - They claim necessity as well; I couldn’t

- Guess:
  - Related to Cartan’s Method of Equivalence
  - Modern treatment – R B Gardner CBMS-NSF-SIAM