Assuming that the $i-\mathrm{th}$ eigen-pair of the beam is $(\omega_i, \varphi_\mathrm{i})$, where $\varphi_i$ satisfying the following condition:

$$\int_0^l m\,\varphi_i^2 dx = \overline{m}l, \qquad i = 1,2,\ldots \tag{1}$$

where $m$ is the mass density of the beam and $\overline{m}$ is the average mass density of the beam.

Also assuming that the an arbitrary eigen-pair of the beam after imposing the constraint is $(\omega', \varphi')$. Expanding $\varphi'$ with respect to $\varphi_i, i = 1,2\ldots$ it yields that

$$\varphi' = \sum_{i=1}^{\infty} \xi_i \varphi_i \tag{2}$$

Substituting (2) into the Rayleigh quotient expression, we have

$$\omega'^2 = \mathrm{s.t}\ \frac{\sum_{i=1}^{\infty}\omega_i^2\xi_i^2}{\sum_{i=1}^{\infty}\xi_i^2} \tag{3}$$

The imposed constraint can be expressed in the following general form:

$$\sum_{i=1}^{\infty} \alpha_i \xi_i = 0 \tag{4}$$

where $\alpha_i$ are some constants. Then the problem is to find the stationary value of (3) under constraint (4). By introducing a Lagrange multiplier λ, the corresponding optimization problem can be expressed in the following equivalent form:

$$\omega'^2 = \mathrm{s.t}\ \frac{\sum_{i=1}^{\infty}\omega_i^2\xi_i^2 - 2\lambda\sum_{i=1}^{\infty}\alpha_i\xi_i}{\sum_{i=1}^{\infty}\xi_i^2} \tag{5}$$

The physical meaning of λ is the reaction force at the added support. From the stationary condition of (5), we have

$$\left(\omega_i^2 - \omega'^2\right)\xi_i - \lambda\alpha_i = 0, \quad i = 1,2,\ldots \tag{6}$$

then

$$\xi_i = \frac{\lambda\alpha_i}{(\omega_i^2 - \omega'^2)}, \qquad i = 1,2,\ldots \tag{7}$$

Substituting (7) into (4), we then obtain a equation for $\omega'$ if $\lambda \neq 0$:

$$f(\omega') = \sum_{i=1}^{\infty}\frac{\alpha_i^2}{(\omega_i^2 - \omega'^2)} = 0 \tag{8}$$

Wenistein pointed out that in some cases $\lambda = 0$. Under this circumstance, the eigen-pair of the beam will not be changed by adding the support.

If $\alpha_i = 0$ for some $i$, then $\omega_i' = \omega_i$ and $\phi_i' = \phi_i$.
If $\alpha_i \neq 0$ for all $i$, then the behavior of $f$ can be shown in Fig.1.

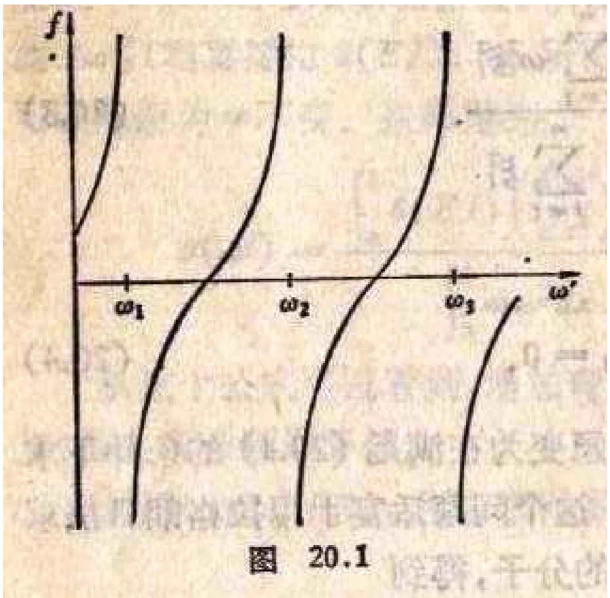


图 20.1

Figure.1

That is $f \to +\infty$ when $\omega' \to \omega_i - 0$; $f \to -\infty$ when $\omega' \to \omega_i + 0$. $f$ is monotonic increasing between $\omega_i$ and $\omega_{i+1}$. It follows that there is a unique root of $f(\omega')$ between $\omega_i$ and $\omega_{i+1}$, which is $\omega_i'$ −the $i-$th eigen value of the constrained beam. In summary we have

$$\omega_i \leq \omega_i' \leq \omega_{i+1},$$

which concludes the proof.