

RHEOLOGICAL MODELS

Rheology is the science of deformation. This science poses a question for every material: Given a history of stress, how do we predict the history of strain, or the other way around?

Rheology answers this question using mathematical models. In constructing a rheological model, we let material evolve through a sequence of homogeneous, but multiaxial, states of stress and strain. We represent a history of stress by a tensor as a function of time, and represent a history of strain by another tensor as a function of time.

For a given material, we fit a rheological model to the experimental data of some histories of stress and strain, and then use the model to predict other histories of stress and strain.

Deformation is often large, and rheological models are usually nonlinear. In this course we have studied several most successful rheological models: linear viscosity, nonlinear viscosity, viscoplasticity, and perfect plasticity. Here we summarize these models, and then construct several others: strain hardening, elastoplasticity, and viscoelasticity. In a separate course, we study large-deformation elasticity, electroelasticity, and poroelasticity (Suo 2013).

We will focus on models in their simplest forms. For deeper and more extensive discussions, see review articles and textbooks (e.g., Tanner 2000; Nemat-Nasser 2004; Xiao, Bruhns, Meyers 2006; Gurtin, Fried, Anand 2010; Irgens 2014).

The Second-Invariant Model

Incompressibility. When a material undergoes a large deformation, the constituents of the material—atoms, molecules and colloidal particles—change neighbors. The large deformation causes large change in shape, but nearly no change in volume. Unless otherwise stated, we neglect the small change in volume, and adopt the idealization that the material is incompressible. Let \mathbf{D} be a rate of deformation. The condition of incompressibility is $D_{kk} = 0$.

Hydrostatic stress does not affect deformation. Most models adopt a further idealization: Superposing a hydrostatic stress on a material does not affect its deformation. For example, such models neglect the effect of pressure on viscosity and yield strength. Let σ_{ij} be a state of stress. The mean stress is $\sigma_m = \sigma_{kk} / 3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$. Unless otherwise stated, we assume that deformation is entirely caused by the deviatoric stress, and is independent of the mean stress.

Isotropy. We will adopt yet another idealization: the material is isotropic. We formulate models in terms of the second invariant of the deviatoric stress, $s_{ij}s_{ij}/2$. We neglect the effect of the third invariant.

In a shearing flow, of the rate of shear $\dot{\gamma}$ and the shearing stress τ , the tensors of the rate of deformation and the deviatoric stress take the forms:

$$\mathbf{D} = \begin{bmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the shearing flow, the second invariant of the deviatoric stress is $s_{ij}s_{ij}/2 = \tau^2$.

This coincidence motivates the following definition. For a flow under a general state of deviatoric stress \mathbf{s} , define the equivalent shearing stress by

$$\tau_e = \sqrt{s_{ij}s_{ij}/2}.$$

The equivalent shearing stress τ_e is just another way to write the second invariant of the deviatoric stress, and is a scalar measure of the amplitude of the state of stress.

The second-invariant model. Here we focus on the model of the form $\mathbf{D} = \lambda \mathbf{s}$, where λ is a scalar. For a linearly viscous material, λ relates to the viscosity by $\lambda = 1/2\eta$. For other types of rheological behavior, including nonlinear viscosity and plasticity, λ depend on the state of stress. The second-invariant model assumes that λ depend on the state of stress through the second invariant, $\lambda(\tau_e)$. We fit this function to the experimental data measured in a simple type of stress, such as shear, or uniaxial tension.

The deviatoric stress is traceless, $s_{kk} = 0$, and ensures that the material is incompressible, $D_{kk} = 0$. Furthermore, the model requires that $\lambda \geq 0$, and thus satisfies the thermodynamic inequality $s_{ij}D_{ij} \geq 0$ for any state of stress.

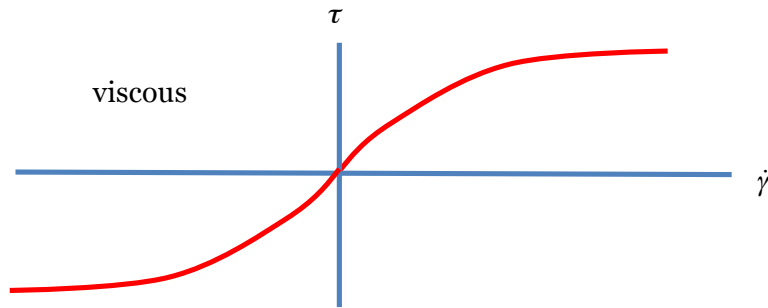
The equation $\mathbf{D} = \lambda \mathbf{s}$ dates back at least to the theory of linear viscosity (Stokes 1845). In describing other types of rheological behavior, this model is known as the J_2 flow theory, the Levy-Mises flow rule, or the generalized Newtonian fluid.

We have seen this model as a special case of Rayleigh's dissipation functions. We have also linked this model to the mathematics of convex functions (Suo 2014, notes on nonlinear viscosity).

Viscous Flow

Viscous flow. The model of viscosity answers the fundamental question of rheology in the purest form: The rate of deformation is a one-to-one function of the deviatoric stress. When the stress vanishes, the flow stops instantly, with no delay. When the stress changes, the rate of shear changes instantly, with no delay.

We test a material under a simple state of stress, such as shear. We plot the experimental data as the *flow curve*, the relation between the shearing stress, τ , and the rate of shear, $\dot{\gamma}$. The flow curve corresponds to a monotonic function, $\dot{\gamma} = h(\tau)$.



The second-invariant model relates two tensors, $\mathbf{D} = \lambda \mathbf{s}$. For a linearly viscous material, λ is a material constant, and relates to the viscosity as $\lambda = 1/2\eta$. For a nonlinearly viscous material, λ is stress-dependent and is a function of the equivalent shearing stress, $\lambda(\tau_e)$.

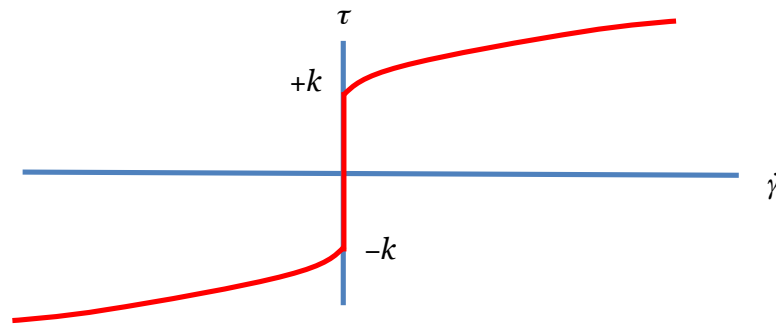
For a given material, we fit the function $\lambda(\tau_e)$ to the flow curve determined experimentally under one type of stress, such as shearing stress. When the fluid is under any state of stress, the second-invariant model predicts that the rate of deformation relates to the deviatoric stress as

$$\mathbf{D} = \frac{h(\tau_e)}{2\tau_e} \mathbf{s}.$$

This general model recovers the flow curve $\dot{\gamma} = h(\tau)$ determined under the shearing stress. So long as the flow curve $\dot{\gamma} = h(\tau)$ is a monotonically increasing function, the second invariant model specifies a one-to-one relation between the two tensors: the rate of deformation \mathbf{D} and the deviatoric stress \mathbf{s}

Viscoplastic flow. The model of viscosity is applicable even when the flow curve $\dot{\gamma} = h(\tau)$ is not a one-to-one relation between the shearing stress and the rate of shear. A viscoplastic fluid has a particular type of flow curve, with a yield stress k . The flow curve is not a one-to-one relation between the rate of

shear and the shearing stress: A range of stress $|\tau| < k$ gives a single value of the rate of shear, $\dot{\gamma} = 0$. When the stress exceeds the yield stress, $|\tau| > k$, the shearing stress and the rate of shear has a one-to-one relation. Thus, $h(\tau) = 0$ when $|\tau| < k$, and $h(\tau)$ is a monotonic function when $|\tau| > k$.



In a multiaxial state of stress, the second invariant model relates the tensor of rate of deformation to the tensor of deviatoric stress:

$$\mathbf{D} = \frac{h(\tau_e)}{2\tau_e} \mathbf{s}.$$

This model of viscoplasticity looks identical to that of nonlinear viscosity. For a viscoplastic material, however, the model no longer prescribes a one-to-one relation between the state of rate of deformation and the state of deviatoric stress. Given a state of stress \mathbf{s} , we calculate the equivalent shearing stress, $\tau_e = \sqrt{s_{ij}s_{ij}}/2$. When the state of stress is inside the yield surface, $\tau_e < k$, the material is rigid, and the rate of deformation vanishes, $\mathbf{D} = 0$. When the state of stress is outside the yield surface, $\tau_e > k$, the material flows, and the rate of deformation relates to the stress one-to-one.

Rigid, Perfectly Plastic Flow

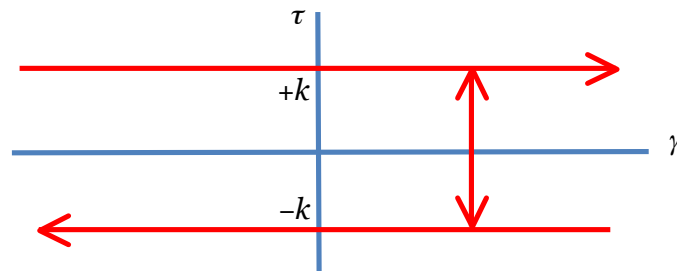
The word “rigid” means that the model neglects elastic deformation. The phrase “perfectly plastic” means that the model neglects strain hardening—that is, the yield strength k remains constant as the material flows.

This model is perhaps the most successful model of plasticity, and is widely used to analyze the forming processes of metals, and the load-carrying

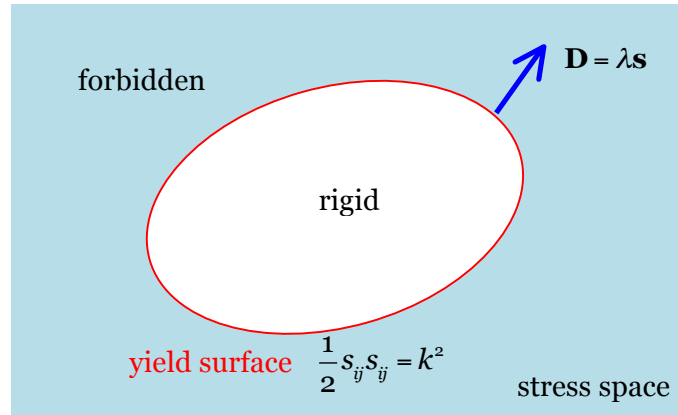
capacity of metallic structures. The model also describes the yield-stress fluids, such as toothpastes and greases. The model requires a single experimental input: the yield strength. Flows under various boundary conditions have been analyzed (Hill 1950, Tanner 2000).

Rigid, perfectly plastic flow in shear. We have studied rigid, perfectly flow in a simple state of stress, such as shear. The stress-strain diagram consists of two horizontal lines and infinitely many vertical lines. Because the model neglects strain hardening, the stress-strain diagram is invariant with the amount of strain. That is, the amount of flow does not affect the *plastic state* of the material.

- When the stress is within the yield strength, $|\tau| < k$, the material is rigid, and the strain does not change, $\dot{\gamma} = 0$. In the stress-strain diagram, this part corresponds to a bi-directional, vertical line.
- When the stress reaches the yield strength in the positive direction, $\tau = +k$, the strain can only increase, $\dot{\gamma} > 0$. The model of perfect plasticity does not specify the amount of strain. In the stress-strain diagram, this part corresponds to a horizontal line directing to the right.
- When the stress reaches the yield strength in the negative direction, $\tau = -k$, the strain can only reduce, $\dot{\gamma} < 0$. In the stress-strain diagram, this part corresponds to a horizontal line directing to the left.



Rigid, perfectly plastic flow in multiaxial state of stress. We have generalized this stress-strain diagram in shear into a model for any state of stress. Given a general state of stress \mathbf{s} , we calculate the equivalent shearing stress, $\tau_e = \sqrt{s_{ij}s_{ij}}/2$. The material obeys the Mises yield condition, $\tau_e = k$, where k is the yield strength under shear. The model neglects strain hardening, and the yield strength k remains constant as the material flows. Consequently, the Mises yield condition $\tau_e = k$ corresponds to a level set in the stress space, known as the *yield surface*.



The material obeys the Levy-Mises flow rule, $\mathbf{D} = \lambda \mathbf{s}$, where λ is a scalar measure of the flow, and depends on the state of stress as follows. The model forbids the stress to go outside the yield surface. When the stress is inside the yield surface, $\tau_e < k$, the material is rigid, and the rate of deformation vanishes, $\lambda = 0$. When the stress is on the yield surface, $\tau_e = k$, the material flows, $\lambda > 0$, but the model does not specify the magnitude of λ .

The flow rule $\mathbf{D} = \lambda \mathbf{s}$ has a pictorial interpretation: the rate of deformation is a “vector” normal to the yield “surface”.

Rigid, Strain-Hardening Flow

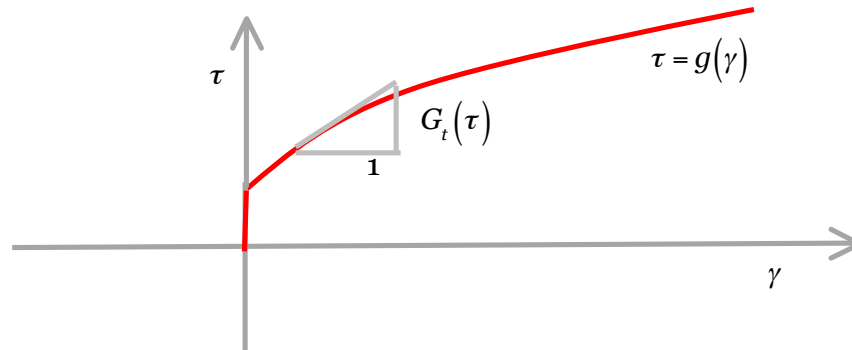
Rigid, strain-hardening flow in monotonic shear. We have studied this model for a material in a simple state of stress, such as shear. We apply a monotonically increasing stress as a function of time, $\tau(t)$, and measure the strain as a function of time, $\gamma(t)$. The material is said to be *rate-independent* if the stress-strain curve is unaffected by how the stress increases with time, so long as $\tau(t)$ is an increasing function. For a rate-independent material, we plot the experimental data as a stress-strain curve, $\tau = g(\gamma)$.

When the stress is below the initial yield strength, the material is rigid, and the stress increases along the vertical line, $\gamma = 0$. We neglect elastic deformation. We then increase the stress again, beyond the initial yield strength, and the material flows. The material is said to *strain-harden* when the stress-strain curve monotonically increases.

For a strain-hardening, rate-independent material, define the tangent modulus by $G_t = d\tau / d\gamma$. The strain-hardening material has a monotonic stress-strain curve. We can regard the tangent modulus either as a function of stress or

as a function of strain. Here we will write the tangent modulus as a function of stress, $G_t(\tau)$. For a rate-independent material, write

$$\dot{\tau} = G_t(\tau)\dot{\gamma}.$$



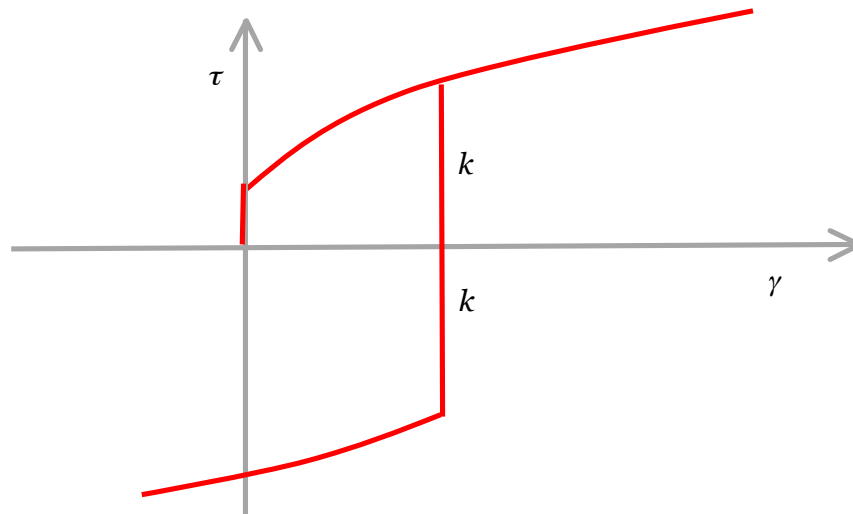
Rigid, strain-hardening flow in arbitrary history of shear.

Whereas the material is rate-independent, the response of the material depends on the *history of stress*. Now let $\tau(t)$ be a history of stress *not* monotonic in time. For example, we first increase the stress and cause the material to flow for some time, and then reduce the stress. Upon unloading, the stress and strain do not trace back along the curve; rather, the material becomes rigid, and the stress decreases along a vertical line.

Upon reloading, the material is initially rigid, and then flows at the previous point on the curve. That is, the previous flow has hardened the material and increased the yield strength to a new level. The bent stress-strain curve in monotonic shearing tells us how the yield strength k increases with flow.

Upon unloading to zero stress, we can shear the material in the opposite direction. When the magnitude of the stress is small, the material is rigid, and the stress goes down along vertical line. At some level of the shearing stress, the material flows, and the stress-strain curve bends. As an idealization, we assume that strain hardening is isotropic: the material yields at the current yield strength established by prior deformation, and the bent stress-strain curve in the negative direction is the same as that in the positive direction.

The above model of isotropic hardening achieves something extraordinary: it uses the experimental record of a single history (i.e., the monotonic loading) to describe all histories of loading, unloading, reloading, and reverse loading. Given a history of the shearing stress, $\tau(t)$, the model co-evolve the history of strain $\gamma(t)$, and the history of the yield strength, $k(t)$.



Internal variable and plastic state. We describe a state of the material using three parameters: the stress, the strain, and the yield strength. We call the *stress* and the *strain* the external variables, which together let the external world do work and change the state of the material. We call the yield strength an *internal variable*.

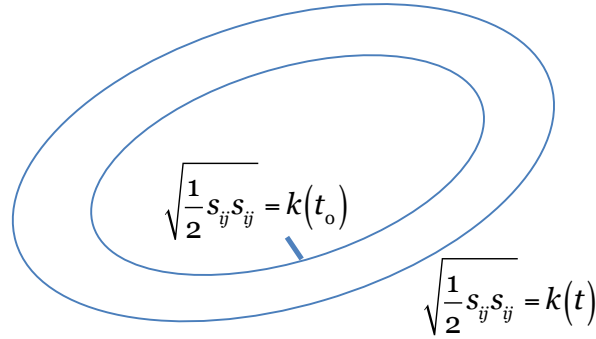
If we only look at the monotonic loading, the yield strength and stress coincide, so that the yield strength plays no distinct role. If we look at an arbitrary history of loading and unloading, the stress and strength play distinct roles, and the histories $\tau(t)$ and $k(t)$ are different. We say that the internal variable—the strength—characterizes the *plastic state* of the material (Prager 1949). As the material strain-hardens, the strength increases. The function $k(t)$ records the history of the plastic state.

We will use a single internal variable, the strength, to construct the isotropic-hardening model. In more complex models of plasticity, we use more than one internal variable to characterize the plastic state of the material.

Family of yield surfaces. We next generalize the model to an arbitrary history of multiaxial stress, $\mathbf{s}(t)$. Given a history of stress, we wish to construct a model that co-evolves the history of multiaxial strain and the history of strength. The model consists of three parts: the yield condition, the hardening rule, and the flow rule.

The second-invariant model assumes that the material obeys the Mises yield condition, $\tau_e = k$. For a strain-hardening material, the strength k increases as the material flows. In the stress space, the yield condition corresponds to a sequence of level sets, namely, a family of yield surfaces. The initial yield surface

is the yield surface of an as-received material, before we cause any flow. We will also speak of subsequent yield surfaces, and the current yield surface. Each yield surface corresponds to a distinct value of the strength.



Hardening rule. We next formulate the hardening rule to evolve the strength k . Given the history of general state of stress, $\mathbf{s}(t)$, we calculate the history of the equivalent stress,

$$\tau_e(t) = \sqrt{\frac{1}{2} s_{ij}(t) s_{ij}(t)},$$

as well as the rate of the equivalent stress, $\dot{\tau}_e = d\tau_e(t)/dt$.

The model evolves strength $k(t)$ incrementally. We know the yield strength at the current time, $k(t)$, we calculate

$$\dot{k} = \begin{cases} 0, & \text{if } \tau_e(t) < k(t) \\ 0, & \text{if } \tau_e(t) = k(t) \text{ and } \dot{\tau}_e(t) < 0 \\ \dot{\tau}_e(t), & \text{if } \tau_e(t) = k(t) \text{ and } \dot{\tau}_e(t) > 0 \end{cases}$$

When the state of stress is inside the current yield surface, $\tau_e(t) < k(t)$, the material is rigid, and the yield strength remains unchanged, $\dot{k} = 0$. When the state of stress is on the current yield surface, $\tau_e(t) = k(t)$ and but unloads, $\dot{\tau}_e(t) < 0$, the material is rigid, and the yield strength remains unchanged, $\dot{k} = 0$. When the state of stress is on the current yield surface $\tau_e(t) = k(t)$ and is loading, $\dot{\tau}_e(t) > 0$, the material flows strain hardens, increasing the strength at a rate $\dot{k} = \dot{\tau}_e(t)$.

Flow rule. The second-invariant model evolves the state of deformation according to the Levy-Mises flow rule, $\mathbf{D} = \lambda \mathbf{s}$. We calculate the scalar λ using the experimentally measured stress-strain curve. The flow rule becomes

$$\mathbf{D} = \frac{\dot{k}}{2kG_t(k)} \mathbf{s}.$$

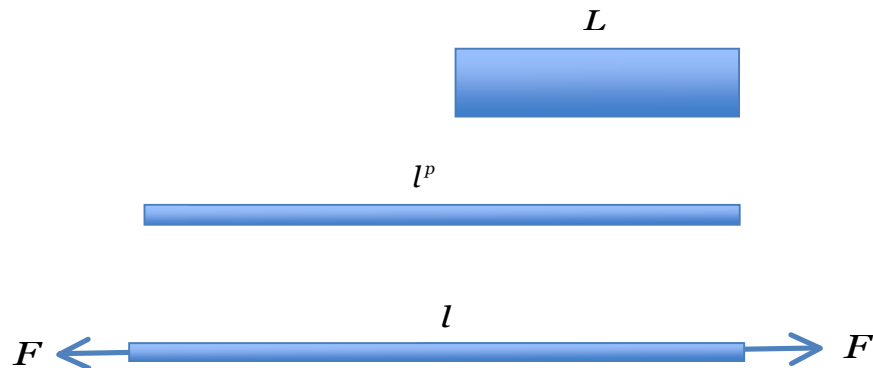
When the history of stress is of a special kind—the shearing stress as a monotonic function of time—the flow rule reduces to $\dot{\tau} = G_t(\tau) \dot{\gamma}$.

The model is rate-independent. The prediction depends on the direction of time, but not on the duration of time. That is, the prediction depends on the sequence of ups and downs in the history of stress, $\mathbf{s}(t)$, but is independent of the durations between ups and downs.

The model achieves something extraordinary: it uses the experimental record of a single history of a single type of stress (e.g., the stress-strain curve under monotonic shearing) to predict all histories and all states of stress. Hill (1950) credited this achievement to Schmidt (1932) and Odquist (1933).

Elastic-Plastic Flow

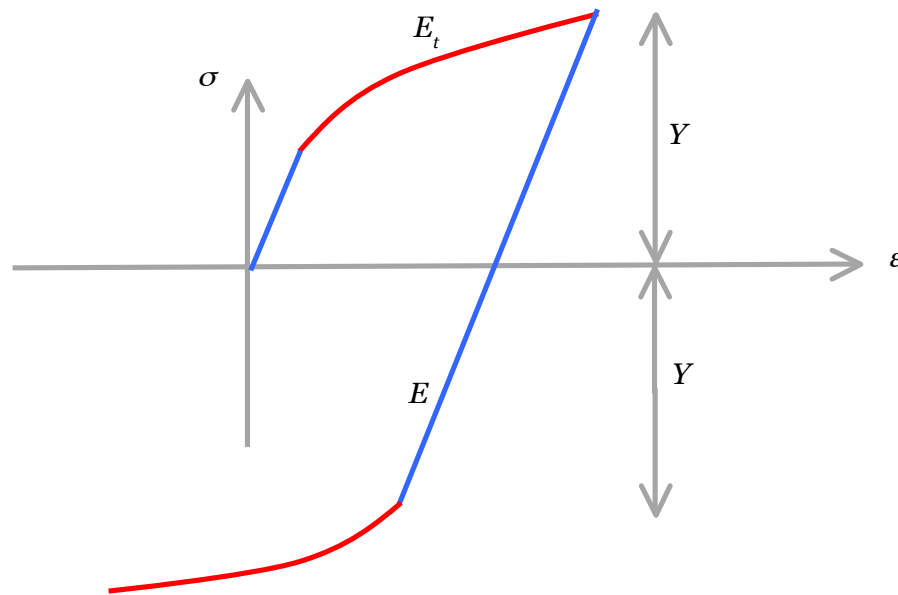
What does $\varepsilon = \varepsilon^e + \varepsilon^p$ mean? A rod is of length L in the initial stress-free state, and is of length l in the current state under stress. The ratio of the two lengths defines the stretch, $\lambda = l/L$.



Starting from the current state, we remove the stress, so that the rod unloads elastically and reaches an intermediate state of length l^p . Both the initial state and the intermediate state are stress-free, but the two states differ by plastic deformation; define the plastic stretch by $\lambda^p = l^p/L$. The intermediate

state and the current state differ by an elastic deformation; define the elastic stretch by $\lambda^e = l/l^p$.

Note an identity: $\lambda = \lambda^e \lambda^p$. Recall the definition of the natural strain, $\varepsilon = \log \lambda$. Similarly, write $\varepsilon^p = \log \lambda^p$ and $\varepsilon^e = \log \lambda^e$. The expression $\lambda = \lambda^e \lambda^p$ is equivalent to $\varepsilon = \varepsilon^e + \varepsilon^p$.



Elastoplasticity under uniaxial stress. We adopt the assumption that the total strain is a sum of the elastic strain and plastic strain:

$$\varepsilon = \varepsilon^e + \varepsilon^p .$$

The elastic strain relates to the stress as

$$\varepsilon^e = \frac{\sigma}{E} .$$

In the stress-strain diagram, we draw a straight line of slope E to represent elastic loading and unloading. The projection of this line on the axis of strain is the elastic strain. Because the stress is usually much smaller than Young's modulus, the straight line is nearly vertical, and the elastic strain is typically less than 1%. The elastic strains in the two transverse directions are $-\nu\sigma/E$, where ν is Poisson's ratio.

If the current stress satisfies the yield condition, $|\sigma| = Y$, and the increment of the stress causes plastic loading, $d|\sigma| > 0$, the metal loads along the

bent curve. The increment of stress changes the yield strength by $dY = |d\sigma|$, and changes the plastic strain by

$$d\varepsilon^p = \left(\frac{1}{E_t(Y)} - \frac{1}{E} \right) d\sigma.$$

This expression gives the increment of strain in the axial direction. The plastic strains in the two transverse directions are one half that in the axial direction.

Elastic, perfectly plastic flow in multiaxial state. We adopt the assumption that the rate of deformation is the sum of elastic and plastic parts (Hill 1959; Prager 1961):

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p.$$

The model calculates the rate of plastic strain as follows. Let σ_{ij} be a state of stress. The mean stress is $\sigma_m = \sigma_{kk} / 3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$. Calculate the equivalent shearing stress, $\tau_e = \sqrt{s_{ij}s_{ij}} / 2$. The material obeys the Mises yield condition, $\tau_e = k$, where k is the yield strength under shear. The model neglects strain hardening, and the yield strength k remains constant as the material flows. The rate of plastic deformation obeys the Levy-Mises flow rule, $\mathbf{D}^p = \lambda \mathbf{s}$. The model forbids the stress to go outside the yield surface. When the stress is inside the yield surface, $\tau_e < k$, the material is rigid, and the rate of deformation vanishes, $\lambda = 0$. When the stress is on the yield surface, $\tau_e = k$, the material flows, $\lambda > 0$, but the model does not specify the magnitude of λ .

The model calculates the rate of elastic deformation according to Hooke's law. For metals, dilatational strain and shearing strain are typically of comparable magnitude. We will assume that elastic deformation is compressible, but plastic deformation is incompressible. Thus, the rate of dilatation is linear in the rate of the mean stress:

$$D_{kk} = \frac{\dot{\sigma}_m}{B},$$

where B is the bulk modulus. The rate of deviatoric deformation is linear in the rate of deviatoric stress:

$$D_{ij}^e - \frac{1}{3} D_{kk} \delta_{ij} = \frac{\partial s_{ij}}{2G},$$

where G is the shear modulus. To preserve frame-indifference, we have used the co-rotational rate:

$$\partial \mathbf{s} = \partial_{in} \mathbf{s} - \mathbf{W} \mathbf{s} + \mathbf{s} \mathbf{W}.$$

Elastic, isotropic hardening flow in multiaxial state. Once again we adopt the assumption that the rate of deformation is the sum of the elastic and plastic parts:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p .$$

We adopt the Mises yield condition $\tau_e = k$, and the same hardening rule as in the model of rigid, isotropic hardening flow.

We also adopt the Levy-Mises flow rule, $\mathbf{D}^p = \lambda \mathbf{s}$, and assume that λ is a function of the second invariant, $\lambda(\tau_e)$. We fit the function $\lambda(\tau_e)$ to the experimental data under monotonic shear. In the shearing flow, the model reduces to

$$\dot{\gamma} = \dot{\gamma}^e + \dot{\gamma}^p, \quad \dot{\gamma} = \frac{\dot{\tau}}{G_t(\tau)}, \quad \dot{\gamma}^e = \frac{\dot{\tau}}{G} .$$

Consequently, the rate of plastic deformation relates to the rate of stress as

$$\dot{\gamma}^p = \left[\frac{1}{G_t(\tau)} - \frac{1}{G} \right] \dot{\tau} .$$

When the material is subject to a general history of multiaxial stress, the flow rule becomes:

$$\mathbf{D}^p = \left(\frac{1}{G_t(k)} - \frac{1}{G} \right) \frac{\dot{k}}{2k} \mathbf{s} .$$

We specify the rate of elastic deformation according to Hooke's law:

$$D_{kk} = \frac{\dot{\sigma}_m}{B} ,$$

$$D_{ij}^e - \frac{1}{3} D_{kk} \delta_{ij} = \frac{\partial s_{ij}}{2G} .$$

To preserve frame-indifference, we adopt the co-rotational rate:

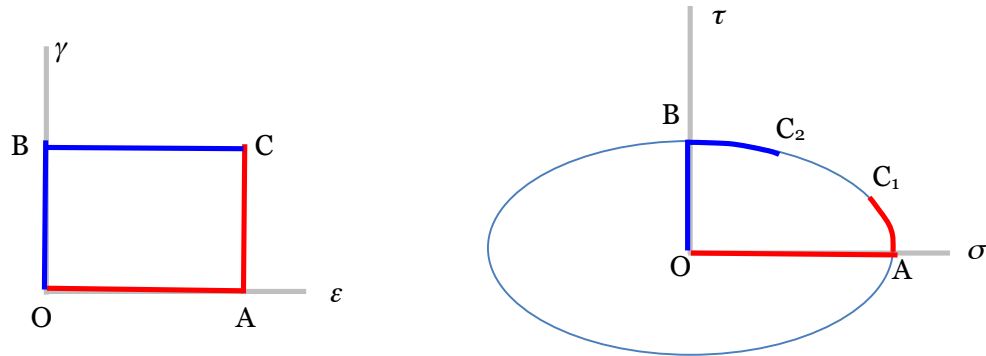
$$\partial \mathbf{s} = \partial_{in} \mathbf{s} - \mathbf{W} \mathbf{s} + \mathbf{s} \mathbf{W} .$$

Elastic-plastic deformation is history-dependent. One can pull and twist a thin-walled tube to achieve various histories of homogeneous deformation. This experimental setup has played a significance role in the development of plasticity (Hill 1950).

The elastic, plastic response is history-dependent. Describe a history of stress by the tensile stress and shearing stress as functions of time, $\sigma(t)$ and $\tau(t)$. Describe a history of deformation by the tensile strain and shearing strain as functions of time, $\varepsilon(t)$ and $\gamma(t)$. For a rate-independent material, a history of stress is a curve on the (σ, τ) plane, and a history of strain is a curve on the (ε, γ)

plane. Each history is fully specified by a curve and its direction. Each point on the curve corresponds a state of stress or strain. How fast the material goes from one state to another does not matter.

Given a history of stress, the elastic, perfectly plastic model cannot predict a unique history of strain. Because once the material reaches the yield condition, the amount of deformation is arbitrary. Given a history of strain, however, the elastic, perfectly plastic model predicts a unique history of stress.



Consider two histories strain. In history OAC, we pull the tube to the state just below the yield condition, hold the tensile strain, and then twist. In history OBC, we twist the tube to the state just below the yield condition, hold the shearing strain, and then pull. The two histories reach the same state of strain at the end, C.

The elastic, perfectly plastic model predicts the two corresponding histories of stress. The two histories will end at two different states of stress, C_1 and C_2 .

Let's calculate. The Mises yield condition is

$$\frac{1}{3}\sigma^2 + \tau^2 = k^2.$$

The yield locus is an ellipse in the stress plane. The stress-strain relation is

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{2}{3}\lambda\sigma, \quad \dot{\gamma} = \frac{\dot{\tau}}{G} + 2\lambda\tau.$$

Here we neglect the rate of rotation, and replace the co-rotational rate with the in-frame rate. When the state of stress is inside the yield locus, the material is elastic and $\lambda=0$. When the state of stress moves on the yield surface, the material flows, and $\lambda>0$. Given a history of strains, $\epsilon(t)$ and $\gamma(t)$, the above three equations determine the histories of $\lambda(t)$, $\sigma(t)$ and $\tau(t)$.

To simplify algebra, assume the material in incompressible, so that $E=3G$. Eliminating λ by combining the two equations in the flow rule, we obtain that

$$\frac{3G\dot{\epsilon} - \dot{\sigma}}{G\dot{\gamma} - \dot{\tau}} = \frac{\sigma}{\tau}$$

Along the segment AC, the tensile strain is fixed, $\dot{\epsilon} = 0$, and the state of stress moves along the yield locus, $\sigma^2/3 + \tau^2 = k^2$.

$$\frac{k^2\dot{\tau}}{k^2 - \tau^2} = G\dot{\gamma}$$

Integrating over the interval $0 < \gamma < k/G$, we get the state C₁: $\sigma = 0.76\sqrt{3}k$, $\tau = 0.65k$. We can similarly get state C₂: $\sigma = 0.65\sqrt{3}k$, $\tau = 0.76k$.

Ratcheting plastic deformation. Describe Huang, Suo, Ma (2002).

Viscoelastic Flow

In a separate course, we study viscoelasticity in infinitesimal deformation (Suo 2008). Many powerful ideas in linear viscoelasticity become invalid when materials are nonlinear. Here we will focus on one idea that is valid for large deformation: representing rheological model using springs and dashpots (Reiner 1945).

For example, consider the Maxwell model, represented by a spring and a dashpot in series. We interpret this picture as

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^v,$$

where \mathbf{D}^e represents the rate of elastic deformation, and \mathbf{D}^v represents the rate of viscous deformation. We calculate the rate of viscous deformation using the second-invariant model:

$$\mathbf{D}^v = \frac{\mathbf{s}}{2\eta}.$$

We calculate the rate of elastic deformation using Hooke's law:

$$D_{kk} = \frac{\dot{\sigma}_m}{B}$$

$$D_{ij}^e - \frac{1}{3}D_{kk}\delta_{ij} = \frac{\partial s_{ij}}{2G}.$$

To preserve the frame-indifference, we adopt the co-rotational rate.

$$\partial \mathbf{s} = \partial_{in} \mathbf{s} - \mathbf{W}\mathbf{s} + \mathbf{s}\mathbf{W}.$$

A commonly used model for metals and ceramics at elevated temperatures is to model viscosity by power law, and model elasticity using Hooke's law. The rate of deformation is often negligible, so that the co-rotational rate is replaced with the in-frame rate.

We can, of course, adopt other arrangements of springs and dashpots, other models of viscosity and elasticity, and other frame-indifferent rates. We get

many viscoelastic models (Tanner 2000, Irgens 2014). Sorting what models predict what phenomena is not easy, and is still an ongoing topic of research.

Inhomogeneous Deformation

When a body of the material undergoes inhomogeneous deformation, we regard the body as a sum of many small pieces. Each small piece undergoes homogeneous deformation, and obeys the rheological model specified above. Different pieces in the body communicate through the compatibility of deformation and the balance of forces. We use the *Eulerian approach*.

Time derivative of a function of material particle. At time t , a material particle \mathbf{X} moves to position $\mathbf{x}(\mathbf{X}, t)$. The velocity of the material particle is

$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}.$$

Here, the independent variables are the time and the coordinates of the material

Let $G(\mathbf{X}, t)$ be a function of material particle and time. For example, G can be the temperature of material particle \mathbf{X} at time t . The rate of change in temperature of the material particle is

$$\frac{\partial G(\mathbf{X}, t)}{\partial t}.$$

This rate is known as the *material time derivative*.

We can calculate the material time derivative by an alternative approach. Change the variable from \mathbf{X} to \mathbf{x} by using the function $\mathbf{X}(\mathbf{x}, t)$, and write

$$g(\mathbf{x}, t) = G(\mathbf{X}, t)$$

Using chain rule, we obtain that

$$\frac{\partial G(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i(\mathbf{X}, t)}{\partial t}.$$

In particular, we replace the in-frame rate of stress with the material derivative, so that the correlational rate becomes

$$\partial \mathbf{s} = \frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial t} + v_i \frac{\partial \mathbf{s}(\mathbf{x}, t)}{\partial x_i} - \mathbf{W} \mathbf{s} + \mathbf{s} \mathbf{W}.$$

Compatibility of deformation. Let \mathbf{x} be the coordinate of a place in space, t be the time, and $v_i(\mathbf{x}, t)$ be the velocity of a small piece of fluid at the place \mathbf{x} and time t . The compatibility of deformation relates the rate of deformation to the field of velocity:

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

This relation between the rate of deformation and velocity holds for deformation of any magnitude.

Balance of forces. Let $b_i(\mathbf{x}, t)$ be the body force per unit volume, and ρ be the mass per unit volume. For an incompressible fluid, ρ is constant. Inside the body, the balance of forces relates the stress to the body force and the inertial force:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right).$$

The right-hand side is the inertial force per unit volume.

Let $n_i(\mathbf{x}, t)$ be the unit vector normal to a small part of the surface of the body. Let $t_i(\mathbf{x}, t)$ be the traction, i.e., the force per unit area acting on the surface. On the surface of the body, the balance of forces relates the stress to the traction:

$$\sigma_{ij} n_j = t_i.$$

Boundary conditions. We divide the surface of the body into two parts. On one part of the surface, S_v , we prescribe the velocity of the fluid. On the other part of the surface, S_t , we prescribe the traction. These boundary conditions, along with the equations listed above, constitute a boundary-value problem that governs the inhomogeneous deformation in the body.

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