SCALAR

We call a one-dimensional vector space a *scalar set*, and call each element in the set a *scalar*. Scalars are scalable. Scalars scale.

Being linear is being scalar. Scalars are fundamental building blocks of linear algebra. We will use scalar sets to construct vector spaces of any dimension. We will use scalar sets to define other objects, such as linear form, bilinear form, quadratic form, and inner product.

Scalar sets are also fundamental building blocks of the models of the world. In physics, we use scalar sets to model mass, charge, energy, space, and time. In economics, we use scalar sets to model population, money, commodity, and labor.

Give us scalars. We build a world.

Number and Scalar

Scalars in linear algebra. Textbooks of linear algebra use the word "scalar" in two ways. They correspond to two different algebraic structures.

In the first usage, the word scalar is a synonym to the word number, an element in a number field. As we have seen before, a number field is a set F closed under two operations: addition and multiplication. Adding two elements in F gives an element in F, and multiplying two elements in F gives an element in F. In linear algebra we will mostly use the field of real numbers, R, and the field of complex numbers, C. Occasionally we will use the field of rational numbers, Q.

In the second usage, the word scalar appears in the definitions of linear form, bilinear form, quadratic form, and inner product. In this second usage, the scalar is an object distinct from the number. Let us denote a scalar set by S. Adding two elements in S gives an element in S, multiplying an element in S with an element in F gives in an element in S, but multiplying two elements in S does not give an element in S. In fact, the last operation is meaningless in the definition of scalar set S.

Scalars in physics. In physics, the word scalar is used to indicate a property like mass, volume, charge, and energy. These properties are scalable and additive, and are known as extensive properties. The usage in physics is consistent with the second usage in linear algebra, but is inconsistent with the first one in several ways:

- A physical property like mass is more than just a number; it has a unit.
- The multiplication defined on a field makes no sense to a physical quantity like mass: the multiplication of two elements in F gives yet another element in F, but the multiplication of two masses does not give another mass.
- If we regard both mass and volume as elements in the field F, then we need to assign a meaning to the addition of mass and volume. What does that even mean?

Do not confuse number and scalar. Using the same word scalar in two ways, textbooks of linear algebra confuse the objects of two distinct types. We will not follow this practice; rather, we will call each element in the field F a number, and will reserve the word scalar for an object in another set, which we call scalar set. As we will see, a number field and a scalar set have different algebraic structures. Numbers and scalars are objects of different types.

Scalar Set

A set S is called a scalar set over a number field F if the following conditions hold.

Adding two elements in *S* gives an element in *S*. To any two elements x and y in S there corresponds an element in S, written as $x+y$, called the addition of x and y. Addition obeys the following rules:

(1) Addition is commutative: $x+y=y+x$ for every x and y in S.

(2) Addition is associative: $(x+y)+z=x+(y+z)$ for every x, y and z in S.

(3) There exists an element 0 (the zero element) in S such that $0+x=x$ for every x in S.

(4) For every x in S there exists an element (the negative element) z in S such that $x+z=0$. We write $z=-x$.

Multiplying an element in *F* and an element in *S* gives an element in *S*. To every element α in F and every element x in S there corresponds an element in S, written as αx, called the multiplication of α and x. The multiplication obeys the following rules:

(5) $1 \cdot x = x$ for every x in S.

(6) $\alpha(\beta x)=(\alpha\beta)x$ for every x in S and for every α, β in F.

Multiplication is distributive over addition. The multiplication of an element in F and an element in S is distributive over two the types of addition: addition of elements in F and addition of elements in S.

(7) $(\alpha+\beta)x=\alpha x+\beta x$ for every x in S and for every α, β in F.

(8) $\alpha(x+y)=\alpha x+\alpha y$ for every x and y in S and every α in F.

Scalar set is one-dimensional. This statement requires two conditions.

(9) The scalar set S has at least one nonzero element.

(10) Any two elements in S are linearly dependent. For every two elements x and y in S, there exist α and β in F, not both of which are zero, such that $\alpha x+\beta y=0$.

Numbers and scalars. We call each element in F a number, and each element in S a scalar.

Remarks

The same word means different things. The element zero in the set F is an object different from the element zero in the set S. The two objects have the same notation, 0. We tell them apart by seeing them in context.

Likewise, we distinguish the addition of two numbers from the addition of two scalars, and distinguish the multiplication of two numbers from the multiplication a number and a scalar.

A scalar set is a commutative group. Axioms (1)-(4) do not mention the set F, and are devoted entirely to the set S. The four axioms define S as a commutative group, with the scalars as the elements, addition as the operation, and 0 as the identity element.

A scalar set is a one-dimensional vector space. For people who know the definition of vector space, it is evident that a scalar set is, by definition, a one-dimensional vector space. Axioms (1)-(8) define a vector space of any dimension, whereas Axioms (9) and (10) make the scalar set one-dimensional. We will talk about vector space in depth later.

Number field and scalar set have different algebraic structures. The definition of number field invokes a single set F and two operations. As stipulated in the definition of a field, the set F is closed under two operations: addition and multiplication. Adding two elements in F gives an element in F:

$$F \times F \xrightarrow{\text{add}} F .$$

Multiplying two elements in F gives an element in F:

$$F \times F \xrightarrow{\text{multiply}} F .$$

By contrast, the definition of scalar set invokes two sets F and S, as well as four operations. We retain the two operations on F:

$$F \times F \xrightarrow{\text{add}} F ,$$

$$F \times F \xrightarrow{\text{multiply}} F .$$

The definition of scalar set invokes two other operations. Adding two elements in S gives an element in S

$$S \times S \xrightarrow{\text{add}} S .$$

Multiplying an element in F and an element in S gives an element in S:

$$F \times S \xrightarrow{\text{multiply}} S .$$

Given two sets S and F, we can think of many possible operations to combine elements in the two sets. Most operations, however, do not appear in the definition of scalar set. In particular, we exclude from the definition of scalar set any operation that might represent the addition of an element in F and an element in S, or represent the multiplication of two elements in S.

Examples

A number field F is a scalar set over itself. The definitions of number field and scalar set have many similarities. Indeed, a number field F is a scalar set over itself. We confirm this statement by confirming the set F also satisfies the rules of the addition of two scalars, and of the multiplication of a scalar and a number.

For any two elements x and y in F, the addition $x+y$ is an element in F. The rules for the addition of two numbers are identical to the rules for the addition of two scalars.

For any two elements α and x in F, the multiplication αx is an element in F. The rules of this multiplication are identical to the rules of the multiplication of a number and a scalar, if we regard α as a number and x as a scalar.

The number field is a peculiar scalar set. We now turn to a more representative scalar set.

Arrows in a line. An arrow drawn in a straight line has a direction and a length. The two ends of the arrow are called the tail and head. The collection of all such arrows forms a set S. Two arrows are regarded as the same element in S if one of them can be translated onto the other. To complete the set S, we include the element zero, which has zero length and unspecified direction.

For two arrows x and y, we define $x+y$ as follows. We make the tail of y coincide with the head of x. We call the tail of x the tail of $x+y$, and the head of y the head of $x+y$.

For a positive real number α and an arrow x, we define αx to be a arrow of the same direction as x, and of length α times that of x. For a negative number α, we define αx to be an arrow in the opposite direction of x, of length $|\alpha|$ times that of x.

The set of arrows S in a straight line is a scalar set over the field of real numbers. This example is of great importance: it is a model of physical quantities like distance, time, volume, mass, energy, matter, and electric charge. It also models population, commodity, labor, and money.

Gold. We have collected pieces and pieces of gold. We can define the addition of the pieces, but we do not have a sensible definition of the multiplication of the pieces. Thus, this set of gold is not a number field.

This set of gold, however, is a scalar set over the field of real numbers. Adding two pieces of gold corresponds to a piece of gold. Multiplying a real number α and a piece of gold corresponds to a piece of gold α times the amount.

Money. The set of all different amounts of money is a scalar set. We regard US dollars and Euros as different scalar sets. Within the set of US dollars, the addition of two amounts means putting the two amounts together. Multiplying a real number α and an amount of money corresponds to an amount of money α times.

Turn numbers into scalars. The set of all numbers of form $p\sqrt{2}$, where *p* is a rational number, is a scalar set over the field of rational numbers.

The set of all numbers of form $p\sqrt{2}+q\sqrt{3}$, where p are q are rational numbers, is not a scalar set over the field of rational numbers. The set violates Axim (10).

The set of all numbers of form bi, where b is a real number and $i=\sqrt{-1}$, is a scalar set over the field of real numbers.

The set of all numbers of form $a+bi$, where a are b are real numbers and $i=\sqrt{-1}$, is not a scalar set over the field of real numbers. The set violates Axim (10).

Unit and Magnitude

Scalars scale. Elements in a scalar set scale with one another. Let *S* be a scalar set over a number field *F*. Axiom (9) states that *S* has at least one none zero element. Let *u* be a nonzero element in *S*, and let *s* be any element in *S*. Axiom (10) states that there exist two elements α and β in *F*, not both of which are zero, such that

$$\alpha s+\beta u=0 .$$

Because u is a nonzero element in *S*, α must be a nonzero element in *F*; otherwise, β would be the element zero in *F*, which would make both α and β the element zero in *F*.

Now that $\alpha\neq 0$, we write equation $\alpha s+\beta u=0$ as

$$s=-\alpha^{-1}\beta u .$$

Denote $s_M=-\alpha^{-1}\beta$ and write

$$s=s_M u ,$$

where s_M is a number in *F*. We call *u* a *unit* of the scalar set *S*, and s_M the *magnitude* of the scalar *s* relative to the unit *u*. We say that elements in *S* scale with one another.

Remark. For people who know about the vector space of any dimension, we note the following. A scalar set is, by definition, a one-dimensional vector space. Nonetheless it is helpful to replace generic terms for a vector space with specific terms for a scalar set. We replace the phrase "one-dimensional vector space" with the phrase "scalar set". We call an element in the vector space a vector, and call an element in the scalar set a scalar. A basis of an n-dimensional vector space consists of n linearly independent elements in the vector space, whereas a unit of a scalar set is a nonzero element in the scalar set. We replace the phrase "components of a vector relative to a basis" with the phrase "magnitude of a scalar relative to a unit".

Unit of money. In the United States, the unit of money is a dollar, with symbol \$. We know that

1 cent = 0.01\$

1 dime = 0.1\$

1 quarter = 0.25\$

Do not confuse a scalar with its magnitude. For a scalar set S over a number field F, given a unit u, a scalar s in S scales with the unit, $s = s_M u$, where the magnitude s_M is a number in F. For the set of gold, the scalar is an object, a piece of gold, and the magnitude is a real number. We do not confuse a piece of gold with a number.

Bijection between scalar set and number field. Once we fix a nonzero element u in S, the equation $s = s_M u$ defines a bijection between the sets S and F. That is, the equation $s = s_M u$ establish a one-one correspondence between the a scalar s and its magnitude s_m. The former is an element in S, and the latter an element in F.

If F is the field of real numbers, F is an ordered set. The bijection between F and S ensures that S is also an ordered set.

If F is the field of complex numbers, F is an unordered set. The bijection between F and S makes S also an unordered set.

Change of Unit

Given a scalar set S over a number field F, once we choose a nonzero element u in S as a unit, any other element s in S scales with the unit, $s = s_M u$. Here s_M denotes the magnitude of the scalar s relative to the unit u. For given u and s, the value of s_M is unique.

We can, of course, choose any nonzero element in S as a unit. Let u and $\tilde{u}$ be two non-zero scalars in a scalar set S over a number field F. The two scalars are proportional to each other:

$$\tilde{u} = pu ,$$

where p is a number in F, and is the magnitude of the scalar $\tilde{u}$ relative to the scalar u.

A scalar s in S scales with either unit:

$$s = s_M u = \tilde{s}_M \tilde{u} ,$$

where the number s_M in F and is the magnitude of the scalar s relative to the unit u, and the number $\tilde{s}_M$ is the magnitude of the scalar s relative to the unit $\tilde{u}$.

A combination of the above expressions gives that

$$s_M = p\tilde{s}_M .$$

This expression relates three numbers in F. The magnitude of a scalar converts in a way opposite to the way in which the unit of the scalar set converts. Thus, the scalar set is contravariant.

Examples. Kilogram and pound are two units of mass. They convert to each other by 1 kilogram = 2.20462 pounds. Thus, a 10-pound turkey is 4.5 kilograms.

Money in the United States comes in many units: cent, nickel, dime, quarter, and dollar. These units are represented by distinct physical objects.

Model vs. Reality*

Apple. We have piles and piles of apples. We perform operations of two types. The addition of any two piles corresponds to another pile having the same quantity of apples as the two piles put together. The multiplication of any pile by any real number α corresponds to another pile α times the quantity of applies. The multiplication requires us to multiply apples by number, but does not require us to multiply apples by apples.

The piles and piles of apples form a set. If we wish to emphasize, we write the set as APPLE. Each element in the set is a pile containing a distinct quantity of apples. We model this set as a scalar set over the field of real numbers. As we have just learned in the formal definition, a scalar set is a set *closed* under two operations. The addition of any two elements in the set corresponds to another element in the set. The multiplication of any element in the set and any real number corresponds to another element in the set.

The piles and piles of apples form a scalar set. Denote a particular pile of apples by u. Any other pile of apples, x, is pile u multiplying a real number α, namely,

$$x = \alpha u .$$

All elements in the scalar set *scale* with one another by real numbers.

Model and reality. The definition of scalar set requires that an element in the set multiplying any real number be an element in the set. If the real number is too large, we do not have that many apples. If the real number is too small, we reach subatomic dimension, in which case the "pile" no longer contains any apple. Also, the definition of the vector space will require that negative quantities of apples be in the set.

In representing the reality with a model, we ignore inconvenient truths. (All models are wrong, but some are useful.) But we do check what the model predicts against the reality. If the model predicts a negative quantity of apples, it means that we are in deficit. If the model predicts a non-integer quantity of apples, we cut apples in pieces, or just round up. If a piece is too small to preserve its appleness, we may approximate the piece as the element "zero".

A model achieves the economics of abstraction. A model is not the reality. Why do we link the reality to a model then? A map of a city is not the city, but the map lets us plan a tour without walking through the city. The map would be useless if it were as large and as detailed as the city. The model abstracts: it subtracts most details, retains a few, and idealizes them. Abstraction is value. A sketch of a shrimp by Qi Baishi is worth a lot more than a photo of a shrimp, or the shrimp itself.

The model of the piles of apples lets us reason something about apples without having apples. The reasoning is about the quantities of apple. This model ignores all other aspects of apples—the smell, the taste, the color, etc.

Economics of scale. This particular model—the scalar set, and its generalization, the vector space—has been reasoned thoroughly in mathematics. The model applies to piles of apples, and to piles of oranges. The model applies to piles containing both apples and oranges. The model applies to durations of time. The model, with one more feature (the metric), applies to displacements in space. The model, with yet one more feature (the constancy of the speed of light), applies to spacetime. The model achieves the economics of abstraction, as well as the economics of scale.

Scalars in Nature*

Like calculus and geometry, algebra has long been used as a part of language for physics. Here we highlight the language of linear algebra in physics.

Extensive property. A piece of a substance, such as gold, has many physical properties, including volume, shape, color, temperature, mass, and energy. A physical property is extensive if it is proportional to the amount of the substance. Volume, mass, and energy are extensive properties. Shape, color, temperature are not extensive properties.

We can use a one-dimensional vector space S to model an extensive physical property such as mass. In this model, F is the field of real numbers. We define the addition of two masses by lumping them together. We define the multiplication of a mass and a real number α by another mass α times the amount.

Mass. Various amounts of mass form a scalar set. You might as regard this statement as a law of physics.

For this scalar set, the unit of mass, kilogram, is the mass of a block metal, called the International Prototype Kilogram (IPK), preserved in a vault located in Sevres, France. Any other mass equals this unit times a real number. For example, 1.7 kg means a mass, which is 1.7 times the mass of the IPK. The mass of the IPK is an element in the scalar set of masses, and by an international convention we agree to call it a unit of mass. The mass 1.7 kg is another element in the set.

Length. The set of points on a straight line is not a scalar set, because it is unclear how we define the addition of two points, or the multiplication of a point and a number. However, we can form a scalar set by the following procedure. Mark a particular point on the line as the origin. The position of any point on the line relative to the origin defines a directed segment. Each directed element has a direction and a length.

The set of all directed segments is a scalar set over the field of real numbers. The addition of two directed segments x and y is a directed segment, formed by placing the tail of x at the origin, and placing the tail of y at the tip of x. The multiplication of a real number α and a segment x is a segment of length α times that of x.

The length of the rod preserved near Paris defines a *meter*. Factors of conversion are never pretty. Consider 1 m = 39.375 inch.

Time. Similarly, the set of all times is not a scalar set, because it is unclear how we define the addition of two times, or the multiplication of a time and a number. However, we can form a scalar set by following a similar procedure. Mark a particular time as the reference. The difference of any other time relative to this reference defines a directed interval. The set of all directed intervals is a scalar set over the field of real numbers. The addition of two directed intervals x and y is a directed interval, formed by placing the tail of x at the reference, and placing the tail of y at the tip of x. The multiplication of a real number α and a n interval x is a segment of length α times that of x.

The unit of time can be, for example, the duration for a full spin of the Earth, which we call a day. The modern unit of time is second, defined as 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

When we change the basis in time from the duration of a full spin of the Earth to some periods of some radiation from some atom, we do not change laws of physics. Any change in unit gives a factor of conversion. Factors of conversion between units are often unsightly numbers. Consider 1 year = 31536000 seconds, for example. Change of unit generates busy work for committees and students, but does not change laws of physics.

Temperature is not a scalar. Since the time of Galileo, people have ordered levels of hotness—temperature—by numbers. Many physicists call the temperature a scalar. This usage is inconsistent with our definition of scalar. The set of all temperatures is *not* a scalar set. The addition of two temperatures is meaningless.

By an unfortunate coincidence of English, a system of numerical labeling of temperatures is called a scale of temperature. Of course, temperatures do *not* scale in the sense of being scalars.