Lecture #4  Objective stress rate (Rong Tian, posted on iMechanica)

The Cauchy stress is related to PK2 stress as
\[
\sigma = J^{-1} \phi [S]
\]  
(43)
The time derivative of PK2 stress \(\dot{S}\) is objective, whereas that of Cauchy stress is not.

4.1 Truesdell rate

Truesdell stress rate is defined by
\[
\sigma^V = J^{-1} \phi \left[ \dot{S} \right]
\]  
(44)
where \(\dot{S}\) is the material time derivative of PK2 stress.

Using \(\sigma^{VT}\) to denote the Truesdell rate, we derive it as
\[
\sigma^{VT} = J^{-1} \phi \left[ \dot{S} \right] = J^{-1} F \frac{d}{dt} \left( J F^{-1} \sigma F^T \right) F^T
\]  
(45)
\[
= J^{-1} \left[ J F^{-1} \sigma F^T + J F^{-1} \sigma F^T + J F^{-1} \sigma F^T + J F^{-1} \sigma F^T \right] F^T
\]
Note that \(\frac{d}{dt}(FF^{-1}) = \dot{F}F^{-1} + F\dot{F}^{-1} = 0\), we have
\[
\dot{F}^{-1} = -F^{-1}\dot{F}F^{-1} = -F^{-1}L
\]  
(46)
Consider
\[
\dot{J} = J \text{trace}(D) = J \text{trace}(L)
\]  
(47)
We obtain
\[
\sigma^{VT} = J^{-1} F \left[ J F^{-1} \sigma F^T - J F^{-1} L \sigma F^T + J F^{-1} \sigma F^T + J F^{-1} \sigma F^T \right] F^T
\]  
= \[
= J^{-1} \left[ J \text{trace}(L) \sigma - J L \sigma + J \dot{\sigma} - J \sigma \dot{L}^T \right]
\]  
(48)
\[
= \dot{\sigma} - L \sigma - \sigma L^T + \text{trace}(L) \sigma
\]
This is the Truesdell stress rate.

4.2 Green-Naghdi rate

Ignore the stretch component of deformation and assume
\[
F = RU \approx R
\]  
(49)
then we obtain
\[
L = \dot{F}F^{-1} = \dot{R}R^T, \quad \text{trace}(\dot{R}R^T) = \text{trace}(\Omega) = 0
\]  
(50)
where \(\Omega\) is the angular velocity matrix, \(\Omega = 0\). Substitute (50) into the Truesdell rate (48), we obtain Green-Naghdi rate as
\[
\sigma^{VG} = \dot{\sigma} - \dot{R}R^T \sigma + \sigma \dot{R}R^T
\]  
(51)

4.3 Jaumann rate
Using polar decomposition $F = RU$ and spin tensor of $W = \frac{1}{2}(L - L^T)$, we can obtain

$$W = \dot{RR}^T - \frac{1}{2}R(\dot{UU}^{-1} + U^{-T}\dot{U})R^T \quad (52)$$

If we assume

$$W \approx \dot{RR}^T \quad (53)$$

and substitute into the Green-Naghdi rate (51), then we obtain Jaumann rate

$$\sigma^{VG} = \dot{\sigma} - W\sigma + \sigma W \quad (54)$$

The relationship among Truesdell, Green-Naghdi, and Jaumann rates are summarized in Figure 3.1. It is noted that the Jaumann and Green-Naghdi stress rates are the approximate of the Truesdell rate; they should not be expected to be the same accurate as the Truesdell rate for a general non-rigid-body motion. This might explain the difference of the shear stress computed by the three different stress rates for the same shear deformation.

Quick question: why are we often using Jaumann rate instead of Truesdell rate?