Lecture #4 Objective stress rate (Rong Tian, posted on iMechanica)

The Cauchy stress is related to PK2 stress as

$$\sigma = J^{-1}\phi_*\left[\mathbf{S}\right] \tag{43}$$

The time derivative of PK2 stress $\dot{\mathbf{S}}$ is objective, whereas that of Cauchy stress is not.

4.1 Truesdell rate

Truesdell stress rate is defined by

$$\sigma^{\nabla} = J^{-1}\phi_*\left[\dot{\mathbf{S}}\right] \tag{44}$$

where $\dot{\mathbf{S}}$ is the material time derivative of PK2 stress.

Using $\sigma^{\nabla\mathrm{T}}$ to denote the Truesdell rate, we derive it as

$$\begin{aligned}\sigma^{\nabla\mathrm{T}} &= J^{-1}\phi_*\left[\dot{\mathbf{S}}\right]\\ &= J^{-1}\mathbf{F}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(J\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathrm{T}}\right)\right]\mathbf{F}^{\mathrm{T}}\\ &= J^{-1}\mathbf{F}\left[\dot{J}\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathrm{T}} + J\dot{\mathbf{F}}^{-1}\sigma\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\dot{\sigma}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\sigma\dot{\mathbf{F}}^{-\mathrm{T}}\right]\mathbf{F}^{\mathrm{T}}\end{aligned} \tag{45}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{F}\mathbf{F}^{-1}\right) = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\dot{\mathbf{F}}^{-1} = 0$, we have

$$\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathbf{L} \tag{46}$$

Consider

$$\dot{J} = J\ \mathrm{trace}\left(\mathbf{D}\right) = J\ \mathrm{trace}\left(\mathbf{L}\right) \tag{47}$$

We obtain

$$\begin{aligned}\sigma^{\nabla\mathrm{T}} &= J^{-1}\mathbf{F}\left[\dot{J}\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathrm{T}} - J\mathbf{F}^{-1}\mathbf{L}\sigma\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\dot{\sigma}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\sigma\dot{\mathbf{F}}^{-\mathrm{T}}\right]\mathbf{F}^{\mathrm{T}}\\ &= J^{-1}\left[J\mathrm{trace}\left(\mathbf{L}\right)\sigma - J\mathbf{L}\sigma + J\dot{\sigma} - J\sigma\mathbf{L}^{\mathrm{T}}\right]\\ &= \dot{\sigma} - \mathbf{L}\sigma - \sigma\mathbf{L}^{\mathrm{T}} + \mathrm{trace}\left(\mathbf{L}\right)\sigma\end{aligned} \tag{48}$$

This is the *Truesdell* stress rate.

4.2 Green-Naghdi rate

Ignore the stretch component of deformation and assume

$$\mathbf{F} = \mathbf{R}\mathbf{U} \approx \mathbf{R} \tag{49}$$

then we obtain

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}},\ \ \mathrm{trace}\left(\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\right) = \mathrm{trace}\left(\mathbf{\Omega}\right) = 0 \tag{50}$$

where $\mathbf{\Omega}$ is the angular velocity matrix, $\Omega_{ii} = 0$. Substitute (50) into the Truedell rate (48), we obtain *Green-Naghdi* rate as

$$\sigma^{\nabla\mathrm{G}} = \dot{\sigma} - \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\sigma + \sigma\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{51}$$

4.3 Jaumann rate

Using polar decomposition $\mathbf{F}=\mathbf{R}\mathbf{U}$ and spin tensor of $\mathbf{W}=\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^{\mathrm{T}}\right)$, we can obtain

$$\mathbf{W}=\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}-\frac{1}{2}\mathbf{R}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}+\mathbf{U}^{-\mathrm{T}}\dot{\mathbf{U}}\right)\mathbf{R}^{\mathrm{T}} \tag{52}$$

If we assume

$$\mathbf{W}\approx\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \tag{53}$$

and substitute into the Green-Naghdi rate (51), then we obtain *Jaumann* rate

$$\sigma^{\nabla G}=\dot{\sigma}-\mathbf{W}\sigma+\sigma\mathbf{W} \tag{54}$$

The relationship among Truesdell, Green-Naghdi, and Jaumann rates are summarized in Figure 3.1. It is noted that the Jaumann and Green-Naghdi stress rates are the approximate of the Truesdell rate; they should not be expected to be the same accurate as the Truesdell rate for a general non-rigid-body motion. This might explain the difference of the shear stress computed by the three different stress rates for the same shear deformation.

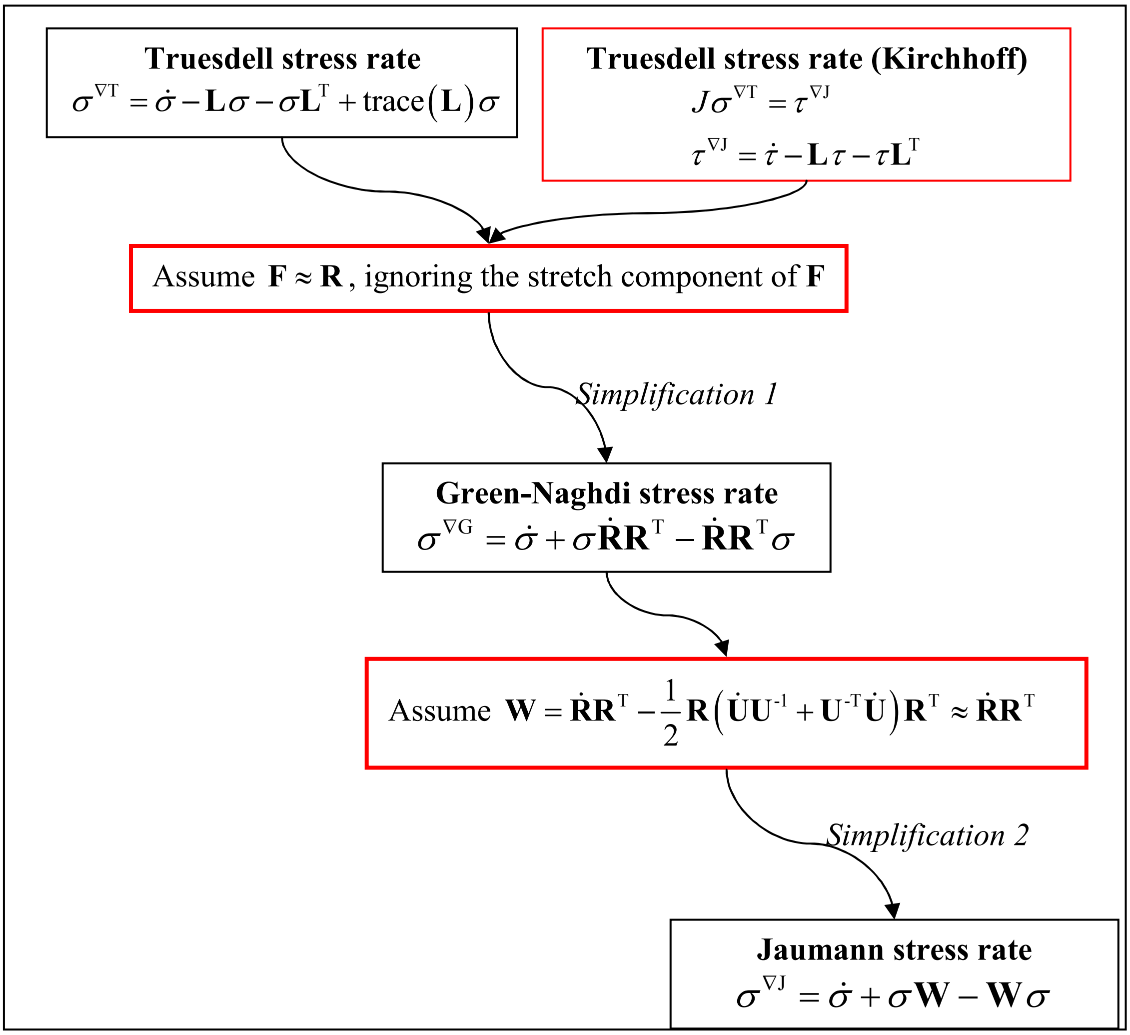

Figure 3.1. Relationship among Truesdell, Green-Naghdi, and Jaumann rates.

Quick question: why are we often using Jaumann rate instead of Truesdell rate?