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A MORE FUNDAMENTAL APPROACH TO PLASTIC STRESS-STRAIN RELATIONS

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ABSTRACT

Essentially thermodynamic definitions (1) of work-hardening and of ideally plastic materials are re-employed to remove an unnecessary assumption in the derivation of the general Mises-Prager plastic potential stress-strain relations $d\epsilon_{ij}^{p} = \lambda \partial f / \partial \sigma_{ij}$. Additional insight is thus ~~ob- is thus~~ obtained into the significance of yield or loading functions. In particular it is shown that the loading surface or any of the customary yield curves for work-hardening or ideally plastic materials must be convex. The meaning of corners in such curves or vertices in the loading surfaces is discussed with especial reference to problems of uniqueness.

DEFINITION OF WORK-HARDENING

As described previously, (1) it is not possible to draw a simple picture to depict work-hardening for general paths of loading. In simple tension, the term means that stress is a monotonically increasing function of strain, Fig. 1. The assumption is made that the curve is unique. In other words, time effects are taken as negligible for the time intervals involved.

This concept may be broadened to include all states of stress and paths of loading by considering the work done by an external agency which slowly applies an additional set of stresses to the already stressed material and then slowly removes the added set. The final geometrical configuration may or may not be the same as the original. The external agency is to be understood as entirely separate and distinct from the agency which causes the existing state of stress and which has produced the existing state of strain.

Work-hardening means that for all such added sets of stresses the material will remain in equilibrium and further that

(a) positive work is done *by the external agency* during the application of the added set of stresses and

(b) the net work performed *by the external agency* over the cycle of application and removal is positive if plastic deformation has occurred in the cycle. The net work is zero if and only if elastic changes in strain alone are produced.

At the risk of being over-repetitious, it is emphasized that the work referred to is not the total work done by all forces acting, it is only the work done by the added set on the displacements which result. Rephrased, work-hardening means that useful net energy can not be extracted from the material *and the system of forces acting upon it* in such a cycle. Furthermore, energy must be put in if plastic deformation is to take place.

Figure 2 provides an illustration of the physical meaning of the definition. A thin-walled circular tube is shown schematically to be under a state of stress produced by an axial force F, twisting moment M, and internal pressure

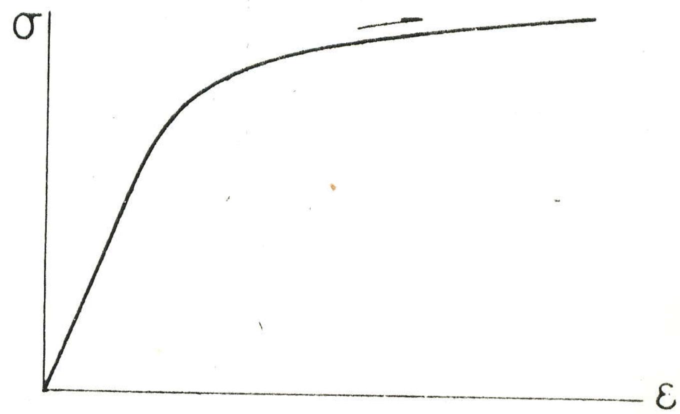

Fig. 1. WORK-HARDENING.

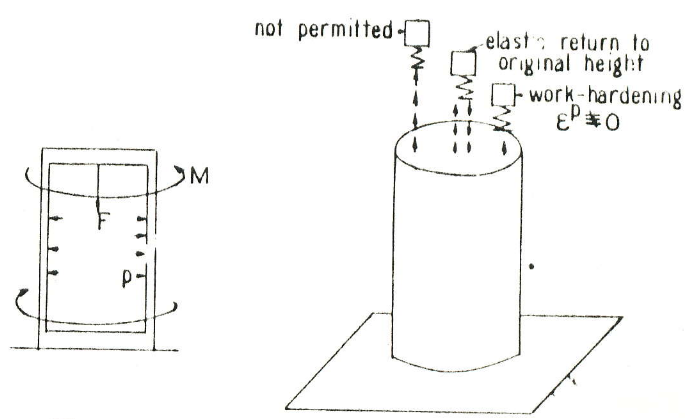

Fig. 2. EXTERNAL AGENCY.

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p. The values of F, M, and p are independent and each may be positive, negative, or zero. The external agency which adds and then removes stress is symbolized by a rigid mass attached to a massless cushioning spring which is "dropped" on the end plate. Velocities are supposed so small that all vibration and wave effects in the tube are negligible. If the added displacements produced by the added stresses are all elastic, then the mass will bounce back to its initial height. If plastic strains, e^p, result and the material is work-hardening, the mass will not rebound completely. The bouncing mass (external agency) does work in the cycle of application and removal of the stress it produces. On the other hand, if the external agency does negative work during the cycle; i.e., has work done on it, then the mass must return to a higher elevation than it had originally. The given definition of work-hardening rules out this type of behavior for any values of F, M, and p for the materials considered in this paper. Non work-hardening models may be constructed. The most obvious is a material with a falling instead of a rising stress-strain curve in simple tension. More subtle types are described in reference (1).

It is important to note that for every quasi-static dissipative system, work-hardening or not, the total potential energy of all the external and internal forces decreases when inelastic action takes place. The work-hardening definition insures that this requirement is met but the converse is clearly not true.

CONVEXITY OF THE LOADING SURFACE

As in the previous paper (1), the reasonable assumption will be made that a loading function exists. At each point of the material at each stage of plastic deformation there exists a number k^2 and a function of the stress $f(\sigma_{ij})$ such that further plastic deformation takes place only for $f(\sigma_{ij}) > k^2$. Both f and k^2 may depend upon the existing state of plastic strain and the prior plastic strain history (2). Most of the salient features of a yield or loading surface appear in a two dimensional representation such as σ_x, σ_y or σ_x, τ_{xy} as illustrated in Fig. 3. Note that the loading surface is drawn in a "stress space."

A pictorial proof of convexity will be given first. Consider any existing state of stress σ_{ij}^* on or inside the loading surface (curve). Then imagine an external agency to add stresses along a path lying inside the surface until a state of stress σ_{ij} is reached which is on the loading surface, Fig. 4. Elastic changes only have taken place so far. As all purely elastic changes are complet ly reversible, the state of the material at σ_{ij} is independ ent of the path from σ_{ij}^* to σ_{ij} providing the path lies in or within the loading surface. Now suppose the external agency to add a very small outward pointing stress incre ment $d\sigma_{ij}$ which produces very small plastic increments in strain, de_{ij}^p, as well as elastic increments. The external agency then releases $d\sigma_{ij}$ and returns the state of stress to σ_{ij}^* along an elastic path. As all the elastic energy is recovered, the positive work done by the external agency over the cycle is

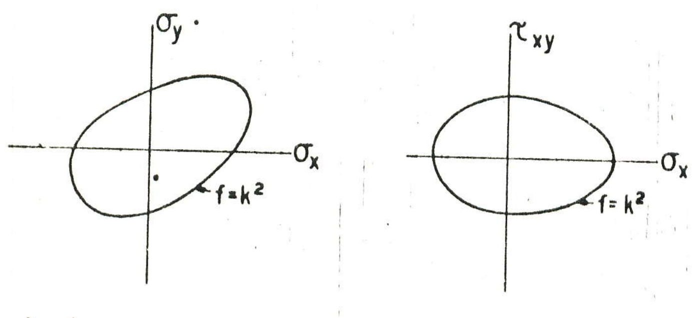


Fig. 3. YIELD OR LOADING CURVES.

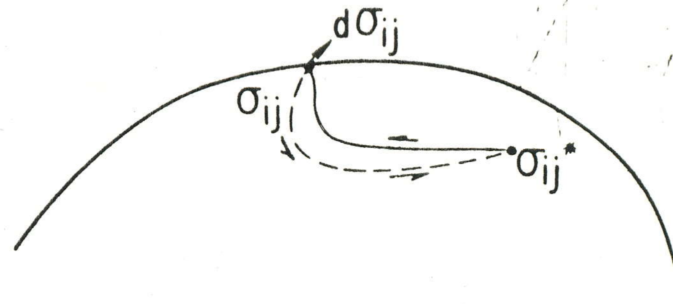


Fig. 4. STRESS PATH PRODUCED BY EXTERNAL AGENCY.

$$\delta W = (\sigma_{ij} - \sigma_{ij}^*)\, de_{ij}^p + d\sigma_{ij}\, de_{ij}^p > 0. \qquad [1]$$

If plastic strain coordinates are superimposed on the stress coordinates as in Fig. 5, δW may be interpreted as the scalar product of the vector $\sigma_{ij} - \sigma_{ij}^*$ and the vector de_{ij}^p plus the scalar product of $d\sigma_{ij}$ and de_{ij}^p.

As σ_{ij}^* may be chosen to be σ_{ij} itself, $d\sigma_{ij}$ and de_{ij}^p must make an acute angle with each other,

$$d\sigma_{ij}\, de_{ij}^p > 0 \qquad [2]$$

as reported in reference (1).* However, if σ_{ij}^* is distinct from σ_{ij}, the magnitude of $\sigma_{ij} - \sigma_{ij}^*$ may be made as large as one pleases compared with $d\sigma_{ij}$. Inequality [1], which may be written

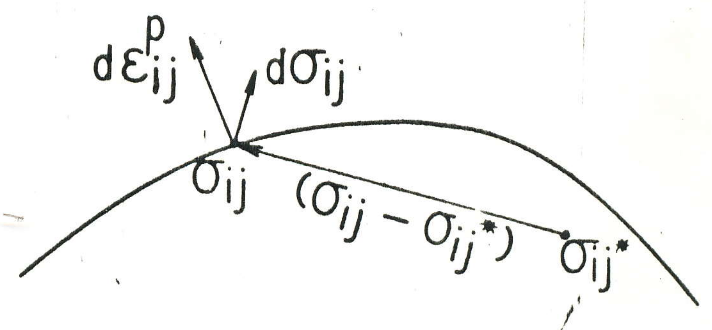


Fig. 5. STRESS AND PLASTIC STRAIN COORDINATES SUPERPOSED

*The usual summation convention is followed. $d\sigma_{ij}\, de_{ij} \equiv d\sigma_{11}\, de_{11} + d\sigma_{22}\, de_{22} + \cdots + d\sigma_{12}\, de_{12} + d\sigma_{21}\, de_{21} + \cdots \equiv d\sigma_x\, de_x + d\sigma_y\, de_y + d\sigma_z\, de_z + d\tau_{xy}\, d\gamma_{xy} + d\tau_{yz}\, d\gamma_{yz} + d\tau_{zx}\, d\gamma_{zx}$

$$(\sigma_{ij} - \sigma_{ij}^{*})\, d\epsilon_{ij}^{p} > - d\sigma_{ij}\, d\epsilon_{ij}^{p}$$

requires

$$(\sigma_{ij} - \sigma_{ij}^{*})\, d\epsilon_{ij}^{p} \geq 0. \qquad [3]$$

The vector $d\epsilon_{ij}^{p}$ is seen to be perpendicular to $\sigma_{ij} - \sigma_{ij}^{*}$ or at an acute angle to this vector *for all* σ_{ij}^{*}. Therefore, all points σ_{ij}^{*} must lie to one side of a plane perpendicular to $d\epsilon_{ij}^{p}$. This must be true for all points σ_{ij} on the loading surface so that no vector $\sigma_{ij} - \sigma_{ij}^{*}$ can pass outside the surface, Fig. 6. The loading surface is convex. Note

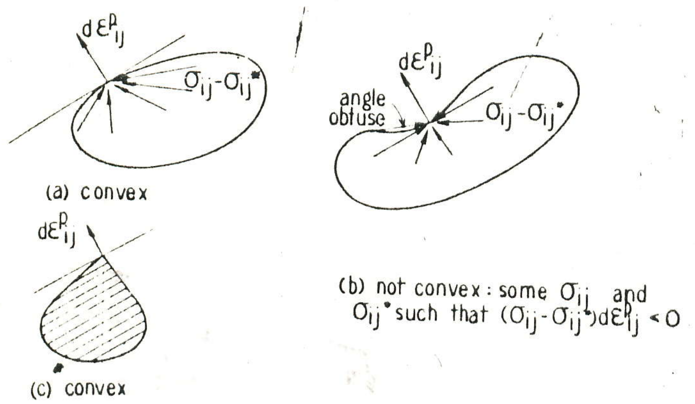


Fig. 6. LOADING SURFACE IS CONVEX.

that convexity does not rule out corners or pointed** vertices.

No formal mathematical proof of convexity is required. Inequality [3] where $f(\sigma_{ij}) = k^2$, $f(\sigma_{ij}^{*}) \leq k^2$, and each surface point, σ_{ij}, has one or more vectors $d\epsilon_{ij}^{p}$ associated with it is a definition of a convex body with interior and surface points σ_{ij}^{*}.

NORMALITY OF THE PLASTIC STRAIN INCREMENT VECTOR

If the loading surface has no corners nor pointed vertices [continuously turning tangent plane], Inequality [3] requires the perpendicular to $d\epsilon_{ij}^{p}$ to be the tangent (plane) to the loading surface. Therefore $d\epsilon_{ij}^{p}$ is the normal to the surface *for all* $d\sigma_{ij}$. The plastic strain increment ratios are thus independent of the stress increment ratios at any 'smooth' point σ_{ij} on the loading surface. In any two dimensional representation, the plane components of $d\epsilon_{ij}^{p}$ are normal to the yield or loading curve, Fig. 6a.

CORNERS AND POINTED** VERTICES

At a point where the loading surface does not have a continuously turning tangent plane, the vector $d\epsilon_{ij}^{p}$ is not necessarily fixed in direction independently of $d\sigma_{ij}$. All that the work-hardening definition, expressed as Inequality [3], requires is that each $d\epsilon_{ij}^{p}$ lies inside the figure formed by the normals to the surface at all points in the neighborhood.

Inequality [2] further requires $d\epsilon_{ij}^{p}$ to make an acute angle with $d\sigma_{ij}$ at a pointed vertex, Fig. 7a. In any two

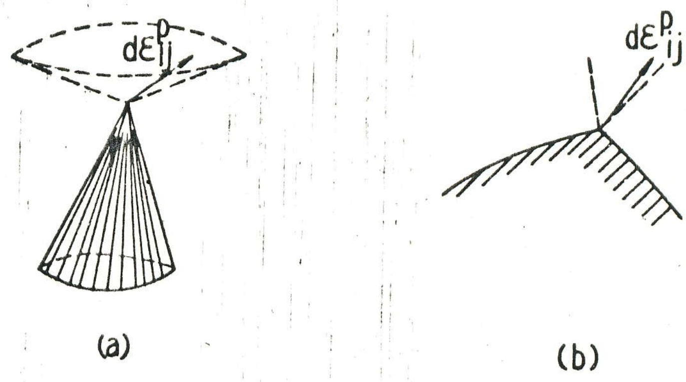


Fig. 7. CORNERS AND POINTED VERTICES.

dimensional representation, the components of $d\epsilon_{ij}^{p}$ in the plane form a vector which lies between the normals on each side of the point, Fig. 7b.

STRESS-STRAIN RELATION

If the loading surface has a continuously turning tangent at a point, then normality of $d\epsilon_{ij}^{p}$ may be expressed mathematically as

$$d\epsilon_{ij}^{p} = \Lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad [4]$$

where Λ is a proportionality factor which must depend upon σ_{ij}, $d\sigma_{ij}$, and may depend upon ϵ_{ij}^{p} and its history. The partial derivative sign is needed to indicate that although the form and value of f may depend upon plastic strain and its history these quantities are to be considered as constant when taking the derivative.

As stated before, when $f(\sigma_{ij}) = k^2$, additional plastic deformation (loading) requires changing the stress so that $f(\sigma_{ij}) > k^2$ or

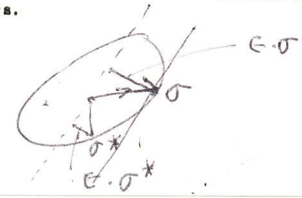
$$\frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} > 0\,. \qquad [5]$$

Therefore it is permissible to write the expression derived in reference (1):

$$d\epsilon_{ij}^{p} = G\, \frac{\partial f}{\partial \sigma_{ij}}\, \frac{\partial f}{\partial \sigma_{kl}}\, d\sigma_{kl} \qquad [6]$$

Here also G is a scalar which may depend upon stress, plastic strain and its history. As yet G may also depend

**The term pointed is used to avoid confusion with the term vertex as applied to conics.



upon $d\sigma_{ij}$ although it must be homogeneous of order zero in the stress increments because time effects are assumed absent. As an example, a permissible G might be

$$g\left[1+\frac{(d\sigma_{ij}\, d\sigma_{jk}\, d\sigma_{ki})^2}{(d\sigma_{mn}\, d\sigma_{mn})^3}\right]$$

where g is a function of stress, plastic strain and its history but not of the stress increments.

However, if at σ_{ij}, $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$ not only determines whether loading is taking place but also measures the extent of loading, say $d\epsilon_{ij}^{p}\, d\epsilon_{ij}^{p}$, G can not depend upon $d\sigma_{ij}$. This assumption is reasonable at this stage because all directions of $d\sigma_{ij}$ produce the same direction for $d\epsilon_{ij}^{p}$. It is implicit or explicit in all existing stress-strain relations based upon the concept of a loading function or plastic potential. Nevertheless it is equivalent to saying that only the component of the stress increment vector normal to the loading surface is of importance in determining the extent of loading. Therefore although resting on much firmer ground than previously (1) it is an assumption of a linear relation between increments of stress and of plastic strain.

It is worth mentioning a special case in which no assumption is needed. If the loading function is dependent upon stress and plastic strain only, $F(\sigma_{ij}, \epsilon_{ij}^{p}) = C^2$ where C is a fixed number, then $dF = 0 = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \epsilon_{ij}^{p}} d\epsilon_{ij}^{p}$. Therefore clearly the linear relation exists.

UNIQUENESS IN THE SMALL

If the loading surface has a corner or a pointed vertex, Fig. 7, there is considerable freedom of choice in the direction of $d\epsilon_{ij}^{p}$. The existing state and the stress increment do not determine $d\epsilon_{ij}^{p}$ uniquely. This difficulty may be overcome by postulating a unique relation or deriving one on some physical basis. It is permissible from this viewpoint to have the direction of $d\epsilon_{ij}^{p}$ depend on $d\sigma_{ij}$ at such vertices. This explains why "slip theory" (3) is self-consistent and why the uniqueness proofs carry through (4).

If the loading surface is 'smooth', the direction of $d\epsilon_{ij}^{p}$ is determined uniquely for each σ_{ij}. However, the converse is not necessarily true. A given $d\epsilon_{ij}^{p}$ does not always determine σ_{ij} even when the shape and position of the loading surface is known. Each $d\epsilon_{ij}^{p}$ gives a unique tangent plane but the tangent plane may have line contact as in the case of the Mises criterion (circular cylinder in principal stress space) or plane contact as for particular normals in the maximum shear stress criterion (hexagonal prism in principal stress space). There must be line contact at least for any plastically incompressible material $d\epsilon_{ii}^{p} = 0$, because addition of hydrostatic tension or compression does not alter $d\epsilon_{ij}^{p}$.

pl on yield surface

IDEAL PLASTICITY

A great deal of the previous discussion carries throu with little change in the case of ideal plasticity. The fundamental difference is that the yield surface $f(\sigma_{ij}) = k$ is fixed in stress space. If no plastic deformation is to take place, it is necessary that $f(\sigma_{ij}) < k^2$. On the othe hand if $f(\sigma_{ij}) = k^2$ flow will occur and furthermore it is not possible for the stress point to move outside the surface; $f(\sigma_{ij}) \not> k^2$.

The extended definition of an ideally plastic material may also be phrased in terms of the work done by an exnal agency. Now, however, the work done by the externa agency over the cycle of application and removal of the added stresses is positive *or zero* if plastic deformation has occurred in the cycle. Just as for the work-hardenin material, no useful net energy may be extracted in such a cycle from the material and the system of forces acting upon it. However, contrary to the work-hardening case it is only necessary to feed energy in to cause plastic deformation if $f(\sigma_{ij}) < k^2$ initially as flow can occur at constant σ_{ij}, $f(\sigma_{ij}) = k^2$.

The "bouncing" mass of Fig. 2 may return to its original height even though $\epsilon^p \not= 0$. It cannot rebound to a higher elevation and will always rebound incompletely if the plastic action is due to the external agency.

Convexity of the yield surface is a direct consequence of the definition of ideal plasticity. As indicated on Fig. 8, consider the external agency to change the state of stress from σ_{ij}^{*} inside the yield surface to σ_{ij} on the surface (where plastic flow takes place) and then to return the state of stress to σ_{ij}^{*}. The external agency work inequality is

$$(\sigma_{ij} - \sigma_{ij}^{*})\, d\epsilon_{ij}^{p} > 0 \qquad [7]$$

and if the original state of stress is also on the yield surface, σ_{ij}^{**}

$$(\sigma_{ij} - \sigma_{ij}^{**})\, d\epsilon_{ij}^{p} \geq 0. \qquad [8]$$

If the surface is smooth, no corners or pointed vertices

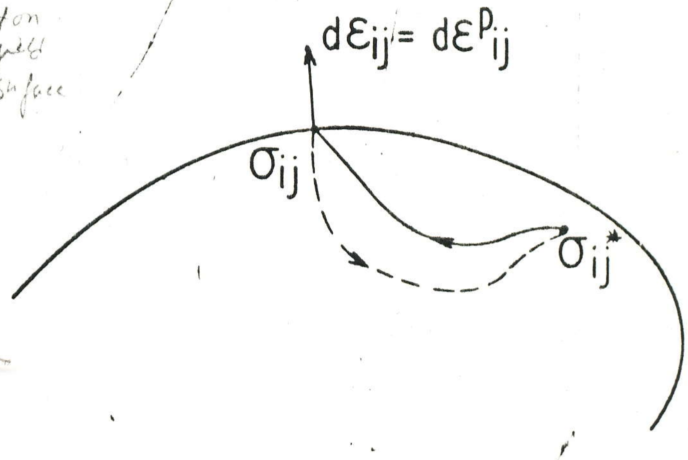


Fig. 8. YIELD SURFACE IS CONVEX.

de_{ij}^{p} must be normal to the surface and the Mises plastic potential expression results (5)

$$de_{ij}^{p} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad [9]$$

Here $\lambda > 0$ is a proportionality factor which depends upon the state of stress and the extent of the deformation. It may be more instructive to consider the expression in the equivalent form

$$\frac{de_{ij}^{p}}{\sqrt{de_{mn}^{p}\, de_{mn}^{p}}} = \frac{\partial f / \partial \sigma_{ij}}{\sqrt{\frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{pq}}}} \qquad [10]$$

If the surface has corners, the vector de_{ij}^{p} is restricted as shown in Fig. 7.

TWO GENERAL COMMENTS

It is of considerable interest to note that combinations of ideal plasticity and work-hardening are permissible.

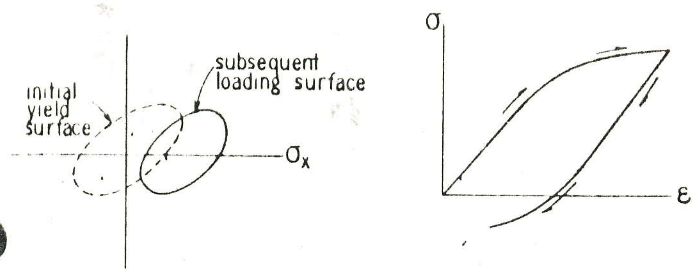


Fig. 9. LOADING SURFACE NEED NOT ENCLOSE ORIGIN.

Furthermore, there is no necessity for the loading surface of a work-hardening material to enclose the origin (stress free state). As shown by the stress-strain curve for uniaxial stress and possible loading surfaces which correspond, Fig. 9, materials which deform plastically upon unloading may be included within the theory.

CONCLUSIONS

Definitions of ideal plasticity and of work-hardening lead without further assumption to the requirements that:

a) all loading or yield surfaces or curves are convex, Fig. 6.

b) if the surface is smooth de_{ij}^{p} is normal to the surface; de_{ij}^{p} is proportional to $\partial f / \partial \sigma_{ij}$, Equations [4] and [9].

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