Advances in Applied Mechanics 7, 55-129 (1962)

# The Mathematical Theory of Equilibrium Cracks in Brittle Fracture

BY G. I. BARENBLATT

*Institute of Geology and Development of Combustible Minerals of the U.S.S.R. Academy of Sciences, Moscow, U.S.S.R.**

*Page*



* Present address: Institute of Mechanics, Moscow State University, Moscow, USSR.

## I. Introduction

In recent years the interest in the problem of brittle fracture and, in particular, in the theory of cracks has grown appreciably in connection with various technical applications. Numerous investigations have been carried out, enlarging in essential points the classical concepts of cracks and the methods of analysis. The qualitative features of the problems of cracks, associated with their peculiar non-linearity as revealed in these investigations, makes the theory of cracks stand out distinctly from the whole range of problems in the present theory of elasticity. The purpose of the present paper is to present a unified view of how the basic problems in the theory of equilibrium cracks are formulated, and to discuss the results obtained.

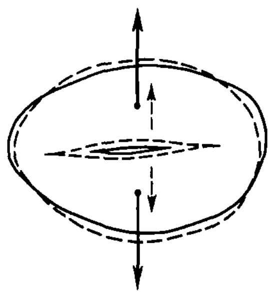

Fig. 1.

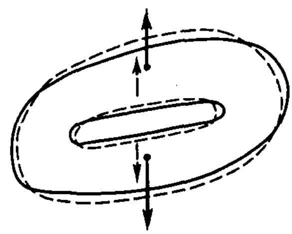

Fig. 2.

The object of the theory of equilibrium cracks is the study of the equilibrium of solids in the presence of cracks. Consider a solid having cracks (Fig. 1) which are in equilibrium under the action of a system of loads. The body, able to sustain any finite stresses, is assumed to be perfectly brittle, i.e. to retain the property of linear elasticity up to fracture. The possibility of applying the model of a perfectly brittle body to real materials will be discussed later.

The opening of a crack (the distance between the opposite faces) is always much smaller than its longitudinal dimensions; therefore cracks can be considered as surfaces of discontinuity of the material, i.e. of the displacement vector. Henceforth, unless the contrary is stated, plane cracks of *normal* discontinuity are considered, i.e. cracks are pieces of a plane bounded by closed curves (crack *contours*), at which only the normal component of the displacement vector has a discontinuity. The case when the tangential component of the displacement vector is discontinuous at the discontinuity surface can be treated in the same manner.

One might think that the investigation of the equilibrium of elastic bodies with cracks can be carried out by the usual methods of the theory of elasticity in the same way as it is done for bodies with cavities (Fig. 2).

However, there exists a fundamental distinction between these two problems. The form of a cavity undergoes only slight changes even under a considerable variation in the load acting upon the body, whereas cracks, whose surface also constitutes a part of the body boundary, can expand a good deal even with small increase of the load, to which the body is subjected. (In Figs. 1 and 2, dotted lines indicate additional loads and the corresponding positions of the body boundaries.)

Thus, one of the basic assumptions of the classical linear theory of elasticity is not satisfied in problems of the theory of cracks, namely the assumption about the *smallness* of changes in the boundaries of a body under loading, which permits one to satisfy the boundary conditions at the surface of the unstrained body. This fact makes the problem of the equilibrium of a body with cracks, unlike traditional problems of the theory of elasticity, essentially non-linear. In the theory of cracks one must determine from the condition of equilibrium not only the distribution of stresses and strains but also the boundary of the region, in which the solution of the equilibrium equations is constructed.

Non-linear problems of this type ("problems with unknown boundaries") have long been known in various fields of continuum physics. Suffice it to mention the theory of jets and the theory of finite-amplitude waves in hydrodynamics, the theory of flow past bodies in the presence of shock waves in gas dynamics, Stefan's problem of freezing in the theory of heat conduction, etc. The main difficulty in all these problems lies in the determination of the boundary of the region in which the solution is sought. Likewise, the basic problem in the theory of equilibrium cracks is the determination of the surfaces of cracks when a given load is applied.

The differential equations of equilibrium and the usual boundary conditions of the theory of elasticity cannot in principle give the solution of this problem without the introduction of some additional considerations. This may be seen from the fact that we can construct a formal solution of the equations satisfying the usual boundary conditions *no matter how we prescribe crack surfaces*. The analysis of these solutions shows that in general the tensile stress $\sigma$ normal to the surface of a crack is infinite at the crack contour. More exactly, near an arbitrary point of the crack contour

$$\sigma = \frac{N}{\sqrt{s}} + \text{ finite quantity,} \tag{1.1}$$

where $s$ is the distance of a point of the body lying in the plane of a crack from the crack contour, $N$ is the *stress intensity factor*, a quantity dependent on the applied loads, the form of the crack contour, and the coordinates of the point considered, but independent of $s$. The form of a normal section of the deformed crack surface near the contour appears in such cases unnaturally rounded (as in Fig. 3 or somewhat different; see details below).

Generally speaking, however, there exist such exceptional contours of cracks for which stresses at the edges are finite ($N = 0$) under a given load; at the same time the opposite faces of cracks close smoothly at the edges. The form of a section of the crack surface near the edge appears then as a cusp, cf. Fig. 4. It can be shown that for such contours, and only for them, the energy released by a small change in the contour of a crack is equal to zero. It follows that only such contours can bound equilibrium cracks.

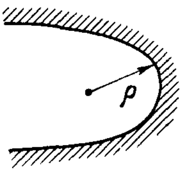


FIG. 3.

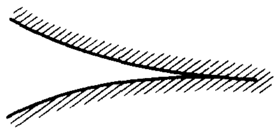

FIG. 4.

Thus, when all loads acting upon a body are given, the problem of the theory of equilibrium cracks may be formulated as follows: for a given position of initial cracks and a given system of forces acting upon the body one requires the determination of the stresses, the strains, and the contours of cracks so as to satisfy the differential equations of equilibrium and the boundary conditions, and to insure finiteness of stresses (or, which is the same, a smooth closing of the opposing faces at the crack edges). If the position of the initial cracks is not given, then, since according to our model the body can sustain any finite stress, the solution of the problem formulated above is not unique. This is only natural because at one and the same load in one and the same body there need not be any cracks, or there may be one crack, or two, and so on.

In the general case of curved cracks, the shape is determined not only by the load existing at a given moment but also by the whole history of loading. If however, the symmetry of the body and the applied monotonically increasing loads assure the development of plane cracks, then the contours of cracks are determined by the current load alone. All the results at present available in the theory of cracks correspond to particular cases of this simplified formulation of the problem.

A given system of forces acting upon the body should in general include not only the loads applied to the body. The following example illustrates what is meant. Let us attempt to determine the contour of an equilibrium crack in the case of the loads represented in Fig. 1. If, in accordance with the usual approach in the theory of elasticity (as in the case of the cavity shown in Fig. 2), the surface of the crack is considered to be free of stresses, the result will be paradoxical: whatever contour of the crack we would take, the tensile stresses at its edge are always infinite. Consequently, there cannot exist an equilibrium crack; however small the force of extension may be, the body that has a crack breaks in two!

Such an obvious lack of agreement with reality can be easily explained: simply using the model of an elastic body, we have not taken into consideration all forces acting upon the body. It appears that — and this is also one of the main distinctions between the problems of the theory of cracks and the traditional problems of the theory of elasticity — for developing an adequate theory of cracks it is necessary to consider molecular forces of cohesion acting near the edge of a crack, where the distance between the opposite faces of the crack is small and the mutual attraction strong.

Although consideration of forces of cohesion settles the matter in principle, it complicates a great deal the analysis. The difficulty is that neither the distribution of forces of cohesion over the crack surface nor the dependence of the intensity of these forces on the distance between the opposite faces are known. Moreover, the distribution of forces of cohesion in general depends on the applied loads. However, if cracks are not too small, there is a way out of the difficulty: with increasing distance between the opposite faces the intensity of forces of cohesion reaches very quickly a large maximum, which approaches Young's modulus and then diminishes rapidly.

Therefore two *simplifying assumptions* can be made. The *first* is that the area of the part of the crack surface acted upon by the forces of cohesion can be considered as negligibly small compared to the entire area of the crack surface. According to the *second* assumption the form of the crack surface (and, consequently, the local distribution of forces of cohesion) near the adges, at which the forces of cohesion have the maximum intensity, does not depend on the applied load.*

The intensity of the forces of cohesion has the highest possible value for a given material under given conditions. This happens for instance at all edge points of a crack formed at the initial rupture of the material as the load increases. For most real materials cracks are *irreversible* under ordinary conditions. If an irreversible crack is produced by an artificial cut without subsequent expansion or is obtained from a crack that existed under a greater load by diminishing the load, then the intensity of forces of cohesion at the crack contour will be lower than the maximum possible one. The forces of cohesion that act at the surface of a crack compensate the applied extensional loads and secure finiteness of stresses and smooth closing of the crack faces. With an increase in extensional loads the forces of cohesion grow, thus adjusting themselves to the increasing tensile stresses, and the crack does not

* Sh. A. Sergaziev very neatly compared cracks for which these assumptions are satisfied with "zippers".

expand further until the highest possible intensity of forces of cohesion is reached. The crack starts expanding[+] only upon reaching the highest possible intensity of forces of cohesion at the edge.

Successive expansion of the crack edge under increasing extensional load is represented schematically in Fig. 5.

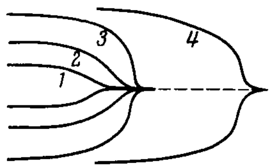


FIG. 5.

1,2. The intensity of forces of cohesion is less than the maximum.
3,4. The intensity of forces of cohesion is equal to the maximum.

If use is made of the first of the above assumptions, molecular forces of cohesion will enter in the conditions that determine the position of crack edges only in the form of the integral

$$K = \int_0^d \frac{G(t)dt}{\sqrt{t}}, \tag{1.2}$$

where $G(t)$ is the intensity of the forces of cohesion acting near the crack edge, $t$ is the distance along the crack surface taken along the normal to the crack edge, and $d$ is the width of the region subject to the forces of cohesion. For those contour points, at which the second assumption applies, this integral represents a constant of the given material under given conditions (temperature, composition, and pressure of the surrounding atmosphere, etc.), which determines its resistance to the formation of cracks. It can be shown that the quantity $K$ is related to the surface tension of the material $T_0$, the modulus of elasticity $E$, and Poisson's ratio $\nu$ by means of the simple equation

$$K^2 = \frac{\pi E T_0}{1 - \nu^2}. \tag{1.3}$$

Furthermore, for all points of the crack edge at which the intensity of forces of cohesion is a maximum, the stress-intensity factor $N$, entering in

+ Quite a similar situation arises when a body moves over a rough horizontal surface under the action of a horizontal force. The motion of the body begins only after the force exceeds the highest possible value of the friction for the given body and the given surface.

(1.1) and calculated without taking into account the forces of cohesion should be equal to $K/\pi$. For all points of the edge at which the intensity of forces of cohesion is below the maximum, the stress intensity factor calculated without considering forces of cohesion is smaller than $K/\pi$.

The foregoing considerations elucidate sufficiently the nature of the forces of cohesion involved in this problem, and it is now possible to formulate the fundamental problem of the theory of equilibrium cracks.* When the symmetry of the body, of the initial cracks, and of the monotonously increasing forces insures development of a system of plane cracks, this problem can be stated as follows.

Let a system of contours of initial cracks be given in a body. It is required to find the stress and displacement fields corresponding to a given load as well as a system of contours of plane cracks surrounding the contours of the initial cracks (and perhaps coinciding with them partly).

Mathematically the problem consists in constructing such a system of contours that the factor of intensity of the tensile stress, calculated without taking into account the forces of cohesion, should be equal to $K/\pi$ at all edge points, not lying on the contours of the initial cracks, and should not exceed $K/\pi$ at all points of contours, lying on the contours of the initial cracks.

The foregoing formulation of the problem eliminates from our direct consideration the molecular forces of cohesion (they enter only through the constant $K$). Therefore stress and strain fields furnished by the solution of this problem will not be realistic in a small neighbourhood of the crack edges.

When cracks are reversible, or when the applied load is great enough to cause the contours of all the cracks to expand beyond the contours of the initial cracks, the form of the latter evidently is no longer of any importance.

The equilibrium state corresponding to the highest possible intensity of forces of cohesion at least at one point of the crack contour can be stable or unstable. Accordingly, further extension of the crack with increasing load proceeds in essentially different ways. In the case of stable equilibrium, a slow quasi-static transition of the crack from one equilibrium state to another takes place, when the load is increased gradually. If the equilibrium is unstable, the slightest excess over the equilibrium load is accompanied by a rapid crack extension that has a dynamic character. In some cases, when there exist no neighbouring stable states of equilibrium, this leads to the complete rupture of the body. The theory of cracks developed in such a way that problems of this latter type were mainly treated until recently.

---

* Such general formulations of problems seem advisable to us despite the fact that their general solution in effective form exceeds by far the possibilities of present mathematics. General statements of problems are a help in realizing the meaning of specific solutions and difficulties arising in developing the theory.

Sometimes the condition for the onset of crack extension is therefore identified with the condition for complete fracture of the body. It should be clearly understood, however, that this is only true in special cases, the practical significance of which must not be exaggerated.

Below, following a brief outline of the development of the mathematical theory of cracks, the fundamentals of the theory of equilibrium cracks are given as well as the results for the most typical special problems treated hitherto. At the end of this review dynamic problems in the theory of cracks are discussed briefly.

When writing this article the author endeavoured to avoid the repetition of available presentations of some aspects of brittle fracture. Thus the review deals with the theory of cracks proper, i.e. with the mathematical theory of brittle fracture. The numerous available experimental investigations are referred to only inasmuch as they are necessary for confirming the theory presented and establishing the limits of its applicability. Experimental investigations of brittle fracture, unlike the mathematical theory, were discussed more than once in reviews and monographs. At the same time, questions concerning exclusively mathematical techniques of solving the problems of elasticity theory are discussed only briefly, if at all. Also the question of the formation of the initial cracks will not be touched.

Trying to preserve a unified point of view in discussing certain results of other investigators, the author permitted himself sometimes a deviation from the original treatment.

## II. The Development of the Equilibrium Crack Theory

Investigations in the field of the theory of cracks were started by C. E. Inglis [1] about fifty years ago. His paper presents the solution of a problem within the classical theory of elasticity concerning the equilibrium of an infinite body with an isolated elliptical cavity (in particular, with a straight-line cut) in a uniform stress field. N. I. Muskhelishvili [2] — also within the classical theory of elasticity — obtained in a simpler and more effective form the solution of a problem concerning the equilibrium of an infinite body having an elliptical cavity in an arbitrary stress field.

However, in spite of their outstanding significance for subsequent investigations, papers [1, 2] did not prepare the foundations for the theory of cracks proper. The fact is that the solutions obtained in these papers possess two properties which were difficult to explain. First, the length of a crack was found to be indefinite at a given load so that it was possible to construct a solution with an arbitrary value of this parameter. Everyday experience suggests nevertheless that the dimensions of cracks existing in a body should be connected somehow with the extensional loads applied to the body. As the load increases, cracks existing in the body do not expand at first when the load is small; upon reaching a certain load they begin to expand, the expansion depending on the manner in which the load is applied. In some cases cracks expand rapidly up to complete rupture of the body with the load maintained constant, in other cases they expand slowly, stopping as soon as the increase of the load is suspended. Since the opening of a crack is usually small compared to its longitudinal dimensions, it is natural to represent a crack as a cut; but then the tensile stresses at the crack edges in Inglis' problem are infinite, and *in general* the same thing happens in the problem treated by Muskhelishvili. Clearly solutions with infinite tensile stresses at the edges of a crack are unacceptable in a physically correct model of a brittle body. Thus, direct application of the classical scheme of the theory of elasticity to the problem of cracks leads to a problem which is incomplete and yields physically unacceptable solutions.

A. A. Griffith's papers [3, 4] are rightly considered fundamental for the theory of cracks of brittle fracture. The important idea, first advanced in these papers is that an adequate theory of cracks requires the improvement of the model accepted for a brittle body by the consideration of molecular forces of cohesion acting near the edge of a crack.

Griffith treated the following problem: An infinite brittle body stretched by a uniform stress $p_0$ at infinity has a straight crack of a certain size $2l$. It is required to determine the critical value of $p_0$ at which the crack begins to expand. The molecular forces of cohesion were considered as forces of surface tension being internal forces for the given body; their effect on the stress and strain field was neglected.

Under this condition the change $\Delta F$ of free energy ("total potential energy" in Griffith's terminology) of a brittle body with a crack, compared to the same body under the same loads but without a crack, is equal to the difference between the surface energy of the crack $U$ and the decrease in strain energy of the body due to formation of the crack $W$. For the crack to expand, the change in free energy of the body must not grow with an increase in the size $2l$ of the crack. Thus, the parameters of the critical equilibrium state are obtained from the condition

$$\frac{\partial(U-W)}{\partial l}=0. \tag{2.1}$$

But the surface energy of the crack $U$ is equal to the product of the surface area of the crack and the energy $T_0$ required to form the unit surface of the crack. Under certain sufficiently general assumptions, the quantity $T_0$, the surface tension, can be considered constant for a given material under given conditions. Therefore, according to Griffith, the determination of the critical load reduces to the determination of the quantity $\partial W/\partial l$, "the elastic energy release rate". Analysing the simplest case, Griffith calculated this quantity by using Inglis' results [1] and obtained relations determining the critical values of tensile stress in the forms

$$(2.2) \qquad p_0 = \sqrt{\frac{2ET_0}{\pi(1-\nu^2)l}}, \qquad p_0 = \sqrt{\frac{2ET_0}{\pi l}},$$

for plane strain and plain stress, respectively.

The theoretical part of Griffith's paper contains also the results of the investigation of the structure of a crack near its ends. This is carried out on the basis of the classical solution of elasticity theory, constructed without considering forces of cohesion, hence with infinite tensile stresses at the ends of the crack, if it has the shape of a cut. Griffith made an attempt to improve this description of a crack by considering it as an elliptical cavity with a finite radius of curvature $\rho$ at the end (Fig. 3). However, according to his own estimate the magnitude of the radius of curvature at the end of the crack was of the order of the intermolecular distance, which clearly indicates the incorrectness of the approach: in any investigation based on the concept of a continuous medium distances of intermolecular order of magnitude cannot be considered as finite.

This part of Griffith's work is inadequate for the following reason. In determining the equilibrium size of a crack, the effect of molecular forces of cohesion on the stress and strain fields can be neglected, but this cannot be done in analysing the structure of a crack near its ends. The distance at which the effect of forces of cohesion is appreciable is comparable to the distance over which the form of a crack varies essentially. Therefore, to a considerable part, Griffith's analysis of the structure of crack edges cannot be accepted as correct, and in particular his conclusion concerning the rounded form of cracks near the ends is wrong, as will be shown in detail later. This aspect of the matter, obviously of prime importance, remained unclarified until recently and led in a number of cases to misinterpretations of Griffith's results [5].

In addition to the basic shortcoming pointed out here, there were some errors in calculations in the theoretical part of the paper [3]. Shortly after it had appeared, A. Smekal [6] published a detailed comment on it, containing also quite an interesting general discussion of the problem of brittle fracture and correcting the errors.

In a subsequent paper by K. Wolf [7] a more precise and simpler account of Griffith's results was given, and similar calculations were made for somewhat different (but also uniform) states of stress. In [7] the relation of Griffith's theory of fracture to previously proposed theories of strength was also discussed.

In connection with his experiments on the splitting of mica I. V. Obreimov investigated [8] the tearing-off of a thin shaving from a body by a splitting wedge that slides over its surface and has a single point of contact with the shaving. Using the approximate methods of thin-beam theory, Obreimov established the relation between the form parameters of a crack and the surface tension by means of an energy method similar to that used in Griffith's paper. The method of paper [8] was continued later by many investigators [9–12].

The determination of the elastic energy release rate $\partial W/\partial l$ for tensile stress fields more complex than a uniform one, as well as for other configurations of cracks encountered considerable mathematical difficulties. The investigations of H. M. Westergaard [13], I. N. Sneddon [14, 15], I. N. Sneddon and H. A. Elliot [16], M. L. Williams [17] clarified the distribution of stresses and strains near the discontinuity surfaces of the displacement. Together with the classical papers by Muskhelishvili [2, 18, 19] the investigations of Westergaard and Sneddon constitute the mathematical basis of subsequent works on the theory of cracks. However, the conditions of equilibrium for new particular cases and, still less, for a somewhat more general case of loading were not obtained in these papers.

In the papers by R. A. Sack [20], T. J. Willmore [21], and O. L. Bowie [22] the conditions of equilibrium were obtained for some new special cases of loading and position of cracks. The energy method was applied directly in these papers, and thus considerable difficulties in the calculations had to be overcome. In view of the fact that the equilibrium states in the problems treated in [20–22] are unstable and unique, the conditions of equilibrium are identical with those for complete fracture of the body.

The papers by G. R. Irwin [23] and E. O. Orowan [24], in which the concept of quasi-brittle fracture was developed, represent an important stage in the theory of cracks. Irwin and Orowan noticed that a number of materials, which behave as highly ductile in standard tensile tests, fracture by a *quasi-brittle* mechanism when cracks are forming. This means that the arising plastic deformations are concentrated in a very narrow layer near the surface of a crack. As was shown by Irwin and Orowan, it is possible in such cases to employ Griffith's theory of brittle fracture, introducing instead of surface tension the effective density of surface energy. This quantity, in addition to the specific work required to produce rupture of internal bonds (= surface tension), includes the specific work required to produce plastic deformations in the surface layer of a crack; it is sometimes several orders of magnitude larger than the surface tension.

The idea of quasi-brittle fracture extended considerably the range of applicability of the theory of brittle fracture and was undoubtedly one of the main reasons for reviving interest in this problem. Irwin, Orowan and other authors published a series of papers [23—32] devoted to the development of the generalized theory of brittle fracture, to the investigation of the limits of its applicability, and to the analysis of experimental data from the view point of this theory. Special notice deserves the paper by

H. F. Bueckner [33] in which a quite general energy analysis of brittle and quasi-brittle fracture was carried out on the basis of the Griffith-Irwin-Orowan scheme.

In all the foregoing papers the question of the structure of a crack near its edge remained without clarification. In a very interesting paper [34] devoted to the physico-chemical analysis of deformation processes, P. A. Rebinder first expressed the thought about the wedge-like form of a crack at its ends and about the necessity of a corresponding development of Griffith's theory. H. A. Elliot [35], N. F. Mott [36], and Ya. I. Frenkel [5], in analysing the form of a crack, proceeded from the idea of a crack of infinite length between two solid blocks of the material, which were at normal intermolecular distance from each other before formation of the crack.

In [35] the blocks were considered to be semiinfinite. Starting from the classical solution for a straight-line crack [1] and a disk-shaped crack [20] having a diameter $2c$ in a uniform tensile stress field $p$, the distributions of normal stresses $\sigma_y$ and lateral displacements $v$ were determined in [35] for points of the planes distant half the normal intermolecular distance from the crack plane. The function $\sigma_y(2v)$ containing $p$ and $c$ as parameters was identified with the relation between molecular forces of cohesion and the distance; by integrating this function, the surface tension was determined, which thus was found to be connected with $p$ and $c$. The author identified this relation with the condition of fracture, which of course differed from Griffith's condition. The distribution of the lateral displacements so obtained was identified with the form of the crack.

Such an approach is inadequate for the following reasons. The formal application of the apparatus of classical elasticity for the determination of stresses and deformations near the edge of a crack is unjustifiable, since in applying this apparatus all distances (even those which are considered small) must be large compared to the intermolecular distance. Moreover, forces of cohesion act not only inside the body but also on a part of the crack surface. If this fact is taken into account, the edges of a crack have a pointed rather than a rounded shape, and there is no infinite stress concentration at the ends. This will be shown below in detail. Thus, stress and displacement distributions near the edge of the crack surface differ essentially from the corresponding distributions obtained according to the solutions of Inglis [1] and Sack [20], in which the surface of cracks was supposed to be free of stress. Note also that the decrease of $\sigma_y(2v)$ with increasing $v$ is very slow in paper [35], much slower indeed than the natural velocity of diminution of the intensity of forces of cohesion.

Ya. I. Frenkel [5] treated the problem of a crack of infinite length cutting through a thin strip in longitudinal direction. The use of the approximate theory of thin beams, which is unsuitable for analysing the form of a crack near its ends, did not permit him to obtain an adequate result. Incidentally, the comments on Griffith's theory contained in this paper cannot be accepted as well justified either. Frenkel criticizes Griffith because of the instability of the equilibrium in the case of a straight crack in a uniform tensile stress field (considered by Griffith) and he ascribes this instability to Griffith's wrong idea about the form of the crack ends. This is not true. The conclusion about stability or instability of equilibrium of a crack does not depend on considerations concerning the structure of the crack ends. As will be shown later, instability of a crack in a uniform field occurs even when allowance is made for smooth closing of cracks at the ends; it is a part of the problem itself rather than a consequence of the peculiar crack shape assumed. Frenkel's conclusion concerning the existence of a stable state of equilibrium in addition to the unstable one is due to his incorrect replacement of the uniform state of stress by another one.*

In a paper by A. R. Rzhanitsyn [**37**] an attempt was made to solve the problem of a circular crack in a body subjected to a uniform tensile stress under consideration of the molecular forces of cohesion distributed over the crack surface and with smooth closing of the crack. Unfortunately the application of inadequate methods (averaging stresses and strains) did not allow the author to obtain the correct conditions of equilibrium.

An idea first suggested by S. A. Khristianovitch [**38**] was of great importance for the proper understanding of the structure of cracks near their ends. Khristianovitch considered, in connection with the theory of the so-called hydraulic fracture of an oil-bearing stratum, an isolated crack in an infinite body under a constant all-round compressive stress at infinity, maintained by a uniformly distributed pressure of a fluid contained inside the crack. The problem was treated in the quasi-static formulation. In solving it, Khristianovitch hit upon the indefiniteness of the crack length. He noticed, however, the following circumstance. Under the assumption that the fluid fills the crack completely, tensile stresses at the end of the crack are always infinite, whatever the size of the crack. But if the fluid fills the crack only partially, so that there is a free portion of the crack surface which is not wetted by the fluid, then at one exceptional value of the crack length tensile stresses at the ends of the crack are finite. It turned out that for this value of the crack length (and only for this one) the opposite faces of the crack close smoothly at its edges. Khristianovitch advanced a hypothesis of finiteness of stresses or, which is the same, of smooth closing of the opposite faces of a crack at its edges as a fundamental condition determining the size of a crack. The use of this hypothesis made it possible to solve a number of problems concerning formation and expansion of cracks in rocks [**38**—**43**].

* Besides these basic shortcomings there are some errors in calculations in [**5**] indicated in [**37**].

In all these papers, however, molecular forces of cohesion were not taken into account directly. Now in dealing with cracks in rock massifs it is quite permissible to neglect forces of cohesion. The estimates show that the effect of rock pressure is far greater here than the action of forces of molecular cohesion, particularly if the natural fissuring of rocks is taken into consideration. Under other conditions (in particular, in many cases when massifs are simulated in laboratories) forces of cohesion play an important part and their consideration is of great significance in analysing the conditions of equilibrium and expansion of cracks.

A very interesting early work by H. M. Westergaard [44] should be mentioned in connection with these investigations (see also [13]). On the basis of the analogy with the contact problem noted by the author, it is stated that there is no stress concentration at the end of a crack in such brittle materials as concrete. The same paper gives formulas which describe correctly stresses and strains near the ends of equilibrium cracks of brittle fracture in the absence of forces of cohesion. However, Westergaard did not connect the condition of finiteness of stress with the determination of the longitudinal dimension of a crack, which he assumed to be given.

In papers [45, 46] by G. R. Irwin (see also [47, 48, 49, 33]) an important formula was established that correlates the strain-energy release rate with the stress intensity factor near the ends of a crack in a problem of the classical theory of elasticity. On the basis of this formula the strain-energy release rate was determined, and the conditions of fracture were obtained for several new cases of loading and position of cracks [47, 50, 32, 51, 52].

Beginning with the work of Griffith, in most of the theoretical investigations problems of a similar type were treated: the equilibrium state, in which the intensity of forces of cohesion at the contour is a maximum, turns out to be unstable, and the condition for the onset of expansion of a crack coincides with the condition for the beginning of complete fracture of the body. Thus the condition for onset of the expansion is identified in some papers with the onset of rapid crack propagation and fracture for all cracks. In general, that is not true. Cracks actually may be stable so that the beginning of crack development is not necessarily connected with the fracture of a body; and one should not imagine that stable cracks are rare, that they are not encountered in practice and are difficult to produce experimentally. As the experimental investigations carried out by numerous authors beginning from I. V. Obreimov [8] show, the extension of cracks is stable in many cases throughout the greater part of the process of fracture. A. A. Wells [30] obtained stable cracks over a certain range of extensional forces in steel plates under combined external tensile stresses and internal stresses due to welded seams. F. C. Roesler [53] and J. J. Benbow [54] investigated stable conical cracks in glass and silica. The same authors [9] obtained stable cracks in wedging a strip of organic glass. Recently

J. P. Romualdi and P. H. Sanders [52] obtained stable cracks within certain limits of loads for a tensile plate stiffened by riveted ribs. References to other investigations in which stable cracks were obtained and analysed can be found in a monograph by B. A. Drozdovsky and Ya. B. Fridman [55]. All these papers confirm strongly the possibility of using the concept of brittle and quasi-brittle fracture for stable cracks.

Consideration of stable cracks greatly extends the problems that can be formulated in the theory of equilibrium cracks. Indeed, in the case of unstable cracks, only the determination of the load at which a crack begins to expand is of interest, since the process becomes dynamic upon reaching this load. In the case of stable cracks, however, one has to investigate the quasi-static expansion of cracks with change in loads.

In papers [56—61] the formulation of problems in the theory of equilibrium cracks of brittle fracture was improved and supplemented in accordance with the foregoing considerations. In these papers a new approach to problems of the theory of cracks was proposed, which is based on the general formulation of problems concerning elastic equilibrium of bodies in the presence of cracks, as it was given in [40]. The further discussions in this review are based on this approach, which is presented in the following chapter. A number of new problems of the theory of cracks were formulated and solved on that basis.

## III. The Structure of the Edge of an Equilibrium Crack in a Brittle Body

### 1. *Stresses and Strains Near the Edge of an Arbitrary Surface of Discontinuity of Normal Displacement*

As has already been pointed out, one can construct a formal solution of the differential equations of the theory of elasticity, which satisfies the boundary conditions corresponding to the applied load, if one prescribes arbitrarily a surface of discontinuity of the displacement. In the present section the behavior of the solutions of the equations of elasticity near the edge of a surface of discontinuity of displacement is investigated. For simplicity we shall restrict ourselves here to surfaces of discontinuity of normal displacement, appearing as plane faces bounded by closed curves (*contours*).

Near an arbitrary point $O$ at the contour of such a surface, let us take a vicinity whose characteristic dimension is small compared to the radius of curvature of the contour at the point $O$. Deformation in this vicinity can be considered as plane and corresponding to a straight infinite cut in an infinite body subjected to a system of symmetrical loads (see Fig. 6; the plane of deformation is a plane normal to the contour of the discontinuity surface at the point $O$; the trace of the cut in the drawing is the intersection of that plane with the discontinuity surface). Loads can be applied at the surface of the cut and inside the body; the loads at the surface can be assumed to be normal without losing the generality of the further analysis. Consider now this configuration in more detail.

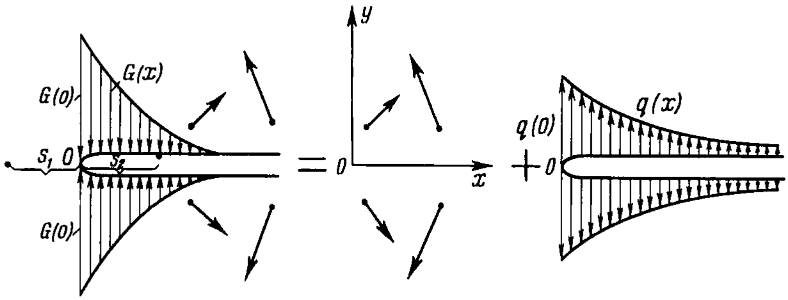


FIG. 6.

The stress and displacement fields can be presented as the sum of two fields (Fig. 6), the first of which corresponds to a continuous body under loads applied inside the body; the second belongs to a body with a cut, symmetrical loads being applied at the surface of the cut only. The shape of the deformed surface of the cut is determined by the second state of stress, since normal displacements at the place of the cut for the first state of stress are equal to zero by symmetry.*

The analysis of the first state of stress can be carried out by the usual methods of the theory of elasticity and is of no special interest; we shall consider this state of stress as given. Let us assume that the line of the cut corresponds to the positive semi-axis $x$; the normal stresses, $g(x)$, applied at the surface of the cut in the second state of stress, represent the difference between the stresses applied at the surface of the cut in the actual field, $G(x)$, and the stresses at the place of the cut, $p(x)$, corresponding to the first state of stress.

Applying Muskhelishvili's method [18] to the analysis of the second state of stress, we obtain the relations determining stresses and displacements

$$\sigma_x^{(2)} + \sigma_y^{(2)} = 4\,\mathrm{Re}\,\Phi(z), \tag{3.1}$$

$$\sigma_y^{(2)} - i\sigma_{xy}^{(2)} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{3.2}$$

* This convenient method of reducing the load to a load distribution over the discontinuity surface was developed in the most general form by H. F. Bueckner [33].

$$2\mu(u^{(2)} + iv^{(2)}) = \varkappa\varphi(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\Phi(z)}, \tag{3.3}$$

$$\varkappa = 3 - 4\nu,$$

where $z = x + iy$; $\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_{xy}^{(2)}$ are the components of the stress tensor of the second state of stress; $u^{(2)}, v^{(2)}$ are the displacement components along the $x$ and $y$ axes corresponding to the second state of stress; $\mu = E/2(1 + \nu)$ is the shear modulus, $E$ is Young's modulus, and $\nu$ is Poisson's ratio. The analytical functions $\varphi, \omega, \Phi, \Omega$ are expressed by formulas

$$\Phi(z) = \Omega(z) = \varphi'(z) = \omega'(z) = \frac{1}{2\pi i\sqrt{z}} \int_0^\infty \frac{\sqrt{t}\, g(t)dt}{t - z}, \tag{3.4}$$

$$\varphi(z) = \omega(z) = \frac{1}{2\pi i} \int_0^\infty g(t) \ln \frac{\sqrt{t} + \sqrt{z}}{\sqrt{t} - \sqrt{z}}\, dt. \tag{3.5}$$

At the cut ($x \geqslant 0$, $y = 0$) and its prolongation ($x \leqslant 0$, $y = 0$) the following relations hold:

$$\sigma_x^{(2)} = \sigma_y^{(2)} = 2 \operatorname{Re} \Phi(z), \qquad \sigma_{xy}^{(2)} = 0, \qquad v^{(2)} = \frac{4(1 - \nu^2)}{E} \operatorname{Im} \varphi(z). \tag{3.6}$$

Using known formulas for limiting values of a Cauchy-type integral at the ends of the contour [19], we obtain an expression for the tensile stresses near the end of the cut along its prolongation,

$$\sigma_y^{(2)} = -\frac{1}{\pi\sqrt{s_1}} \int_0^\infty \frac{g(t)dt}{\sqrt{t}} + g(0) + O(\sqrt{s_1}), \tag{3.7}$$

where $s_1$ is the small distance of the point considered from the end of the cut (Fig. 6). Similarly, we have for the distribution of normal displacements of points at the surface of the cut near its end

$$v^{(2)} = \mp \frac{4(1 - \nu^2)}{\pi E} \sqrt{s_2} \int_0^\infty \frac{g(t)dt}{\sqrt{t}} + O(s_2^{3/2}), \tag{3.8}$$

where $s_2$ is the distance of a surface point of the cut from its end, and negative and positive signs correspond to the upper and lower faces of the cut, respectively (Fig. 6).

This result also fully elucidates the distribution of normal tensile stresses and normal displacements near the contour of an arbitrary surface of normal discontinuity. Indeed, the following formulas are readily obtained from relations (3.7) and (3.8):

$$\sigma_y = \frac{N}{\sqrt{s_1}} + G(0) + O(\sqrt{s_1}), \qquad v = \mp \frac{4(1-\nu^2)N\sqrt{s_2}}{E} + O(s_2^{3/2}), \tag{3.9}$$

where $\sigma_y$ is the tensile stress at a point of the body a small distance $s_1$ away from the contour of the discontinuity surface, lying in the osculating plane to the contour of the discontinuity surface through the point $O$; $N$ is *the*

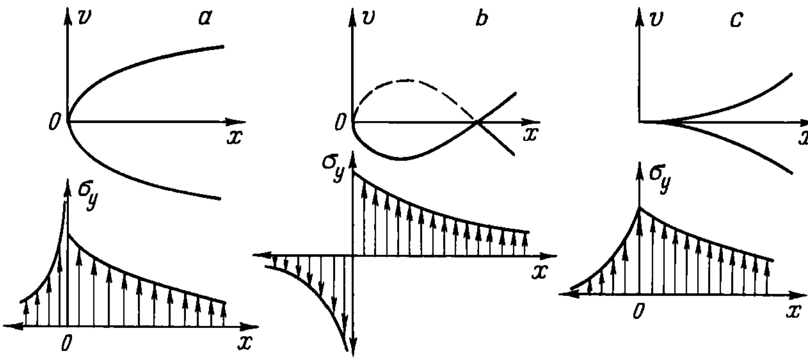


FIG. 7.

*stress intensity factor*, a quantity dependent on the acting loads, on the the configuration of the body and of the discontinuity surfaces in it, and on the coordinates of the point $O$ considered; $G(0)$ is the magnitude of the normal stress applied to the discontinuity surface at $O$ (Fig. 6); $s_2$ is the small distance of a point of the discontinuity surface from its contour. Depending on the sign of $N$, there are in general three possibilities.

If $N > 0$, an infinite tensile stress acts at the point $O$. The shape of the deformed discontinuity surface and the distribution of normal stresses $\sigma_y$ near the point $O$ are represented in Fig. 7a.

If $N < 0$, then an infinite compressive stress acts at the point $O$; the shape of the deformed discontinuity surface and the distribution of stresses near $O$ are represented in Fig. 7b. The opposite faces of the crack overlap in this case, and it is quite evident that this case is physically unrealistic.

Finally, if $N = 0$, the stress acting near the contour is finite and tends to the normal stress applied at point $O$ of the contour if $O$ is approached. Thus the stress $\sigma_y$ is continuous at the contour, and the opposite faces of the discontinuity surface close smoothly (Fig. 7c).

The investigation of the stress and strain distribution near the edge of the surface of normal discontinuity was begun by Westergaard [44, 13] and Sneddon [14, 15] and continued later by the author [40], by Williams [17], and by Irwin [45—47]. In view of the character of the stress states considered in [14, 15, 45—47] results were obtained only for the case $N > 0$.

## 2. *Stresses and Strains Near the Edge of an Equilibrium Crack*

The results obtained in the preceding section pertain to an arbitrary surface of discontinuity of normal displacement. We now show that, for an equilibrium crack, $N = 0$ at all points of its contour.

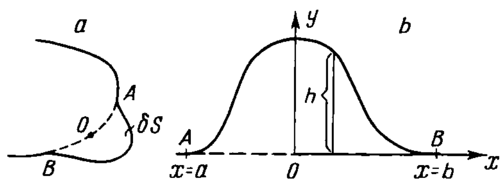


FIG. 8.

Consider a possible state of the elastic system, which differs from the actual state of equilibrium only by a slight variation in the form of the crack contour in a small vicinity of the arbitrary point $O$ (Fig. 8). The new contour is a curve that encloses the point $O$ lying in the plane of the crack. This curve is tangential to the former contour of the crack at points $A$ and $B$ close to $O$; everywhere else the contours of all the cracks remain unchanged. In view of the closeness of the points of tangency $A$ and $B$ to the point $O$, the initial contour of the crack at the portion $AB$ can be considered as straight. The distribution of normal displacements of the points of the new crack surface and the distribution of tensile stresses at these points prior to the formation of the new crack surface are, according to the above, given, to within small quantities, by

$$v = \mp \frac{4(1-\nu^2)N}{E}\sqrt{h-y}, \qquad \sigma_y = \frac{N}{\sqrt{y}}, \tag{3.10}$$

where $N$ is the stress intensity factor at the point $O$.

The energy released in the formation of the new crack surface, which is equal to the work required to close this new surface, is given by

$$\delta A = \tfrac{1}{2} 2 \int_{\delta S} \sigma_y |v| dS = \frac{4(1-\nu^2)N^2}{E} \int_a^b dx \int_0^h \sqrt{\frac{h-y}{y}}\, dy$$

$$= \frac{2(1-\nu^2)\pi N^2}{E} \int_a^b h dx = \frac{2(1-\nu^2)\pi N^2 \delta S}{E}, \tag{3.11}$$

where $\delta S$ is the area of the projection of the new crack surface on its plane.

The condition of equilibrium of the crack requires that $\delta A$ vanishes; this together with (3.11) implies that $N = 0$. Thus we arrive at a very important result characterizing the structure of cracks near their contours:

1. *The tensile stress at the contour of a crack is finite.*
2. *The opposite faces of a crack close smoothly at its contour.*

It appears, therefore, that contrary to Griffith's conception the form of a crack near its edge is as represented in Fig. 4. Since the only acting forces at the surface of a crack near its contour are forces of cohesion, it follows from (3.9) that the tensile stress at the crack contour is equal to the intensity of forces of cohesion at the contour. In particular, if there are no forces of cohesion, the tensile stress at the crack contour is equal to zero.

The condition of finiteness of stresses and smooth closing of the opposite faces at the edges of a crack was first suggested as a hypothesis by S. A. Khristianovitch [38], to serve as a basic condition that determines the position of the crack edge. The proof of this condition given above follows [60] mainly. Formula (3.11) for the case of plane stress was first proved by Irwin [45, 46] irrespective of finiteness of stresses and smoothness of closing (see also the review by Irwin [47] and the paper by Bueckner [33]). The early paper by Westergaard [44] contains a statement concerning the absence of stress concentration at the end of a crack in brittle materials like concrete, but the condition of finiteness of stress that appears in this work was not connected with the determination of the size of the crack.

We have confined ourselves here to the examination of cracks of normal discontinuity only for simplicity of treatment. Analogous reasoning, in particular the proof of finiteness of stress at the crack edge, can be extended without any substantial changes to cover the general case in which also the tangential displacement components have a discontinuity at the crack surface.

### 3. *Determination of the Boundaries of Equilibrium Cracks*

The conditions of finiteness of stresses and smooth closing of a crack at its contour permit us to formulate the problem of equilibrium cracks for a given system of loads acting upon the body: for a given position of initial cracks and a given system of forces acting upon the body, it is required to find stresses, deformations, and crack contours in the elastic body so as to satisfy the differential equations of equilibrium and the boundary conditions, and to insure finiteness of stresses and smooth closing of the opposite faces at the crack contours.

We shall illustrate the solution of this problem by an elementary example of an isolated straight crack in an infinite body under all-round compressive stress $q$ at infinity and with concentrated forces $P$ applied at opposite points of the crack surface (Fig. 9).

The solution of the equilibrium equations satisfying the boundary conditions can be obtained by Muskhelishvili's method [18] for an *arbitrary crack length* $2l$. Stresses and displacements are expressed by formulas (3.1)—(3.3) with

$$\Phi(z) = \frac{2\zeta^2}{l(\zeta^2-1)}\left\{\frac{P}{\pi(\zeta^2+1)} - \frac{ql(\zeta^2+1)}{4\zeta^2}\right\},$$

(3.12)

$$z = \frac{l}{q}\left(\zeta + \frac{1}{\zeta}\right).$$

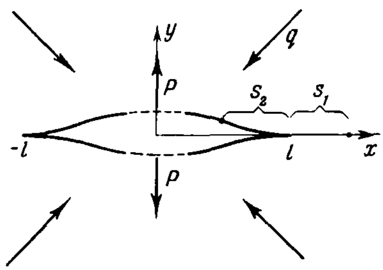


FIG. 9.

Evidently, equilibrium equations and boundary conditions do not determine the length of the crack. The distributions of stresses $\sigma_y$ at the prolongation of the crack and normal displacement $v$ of points of the crack surface near its edge are given by

$$\sigma_y = \left(\frac{P}{\pi l} - q\right)\sqrt{\frac{l}{8s_1}} + O(1), \qquad v = \mp\frac{(1-\nu^2)}{E}\left(\frac{P}{\pi l} - q\right)\sqrt{8s_2 l} + O(s_2{}^{3/2}).$$

(3.13)

Finiteness of stress and smooth closing of the crack at its ends are assured simultaneously by the condition

$$l = \frac{P}{\pi q}, \tag{3.14}$$

which determines the crack size under given loads $P$ and $q$.

Let us now attempt to determine the size $2l$ of an isolated straight crack in an infinite body stretched by uniform stress $p_0$ at infinity in the direction perpendicular to the crack. If the crack surface is assumed to be free of stress, then one can easily show that the tensile stress at the prolongation of the crack near its edge depends on the distance $s_1$ as follows:

$$\sigma_y = \frac{p_0\sqrt{l}}{\sqrt{2s_1}}; \tag{3.15}$$

hence it appears that for no $l$ the stress $\sigma_y$ will be finite at the crack end and there does not exist an equilibrium crack! This paradoxical result is due to the fact that we did not take into account the molecular forces of cohesion acting near the crack edges and thus did not completely account for the loads acting upon the body. The consideration of these forces and the definitive formulation of problems in the theory of equilibrium cracks of brittle fracture are discussed in the following section.

## IV. Basic Hypotheses and General Statement of the Problem of Equilibrium Cracks

### 1. *Forces of Cohesion; Inner and Edge Regions; Basic Hypotheses*

In order to construct an adequate theory of cracks of brittle fracture, it is necessary to supplement the model of a brittle body by considering the molecular forces of cohesion acting near the edge of a crack at its surface. It is known that the intensity of forces of cohesion depends strongly on the distance. Thus, for a perfect crystal the intensity $f$ of forces of cohesion acting between two atomic planes at the distance $y$ from each other is zero if $y$ is equal to the normal intermolecular distance $b$. With $y$ increasing up to about one and a half of $b$, the intensity $f$ grows and reaches a very high maximum $f_m \sim \sqrt{ET_0/b} \sim E/10$; after that it diminishes rapidly with further increase of $y$ (Fig. 10). Here $E$ is Young's modulus, and $T_0$ is the surface tension related to $f(y)$ by the formula

$$2T_0 = \int_b^\infty f(y)dy. \tag{4.1}$$

The maximum intensity $f_m$ defines the theoretical strength, i.e. the strength of a solid if it were a perfect crystal. The actual strength of solids is usually several orders of magnitude lower because of defects of crystal structure. For amorphous bodies the relation between the intensity of forces of cohesion and the distance has qualitatively the same character.

Data at present available, which confirm the above character of the relation between the intensity of forces of cohesion and the distance, lead to the following conclusion. It has long been known that the strength of thin fibers exceeds considerably that of large specimens of the same material [62, 63]. Experiments carried out recently with filamentary crystals of some metals revealed an exceptionally high strength approaching the theoretical value [63]. It is supposed that this phenomenon is due to the relatively small amount of structural defects in thin fibers and filamentary crystals. Furthermore, numerous direct measurements of the intensity of molecular forces of cohesion for glass and silica [64—66] were made recently. The

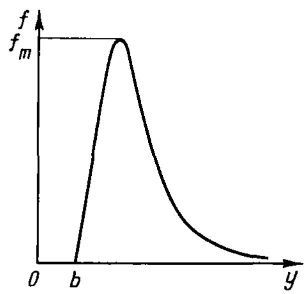


FIG. 10.

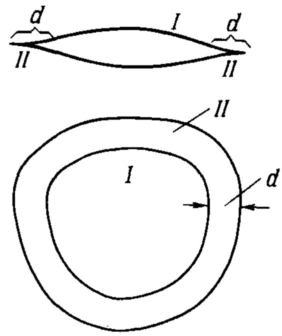


FIG. 11. *I*-inner region, *II*-edge region.

very elegant method used in this kind of measurements is based on the application of a regenerative microbalance and was suggested and employed by B. V. Deryagin and I. I. Abrikosova [64, 65]. However, these direct measurements deal with very great distances $y$ compared to the normal intermolecular distance and thus determine only the end of the falling branch of the curve $f(y)$. A macroscopic theory for forces of cohesion at such distances was developed by E. M. Lifshitz [64] and was found in good agreement with the results of these aforementioned measurements. The relation $f(y)$, if $y$ equals several normal intermolecular distances, seems to be beyond any strict quantitative theory and difficult for the direct experimental determination at present. A description of available attempts to estimate the relation $f(y)$ at such distances and, consequently, the theoretical strength can be found in [67, 63, 68].

The distance between the opposite faces of a crack varies from magnitudes of the order of the intermolecular distance near the crack edge to sometimes rather great magnitudes far from the edge. It is therefore convenient to divide the crack surface into two parts (Fig. 11). The opposite faces in the first part — *the inner region of the crack* — are a great distance apart, hence their interaction is vanishingly small, and the crack surface can be considered free of stresses caused by the interaction of the opposite faces. The opposite faces of a crack in the second part are adjacent to the crack contour — *the edge region of the crack* — and come close to each other so that the intensity of the molecular forces of cohesion acting on this part of the surface is great. Of course, the boundary between the edge and inner region of the crack surface is conventional to a certain extent. For very small cracks there may be no inner region of the crack at all.

Since the distribution of the forces of cohesion over the surface of the edge region is not known beforehand, a substantial part of the loads applied to the body is not known. It is thus impossible to handle the problem of cracks directly in the way it was stated in Chapter III. But the following method of solving problems of cracks is possible in principle: the distance between the opposite faces of a crack is found at each surface point as a function of the unknown distribution of forces of cohesion over the surface. Assuming the relation $f(y)$ between forces of cohesion and distance as given, a relationship can be obtained which determines the distribution of forces of cohesion over the crack surface.

Such an approach is not practicable. First, the relation $f(y)$ is not known to a sufficient extent for a single real material. Even if it were known, the problem would constitute a very complex non-linear integral equation, the effective solution of which presents great difficulties even in the simplest cases.*

Attempts were made to prescribe the distribution of forces of cohesion over the crack surface in a definite manner, but these attempts cannot be considered sufficiently well founded.

For sufficiently large cracks, consideration of which is of principal interest, the difficulty connected with our lack of knowledge of the distribution of forces of cohesion can be avoided without making any definite assumptions concerning this distribution. In this case the general properties of the relation between forces of cohesion and distance allow the formulation of two basic hypotheses which not only simplify essentially the further analysis, but permit the determination of contours of cracks, although the forces of cohesion are finally altogether excluded from consideration as loads acting upon the body.

*First hypothesis: The width $d$ of the edge region of a crack is small compared to the size of the whole crack.*

This hypothesis is acceptable because of the rapid diminution of forces of cohesion with the increase in the distance between the opposite faces of

---

* In papers of M. Ya. Leonov and V. V. Panasyuk [69, 70] the relation $f(y)$ is approximated by a broken line, and on the basis of this approximation a linear integral equation for the normal displacements of the crack surface points is derived. It is solved approximately, the representation of the solution being not quite successfully selected so that the form of the crack at its end appears wedge-shaped with a finite edge angle. In fact, as was shown above, the edge angle must be zero. The shortcoming of these papers lies also in the application of the results obtained by the methods of mechanics of continua to cracks whose longitudinal dimensions are only of the order of several intermolecular distances.

a crack. Of course, there exist micro-cracks to which this hypothesis cannot be applied. However, as the width $d$ of the edge region is quite small, the hypothesis is already valid for very small cracks and certainly for all macro-cracks. Nevertheless, the width $d$ is considered to be sufficiently great compared to micro-dimensions (for instance, compared to the lattice constant in a crystalline body), so that it is permissible to employ the methods of continuum mechanics over distances of the order of $d$.

*Second hypothesis: The form of the normal section* of the crack surface in the edge region (and consequently the local distribution of the forces of cohesion over the crack surface) does not depend on the acting loads and is always the same for a given material under given conditions (temperature, composition and pressure of the surrounding atmosphere and so on).*

When the crack expands, the edge region near a given point, according to the second hypothesis, moves as if it had a motion of translation, and the form of its normal section remains unchanged. This hypothesis is applicable only to those points of the crack contour where the maximum possible intensity of forces of cohesion is reached; an expansion of the crack occurs then at this point with an arbitrarily small increase in the loads applied to the body.

Equilibrium cracks, on whose contour is at least one such point, will be called *mobile-equilibrium* cracks to distinguish them from *immobile-equilibrium* cracks which do not possess this property, i.e. do not expand with an infinitesimal increase in the load. Thus the second hypothesis and all conclusions based on it are applicable to reversible cracks as well as to irreversible equilibrium cracks, which formed at the initial rupture of a brittle body in the process of increasing the load. It is not applicable to cracks which result from equilibrium cracks existing at some greater load by diminishing that load; nor can it be applied, to artificial cuts made without subsequent expansion.

The second hypothesis is suggested by the fact that the maximum intensity of the forces of cohesion is so very great and exceeds by several orders of magnitude the stresses which would arise under the same loads in a continuous body without a crack. Therefore it is possible to ignore the change of stress in the edge region when loads vary and, consequently, the corresponding variation of the normal sections.

These two hypotheses reformulate the results of the qualitative analysis of the brittle-fracture phenomenon carried out by a number of investigators beginning with Griffith. They are the only assumptions concerning the forces of cohesion which underlie the theory presented below and appear in this explicit form in [56, 57].

---

* = intersection with a plane normal to the crack contour.

## 2. *Modulus of Cohesion*

The body considered is assumed to be linearly elastic up to fracture. The elastic field in the presence of cracks can then be represented as the sum of two fields: a field evaluated without taking into account forces of cohesion and a field corresponding to the action of forces of cohesion alone. Therefore the quantity $N$ entering in formulas (3.15) and, as was proved, equal to zero can be written as $N = N_0 + N_m$, where the stress intensity factor $N_0$ corresponds to the loads acting upon the body and to the same configuration of cracks *without* considering forces of cohesion, and the stress intensity factor $N_m$ corresponds to the same configuration of cracks and forces of cohesion only.

According to the first hypothesis the width $d$ of the edge region acted upon by forces of cohesion is small compared to the crack dimensions on the whole and, in particular, to the radius of curvature of the crack contour at the point considered. In determining the value of $N_m$ we may thus assume that the field belongs to the configuration discussed in Section III,1, i.e. to an infinite body with a semi-infinite cut, with symmetrical normal stresses being applied to the surface of the cut. Hence it follows from (3.7) that

$$N_m = -\frac{1}{\pi}\int_0^\infty \frac{G(t)dt}{\sqrt{t}} = -\frac{1}{\pi}\int_0^d \frac{G(t)dt}{\sqrt{t}}, \tag{4.2}$$

where $G(t)$ is the distribution of forces of cohesion different from zero only in the edge region $0 \leqslant t \leqslant d$.

According to the second hypothesis, the distribution of forces of cohesion and the width $d$ of the edge region at those points of the contour, where the intensity of forces of cohesion is a maximum, do not depend on the applied load; the integral in (4.2) represents then a constant characterizing the given material under given conditions. This constant will be denoted by $K$:

$$K = \int_0^d \frac{G(t)dt}{\sqrt{t}}. \tag{4.3}$$

It was termed *the modulus of cohesion* since this quantity characterizes the resistance of the material to an extension of its cracks, caused by the action of forces of cohesion. As will be shown below, the quantity $K$ is the only characteristic of the forces of cohesion, that enters in the formulation of the problem of cracks.

The dimension of the modulus of cohesion is:

$$[K] = [F][L]^{-3/2} = [M][L]^{-1/2}[T]^{-2}, \tag{4.4}$$

where $[F]$, $[L]$, $[M]$, and $[T]$ denote the dimensions of force, length, mass, and time, respectively. Constants of a similar dimension are encountered in the contact problem of the theory of elasticity [71, 72, 73]. It is no coincidence, that there exists a profound connection between the contact problem and problems in the theory of cracks of brittle fracture; it seems that this was first pointed out in the papers of Westergaard [44, 13].

### 3. *The Boundary Condition at the Contour of an Equilibrium Crack*

For points of the contour of an equilibrium crack, at which the maximum intensity of cohesion is reached, the second hypothesis is applicable, and (4.2) may be written as

$$N_m = -\frac{1}{\pi} K; \tag{4.5}$$

considering that $N = 0$, we obtain

$$N_0 = \frac{1}{\pi} K. \tag{4.6}$$

The boundary condition at contour points of an equilibrium crack, at which the intensity of forces of cohesion is maximal, can also be formulated as follows: on approaching these points, the normal tensile stress $\sigma_y$ at the points of the body lying in the crack plane, if calculated without taking into account forces of cohesion, tends to infinity according to the law

$$\sigma_y = \frac{K}{\pi \sqrt{s}} + O(1), \tag{4.7}$$

where $s$ is the (small) distance from the contour point considered. Satisfying (4.6) at least at one point of the contour is the condition that the crack is in the state of mobile equilibrium.

One should not connect, in general, the reaching of the state of mobile equilibrium by the crack with the onset of its unstable rapid growth and still less with complete fracture of the body. A mobile-equilibrium crack may be either stable or unstable. Only in case of instability is the condition for the onset of rapid crack propagation given by (4.6). However, not even in this case is complete fracture of the body unavoidable, since the transition from the unstable state of equilibrium to the other, stable one, is possible. Numerous examples illustrating various possibilities will be discussed in the following chapter.

If a crack is irreversible and there are points on its contour where the intensity of forces of cohesion is less than maximal,* then the second hypothesis is not applicable at such points. Since cohesive forces that act in the edge region of the crack surface are smaller near such points than those acting near points of the type considered above, it follows from (4.2) that $-N_m < K/\pi$; and since $N_0 = -N_m$, we have for these points

$$N_0 < \frac{K}{\pi}. \tag{4.6 a}$$

As the load increases, forces of cohesion in the edge region grow; they compensate the increase in the load and insure finiteness of stress and smooth closing at the crack contour. However, the crack does not expand at a given contour point until the forces of cohesion become maximal. The second hypothesis now becomes applicable, and condition (4.6) is satisfied.

In determining the form of contours of equilibrium cracks, conditions (4.6) and (4.6a) permit us to exclude the forces of cohesion altogether from the consideration of the loads acting upon the body. Instead, we work with their overall integral characteristic, the modulus of cohesion. Special estimates show [57, 58] that the influence of molecular forces of cohesion on the stress and displacement field is essential only in the neighbourhood of the edge in a region of the order of magnitude $d$. Forces of cohesion thus determine the structure of cracks near their ends, and the forms of crack contours depend on them only through the integral characteristic $K$.

### 4. *Basic Problems in the Theory of Equilibrium Cracks*

The basic problem in the theory of equilibrium cracks can be stated in its most general form as follows. A certain system of initial cracks and a process of loading the body, i.e. a system of loads acting upon the body, dependent on one monotonously increasing parameter $\lambda$, are given. The value of $\lambda$ for the initial state may be assumed as zero. It is required to determine the form of the crack surfaces and to find the distribution of stresses and strains in the body corresponding to any $\lambda > 0$. The process of varying the load is supposed to be sufficiently slow so that dynamic effects need not be considered.

When the symmetry of body, loads, and initial cracks insures the possibility of developing a system of plane cracks and the extensional loads grow monotonously with increasing $\lambda$, the configuration of cracks in the body is determined by the current load only and not by the whole history of the

* For instance, contour points of non-expanded cuts or of cracks formed from cracks which existed under a greater load when the load is diminished.

process of loading, as it is in the general case. In this case the problem is formulated as follows (it will be called problem $A$). In a body bounded by a surface $\Sigma$ contours of an initial system of plane cracks $\Gamma_0$ are given (Fig. 12; the plane of the drawing is the plane of the cracks). It is required to find the elastic field and the contours of a system of plane cracks $\Gamma$ enclosing the contours $\Gamma_0$ (and perhaps coinciding with them partially) corresponding to a given load, i.e. to a given value of $\lambda$.

This problem reduces mathematically to the following one. It is required to construct the solution of the differential equations of equilibrium of elasticity theory in the region bounded by plane cuts with contours $\Gamma$ and by the body boundary $\Sigma$ under boundary conditions corresponding to the given load. The contours $\Gamma$ must be determined so that condition (4.6) is satisfied at points of these contours not lying on $\Gamma_0$, and condition (4.6a) at points of $\Gamma$ lying on $\Gamma_0$.

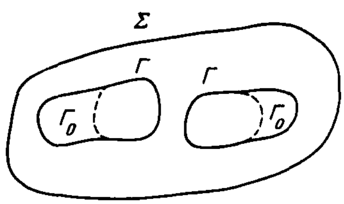

FIG. 12.

If the cracks are reversible or if the applied loads are sufficiently great so that the contours $\Gamma$ do not coincide with $\Gamma_0$ at any point, then the form of the initial contours is of no importance. It is then possible, without prescribing the initial cracks, to formulate directly the problem of determining the contours $\Gamma$ of equilibrium cracks of a given configuration so that condition (4.6) is satisfied at each point of $\Gamma$. Here we assume that the initial cracks are such that they are compatible with the realization of the given configuration of cracks when the load increases. This problem will be called problem B.

It may happen that a solution of either of the above stated problems does not exist. If this happens, it has quite a different significance for the problems A and B. If no solution of problem A exists this means that the applied load exceeds the breaking load, hence its application causes fracture of the body. The limiting value of the parameter $\lambda$ up to which the solution of problem $A$ exists, corresponds to the breaking load. The determination of the breaking load for a given configuration of the initial cracks and a given process of loading presents an important problem in the theory of cracks. Non-existence of the solution of problem B signifies that, whatever initial cracks may be within a given configuration, they will not expand under a given load, hence the applied load is too small. In such cases the conventional description of the state would be that mobile-equilibrium cracks do not form under the given load.

## 5. *Derivation of the Boundary Condition at the Contour of an Equilibrium Crack by Energy Considerations*

Molecular forces of cohesion so far have been considered as external forces applied to the surface of the body. This was necessary for analysing the structure of cracks near their ends.

If only boundary conditions are to be obtained, another approach can be employed which considers the forces of cohesion as internal forces of the system. On the basis of this approach, the idea of which goes back to Griffith [**3**, **4**], a relation between the modulus of cohesion and other characteristics of the material will be obtained.

As before, let there be a certain configuration of equilibrium cracks in a brittle body and consider as in Section III,2 a possible state of the elastic system, which differs from the real one only by a variation in the crack contour near a certain point $O$ (Fig. 8). However, unlike Section III,2, the characteristic size of the new area of the crack surface is assumed to be large compared to the dimension $d$ of the edge region, though small compared to the size of the crack as a whole; according to the first hypothesis (Section IV,1) such an assumption is permissible. Under this assumption forces of cohesion can be considered merely as forces of surface tension, and a certain amount of work must be done to overcome these forces in increasing the crack surface. The influence of forces of cohesion on the stress and strain fields can be neglected since it is essential only in the neighbourhood of the crack edge, whose dimension is of the order of the width of the edge region.

The work $\delta A$ required for the transition from the actual state to a virtual one is equal to the difference between the corresponding increment in surface energy $\delta U$ and released elastic energy $\delta W$:

$$\delta A = \delta U - \delta W. \tag{4.8}$$

For the actual state of an elastic system to be an equilibrium state, $\delta A$ must vanish, hence

$$\delta U = \delta W. \tag{4.9}$$

Quite similarly to Section III,2 an expression for $\delta W$ is obtained:

$$\delta W = \frac{2(1-\nu^2)\pi N_0{}^2 \delta S}{E}, \tag{4.10}$$

where $N_0$ is the value of the stress intensity factor at the point $O$ calculated without taking into consideration the forces of cohesion. Formula (4.10) in a somewhat different form was established by Irwin [**45–47**].

If the form of the edge region near a given point of the contour corresponds to the maximum intensity of forces of cohesion, then, according to the above, in forming a new crack surface the edge region is displaced without deformation; the work against the forces of cohesion per unit of newly formed surface is then constant and equal to the surface tension $T_0$. Therefore, $\delta U = 2T_0\delta S$, because two surfaces form in rupture. Together with (4.9) and (4.10), we have

$$N_0 = \sqrt{\frac{ET_0}{\pi(1-\nu^2)}}. \tag{4.11}$$

Comparing (4.11) and (4.6), we obtain a relationship correlating the modulus of cohesion $K$, defined independently by (4.3), with the surface tension $T_0$ and the elastic constants of the material $E$ and $\nu$:

$$K^2 = \frac{\pi E T_0}{1-\nu^2}. \tag{4.12}$$

### 6. *Experimental Confirmation of the Theory of Brittle Fracture; Quasi-Brittle Fracture*

After Griffith's work [3, 4] many investigators attempted to carry out experimental verifications of the theory of brittle fracture. We cannot

| Spherical bulbs | | | | Cylindrical tubes | | | |
|---|---|---|---|---|---|---|---|
| $2l$ inches | $D$ inches | $p_0$ psi | $p_0\sqrt{l}$ | $2l$ inches | $D$ inches | $p_0$ psi | $p_0\sqrt{l}$ |
| 0.15 | 1.49 | 864 | 237 | 0.25 | 0.59 | 678 | 240 |
| 0.27 | 1.53 | 623 | 228 | 0.32 | 0.71 | 590 | 232 |
| 0.54 | 1.60 | 482 | 251 | 0.38 | 0.74 | 526 | 229 |
| 0.89 | 2.00 | 366 | 244 | 0.28 | 0.61 | 655 | 245 |
| | | | | 0.26 | 0.62 | 674 | 243 |
| | | | | 0.30 | 0.61 | 616 | 238 |

analyse all this work here in detail and shall dwell only on several of the most characteristic papers, referring for details and discussion of other numerous investigations to the special publications [62, 55, 74–78].

Griffith's paper [3] gives descriptions and results of the following experiments. Cracks of various length $2l$ were placed on spherical glass bulbs and cylindrical tubes, whose diameter $D$ was sufficiently great so that a special verification showed no influence of the diameter on the experimental results. After the tubes and bulbs had been annealed to relieve residual stresses produced by making the cracks, they were loaded from the inside by hydraulic pressure up to fracture. The breaking stress $p_0$ corresponding to each crack length $2l$ was measured.

According to the foregoing theory it appears that the breaking stress $p_0$ at which a given crack becomes unstable (onset of mobile equilibrium) can depend only on the crack length $2l$ and the modulus of cohesion $K$. From dimensional analysis [79] it follows that $p_0 = \alpha K/\sqrt{l}$, where $\alpha$ is a dimensionless constant. Consequently, $p_0\sqrt{l}$ must be constant for a given material (in full accord with (2.1)).

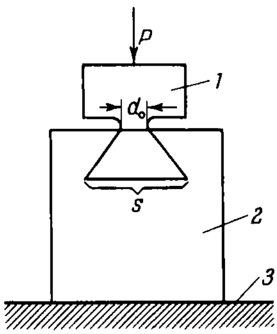


FIG. 13.
1. Steel indentor.
2. Specimen.
3. Steel support.

Griffith's experiments, which are tabulated here, confirm the constancy of this quantity and thus the foregoing theoretical scheme.

The remarkably elegant experiments of Roesler [53] and Benbow [54], in which stable conical cracks were produced, are of special interest for the confirmation of the theory of brittle fracture. The scheme of these experiments is presented in Fig. 13; the photograph of conical cracks in fused silica, borrowed

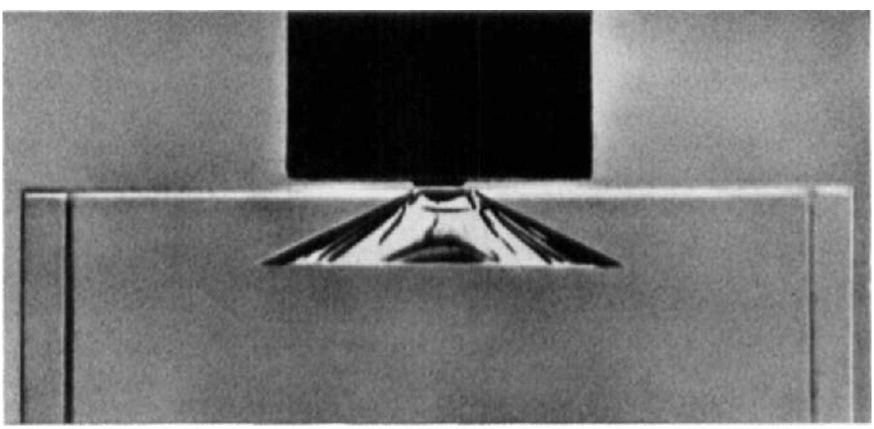

FIG. 14.

from Benbow's paper [54], is given in Fig. 14. The cracks were formed by the penetration into a specimen of glass [53] and fused silica [54] of a cylindrical steel indentor with a flat end. In accordance with the above, the diameter $s$ of the base of a conical crack can depend only on the diameter $d_0$ of the indentor base, the force $P$ pressing the indentor, the modulus of cohesion $K$, and Poisson's ratio $\nu$. Since the correct formulation of the corresponding problem of elasticity theory does not include Young's modulus, it should not be included in the number of determining parameters of the crack problem. Dimensional analysis yields

$$s = \left(\frac{P}{K}\right)^{2/3} \varphi \left[\frac{K^{2/3}d_0}{P^{2/3}}, \nu\right], \tag{4.13}$$

where $\varphi$ is a dimensionless function of its arguments.

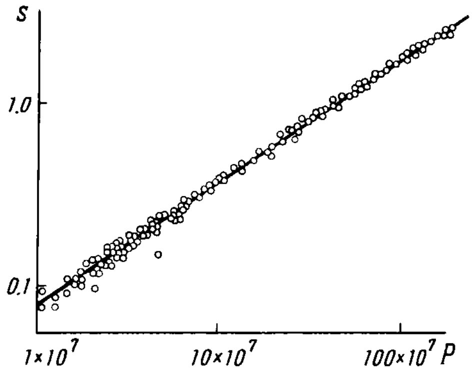


FIG. 15.

Experiments carried out with indentors of three diameters on eleven glass specimens [53] confirm well the existence of the universal relation (4.13). At large values of $P$, when the first argument of the function $\varphi$ becomes vanishingly small, *self-similarity* takes place, and the following relationship holds:

$$s = \left(\frac{P}{K}\right)^{2/3} \varphi_1(\nu). \tag{4.14}$$

Fig. 15 represents a graph, taken from Benbow's paper [54], of the $s(P)$ relation according to data from experiments with fused silica carried out under conditions corresponding to the self-similar regime. As can be observed, these experiments give a conclusive proof of the validity of relation (4.14) and confirm thereby the above scheme.

The experiments described were carried out with materials which can be considered as perfectly brittle. This refers especially to fused silica. Benbow [54] presents certain facts indicating that the mechanism of crack formation in fused silica is closer to being perfectly brittle than it is in glass: cracks in glass grow for a long time under constant load, whereas in fused silica their size is established quickly and then remains unchanged; after removal of the load, cracks in glass remain distinctly visible, but in silica they are imperceptible, etc. However, the significance of the theory of brittle fracture greatly exceeds what should be the limits of its applicability to those comparatively rare materials that are perfectly brittle. Experimental investigations show that when cracks appear some materials, which behave as highly plastic bodies in common tensile tests, fracture in such a way that plastic deformations, though present, are concentrated in a thin layer near the crack surface.

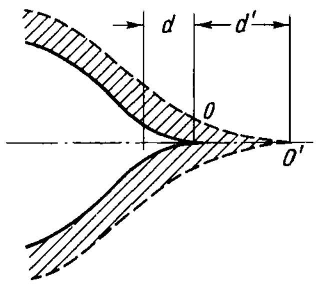

FIG. 16.

D. K. Felbeck and E. O. Orowan [28] carried out experiments on fracture of low-carbon steel plates with a saw-cut crack under conditions corresponding to Griffith's scheme of uniform extension. Experimental results are in good agreement with Griffith's formula, but the surface-energy density exceeds by about three orders of magnitude the surface tension of the material investigated. It was found in good agreement with the specific work of plastic deformations in the layer near the crack surface, which was determined by independent measurements.

On the basis of this and similar experimental results Irwin [23] and Orowan [24] advanced the concept of quasi-brittle fracture, which permitted an important extension of the limits of applicability of the theory of brittle fracture. Here the theory of brittle fracture covers the case when the plastic deformations are concentrated in a thin layer near the crack surface. The energy $T$ required to form the unit surface of a crack is expressed as the sum of the specific work against the forces of molecular cohesion $T_0$ (= surface tension) and the specific work of plastic deformation $T_1$:

$$T = T_0 + T_1. \tag{4.15}$$

A formal extension to quasi-brittle fracture is made as follows (Fig. 16, the plastic deformation zone near the surface is shaded). Imagine the whole plastic region cut out and shift the crack end to the end of the plastic region. This can be done, if the forces exerted by the plastic zone upon the elastic zone are considered as external forces applied to the crack surface. After that the previous reasoning remains unchanged, if the plastic zone is assumed as thin and use is again made of the hypothesis concerning the invariability of the edge region (which here includes the boundary of the elastic and plastic zones). The modulus of cohesion is now expressed as

$$K = \int_0^{d+d'} \frac{G_1(t)dt}{\sqrt{t}} = \frac{\pi E T}{1 - \nu^2}, \tag{4.16}$$

where $G_1(t)$ is the distribution of normal stresses acting on the boundary of the elastic and plastic zones.

When the contribution of molecular forces of cohesion to integral (4.16) can be ignored in comparison to the contribution of stresses that act in the region ahead of the actual crack end and have the order of magnitude of the yield point stress $\sigma_0$, we obtain an estimate for the modulus of cohesion:

$$K = \frac{\pi E T_1}{1 - \nu^2} \sim 2\sigma_0 \sqrt{d'}. \tag{4.17}$$

Note that the value of $\sigma_0$ at the yield point near the crack end may differ from that at the yield point obtained in tensile tests with large specimens.

The concept of quasi-brittle fracture is somewhat related to the concept of the "plastic particle" at the ends of notches with a zero radius of curvature, advanced in a classical monograph by H. Neuber [80].

In the following we shall speak of cracks of brittle fracture, bearing in mind the possibility of extending the results to the case of quasi-brittle fracture. Of course, in this latter case it is necessary to take into consideration the irreversibility of cracks of quasi-brittle fracture.

### 7. *Cracks in Thin Plates*

If the state of stress can be assumed to be plane, then all relations derived previously for the case of plane strain hold also for thin plates, if only $E$ is replaced by $E(1 - \nu^2)$ and the modulus of cohesion is assumed to have some other value $K_1$. Repeating the derivation of formula (4.12) for the plane stress state we obtain

$$K_1^2 = \pi E T. \tag{4.18}$$

The experiments show that the surface energy density $T$ in the case of quasi-brittle fracture increases with a reduction in the plate width [48], which is due to a broadened plastic-strain zone near the crack surface. An approximate theoretical analysis of this phenomenon was attempted by I. M. Frankland [81].

Bearing in mind the complete analogy of the analysis of plane stress and plane strain we shall in the following consider only plane strain.

## V. Special Problems in the Theory of Equilibrium Cracks

This Chapter deals with solutions of special problems in the theory of cracks available at present. A few of the examples have illustrative character, but most problems presented are interesting in themselves.

### 1. *Isolated Straight Cracks*

In this and the following section isolated mobile-equilibrium cracks are examined, and all along the contour the maximum intensity of forces of cohesion is assumed to prevail. The problem reduces here to the determination of crack contours corresponding to a given load so that condition (4.6) is satisfied at these contours, and it represents a particular case of problem B formulated above. It is supposed that the initial cracks guarantee the possibility of producing such cracks; the necessary requirements for the initial cracks in the cases of reversible and irreversible cracks follow readily from the solutions obtained.

Let us consider an isolated straight mobile-equilibrium crack extending along the $x$-axis from $x = a$ to $x = b$ in an infinite body subject to plane strain. Let $p(x)$ be the distribution of normal stresses, which arise at the place of the crack in a continuous body under the same loads. This distribution is computed by the usual methods of elasticity, and we may consider it as given. It may be shown by using Muskhelishvili's solution [2, 18] that tensile stresses near the crack ends calculated without taking into account forces of cohesion become infinite according to the law $\sigma_y = N/\sqrt{s} + \ldots$, where

$$N_a = \frac{1}{\pi\sqrt{b-a}}\int_a^b p(x)\sqrt{\frac{b-x}{x-a}}\,dx, \qquad N_b = \frac{1}{\pi\sqrt{b-a}}\int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx \tag{5.1}$$

are the values of the stress intensity factors for points $a$ and $b$, respectively. Satisfying condition (4.6) at these points, we obtain relations that determine the coordinates of the crack ends $a$ and $b$:

$$\int_a^b p(x)\sqrt{\frac{b-x}{x-a}}\,dx = K\sqrt{b-a}, \qquad \int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx = K\sqrt{b-a}. \tag{5.2}$$

In particular, if the applied load is symmetrical with respect to the crack middle, where we place the origin of coordinates, then $-a = b = l$, and Eqs. (5.2) become one relation determining the half-length of the crack $l$:

$$\int_0^l \frac{p(x)dx}{\sqrt{l^2 - x^2}} = \frac{K}{\sqrt{2l}}. \tag{5.3}$$

Note that (5.2) and (5.3) represent finite equations, since $p(x)$ is a given function. These equations determine the position of the ends of an isolated straight-line mobile-equilibrium crack under a given load, if this load guarantees that such a crack can exist.

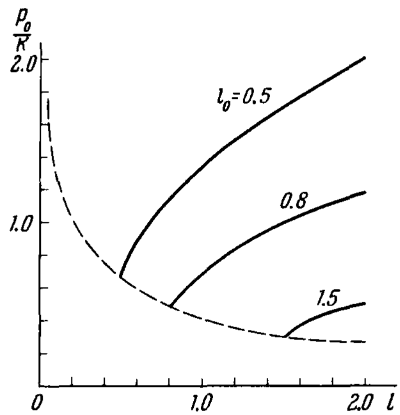


FIG. 17.

A method to calculate the strain-energy release rate $\partial W/\partial l$ for a symmetrical isolated crack was indicated by K. Masubuchi [82]. He proposed a trigonometrical representation of stresses $p(x)$ and displacements $v$ of points of the crack surface,

$$p(x) = \frac{E}{4l} \sum_{n=1}^{\infty} nA_n \frac{\sin n\theta}{\sin \theta}, \qquad v = \tfrac{1}{2} \sum_{n=1}^{\infty} A_n \sin n\theta, \qquad x = l \cos \theta. \tag{5.4}$$

As was shown by Masubuchi,

$$\frac{\partial W}{\partial l} = \frac{\pi E}{8(1 - \nu^2)l} \sum_{n=1}^{\infty} (nA_n)^2. \tag{5.5}$$

Equating this expression to $4T$, where $T$ is the surface energy density, a relation between the applied stresses and the crack size can be obtained, though in a form far more complicated than (5.3).

Let us now look at a few examples. A crack may be kept open by a uniform tensile stress $p_0$ applied at infinity. As already pointed out, this problem was first treated by Griffith [3, 4]. In this case $p(x) \equiv p_0$ and equation (5.3) yields

$$l = \frac{2K^2}{\pi^2 p_0^2}. \tag{5.6}$$

Relation (5.6) appears in Fig. 17 as the dotted line. One sees that the size of a mobile-equilibrium crack diminishes with increasing tensile stress, which is indicative of the instability of mobile equilibrium in this case. Despite this instability the size $l$ defined by (5.6) has a physical meaning: If there is a crack of length $2l_0$ in a body, to which constant tensile stress $p_0$ is applied at infinity, then at $l_0 < l$ this crack does not expand (and closes if it is a reversible crack) while at $l_0 > l$ it grows indefinitely. Thus, the equilibrium size is in a certain sense critical (this will be discussed in more detail in Section V,3). It is obvious that instability of mobile equilibrium in this case fully corresponds to the substance of the matter and, contrary to the opinion expressed by Frenkel [5], is not connected with Griffith's incorrect ideas about the geometry of the crack ends.

If stresses vanish at infinity, and if a crack is maintained by a uniformly distributed pressure applied over a part of its surface ($0 \leqslant x \leqslant l_0$) while the remaining part of the crack surface ($l_0 \leqslant x \leqslant l$) is free of stress, then the half-length of the mobile-equilibrium crack $l$ is given by the relation [58]

$$\sqrt{\frac{l}{l_0}} \arcsin\left(\frac{l_0}{l}\right) = \frac{K}{p_0\sqrt{2l_0}}. \tag{5.7}$$

This relation is shown in Fig. 17 by the solid lines which may be obtained from each other by a similarity transformation. It is evident that the opening of a crack, i.e. the appearance of a free segment, is possible provided $l_0$ is not less than the corresponding size of a mobile-equilibrium crack kept open by a uniform tensile stress at infinity, $p_0$, which is determined by (5.6). Therefore all the solid lines (Fig. 17) must start from the dotted line.

A limiting case of (5.7) is of interest. It occurs when $p_0$ tends to infinity and $l_0$ tends to zero, while $2p_0l_0 \equiv \text{Const} = P$. This corresponds to a crack kept open by concentrated forces applied at opposite points of its surface. The half-length of the crack is then given by

$$l = \frac{P^2}{2K^2}. \tag{5.8}$$

Note that (5.6) and (5.8) may be obtained, disregarding the value of the numerical factor, by dimensional analysis. For example, the size of a crack maintained by concentrated forces is determined only by the magnitude $P$ of these forces and the overall characteristic of the forces of cohesion, $K$. It is obvious that the modulus of elasticity and Poisson's ratio do not enter in the number of determining parameters, since the corresponding problem of the theory of elasticity is naturally formulated only in terms of stresses. Considering the dimensions of $P$ and $K$, we see that it is possible to set up only one combination with the dimension of length from these quantities, namely the ratio $P^2/K^2$, and no dimensionless combination exists. Thus the length of a mobile-equilibrium crack must be proportional to $P^2/K^2$, and the coefficient of proportionality a universal constant [cf. 79].

Let now a crack be maintained by two equal and opposite concentrated forces $P$, whose points of application are separated by $L$ along the common line of action of the forces; the crack is supposed to be perpendicular to the line of action of the forces and located symmetrically [58].

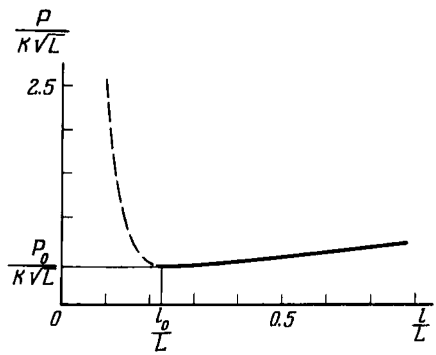


FIG. 18.

The distribution of tensile stresses at the place of the crack in a continuous body is in this case given by

$$p(x) = \frac{PL}{2\pi(x^2+L^2)}\left[1-\nu+2(1+\nu)\frac{L^2}{x^2+L^2}\right] \tag{5.9}$$

(the origin of coordinates is taken in the middle of the crack). Using (5.3), we obtain the relation determining the crack size in the form

$$\frac{P}{K\sqrt{L}} = \left(1+\frac{L^2}{l^2}\right)^{3/2} \frac{\sqrt{2}}{[2+(3+\nu)L^2/l^2]\sqrt{L/l}}\,. \tag{5.10}$$

A plot of $P/K\sqrt{L}$ versus the relative length of the crack $l/L$ for $\nu = 0.25$ is shown in Fig. 18. As can be seen, at $P > P_0$ two lengths of a mobile-equilibrium crack correspond to each value of $P$, the smaller decreasing and the greater increasing with increasing $P$. States of mobile equilibrium corresponding to the smaller equilibrium length are unstable; the corresponding branch of the load-length diagram in Fig. 18 is shown by the dotted line. States corresponding to the greater length are stable (solid line in Fig. 18).

The smaller size $l_1$ is the critical size at a given load $P$; initial cracks present in the body and smaller than $2l_1$ do not expand under the action of applied loads of magnitude $P$ (in case of reversible cracks they close), and those which are greater expand until the crack reaches the second (stable) equilibrium size.* At $P < P_0$ equation (5.10) has no solution. This means that, whatever length of the initial crack we take, it will not develop into a mobile-equilibrium crack at the given load. The size of a mobile-equilibrium crack $l_0$ different from zero corresponds to the critical value of forces $P_0$.

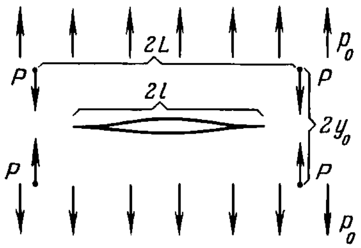


FIG. 19.

An interesting problem concerning the influence of riveted stiffeners on crack propagation was treated by J. P. Romualdi and P. H. Sanders [52]. This problem is schematized by the authors as follows (Fig. 19). An infinite plate is stretched by a uniform stress $p_0$ in the direction perpendicular to a crack. The action of the rivets and the stiffeners is represented by two symmetrically located pairs of opposite concentrated forces equal in magnitude to $P$; they are considered as given.

Substituting the corresponding stress distribution in (5.3) and working out† the elementary though somewhat cumbersome integrals, we obtain the relation between the applied load and the half-length of an equilibrium crack $l$:

$$\frac{p_0\sqrt{L}}{K} = \frac{\sqrt{2}}{\pi}\frac{P}{K\sqrt{L}}\bar{y}_0\left[\frac{1-\nu}{A\sqrt{A-B+2}} + \frac{12(1+\nu)\bar{y}_0^2}{A^2(A+B-2)\sqrt{A-B+2}} + \right.$$

$$\left. \frac{2(1+\nu)(2B-A-4)}{A^2\sqrt{A-B+2}} + \bar{y}_0^2\frac{(1+\nu)(B+A)(2B-A-4)}{A^3(A+B-2)\sqrt{A-B+2}}\right] + \frac{\sqrt{2}}{\pi\sqrt{l}}; \tag{5.11}$$

$$\bar{y}_0 = \frac{y}{L}, \quad l = \frac{l}{L}, \quad B = \bar{y}_0^2 + l^2 + 1, \quad A = \sqrt{B^2 - 4l^2}.$$

* Note that, because of dynamic effects accompanying the expanding of the initial cut, the crack actually may "overshoot" the stable equilibrium state to some extent. This will be discussed later in more detail.

† Computation of the integrals and numerical calculations for the graphs in Fig. 20 were made by V. Z. Parton and E. A. Morozova.

The results of the calculations are plotted in Fig. 20 for $\nu = 0.25$, $P/K\sqrt{L} = 0.2$ and for several values of the parameter $y_0/L$. As is seen, mobile-equilibrium cracks are unstable in the absence of stiffeners. The influence of the stiffeners shows itself first of all in an increase of the size of a mobile-equilibrium crack at a given load and, as an especially important feature, in the appearance of stable states of mobile equilibrium at sufficiently small $y_0/L$, i.e. when rivets are spaced closely enough. The appearance of stable states of mobile equilibrium changes considerably the character of the crack expansion (see details below).

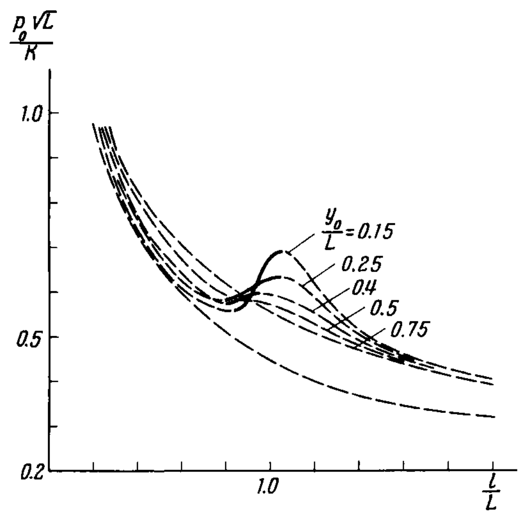


FIG. 20.

The authors observed experimentally the transition of cracks from unstable mobile-equilibrium states to stable ones; their experiments, carried out with aluminium alloy plates in the presence and absence of stiffeners, reveal a considerable increase in size of mobile-equilibrium cracks in the presence of stiffeners at the same value of $p_0$. In [52] the stress intensity factor at the crack ends was also determined experimentally for several stable and unstable mobile-equilibrium states. In the absence of stiffeners, measurements of the stress intensity factor were made by the direct method, i.e. by diminution of tensile stresses near the crack ends (at distances obviously large compared to the size of the crack-edge region). In the presence of stiffeners the stress intensity factors were measured indirectly. The values of these factors were found to coincide except in two cases when they were smaller by approximately 15 per cent. However, these two tests carried out with one and the same specimen, with a stable crack in one case and unstable crack in the other, gave values of the stress intensity factor close to each other. (A somewhat lower value of this factor at the end of the stable crack can be explained by the considerable dynamic effects which, according to the authors, occur in the transition from the unstable state to the stable one.) Thus it may be supposed that the deviation observed is due to some peculiarity of the specimen. Altogether, these experiments confirm directly the proposed general scheme.

This discussion can be readily extended to straight cracks in an anisotropic medium, placed in the planes of elastic symmetry of the material. The problem of a straight crack in an orthotropic infinite body subjected to a uniform stress field was treated by T. J. Willmore [21] and A. N. Stroh [83]. In [83], the results of [16] were also extended to cover the case of a straight crack in an anisotropic body under an arbitrary stress field, and the stress intensity factors at crack ends were found for this problem. Paper [84] brings the solution of the general problem concerning a straight mobile-equilibrium crack in an orthotropic body subjected to an arbitrary stress field symmetrical with respect to the line of the crack.

## 2. *Plane Axisymmetrical Cracks*

If a disk-shaped mobile-equilibrium crack of radius $R$ is maintained in an infinite body by an axisymmetrical load, tensile stresses near the crack contour calculated without taking into account forces of cohesion tend to infinity according to the law

$$\sigma_y = \frac{N}{\sqrt{s}}, \qquad N = \frac{1}{\pi\sqrt{R/2}} \int_0^R \frac{r p(r) dr}{\sqrt{R^2 - r^2}}, \tag{5.12}$$

where $p(r)$ is the tensile-stress distribution at the place of the crack in a continuous body subjected to the same loads. According to the general condition (4.7), the equation determining the radius of a mobile-equilibrium crack $R$ is

$$\int_0^R \frac{r p(r) dr}{\sqrt{R^2 - r^2}} = K \sqrt{\frac{R}{2}}. \tag{5.13}$$

This equation was established in [56, 57]. Its derivation is based on the application of the method of Fourier-Hankel transforms, developed by I. N. Sneddon [14, 15] for solving axisymmetrical problems of elasticity. In particular, if a mobile-equilibrium crack is kept open by a uniform tensile stress at infinity $p_0$, then $p(r) \equiv p_0$ and the radius of an equilibrium crack is given by

$$R = \frac{K^2}{2p_0^2}. \tag{5.14}$$

This problem was first solved by R. A. Sack [20] by the energy method; his method is quite similar in principle to Griffith's [3, 4] treatment of the corresponding plane problem.

If there is no tensile load at infinity, and if the crack is kept open by a uniformly distributed pressure $p_0$ over a part of its surface $(0 \leqslant r \leqslant r_0)$ while the remaining part of the crack surface $(r_0 \leqslant r \leqslant R)$ is free, then the radius of the mobile-equilibrium crack is found from the relation

$$\frac{p_0 \sqrt{r_0}}{K} = \frac{1}{\sqrt{2}} \left(\frac{r_0}{R}\right)^{-3/2} \left[1 + \sqrt{1 - \left(\frac{r_0}{R}\right)^2}\right]. \tag{5.15}$$

Here, just as in the plane case, the radius of the loaded part of the crack surface $r_0$ must not be less than the critical radius for a given pressure $p_0$, which is defined by (5.14). In particular, if a disk-shaped crack is maintained open by equal and opposite concentrated forces $P$ applied at its surface, then the radius of a mobile-equilibrium crack is determined by the formula

$$R = \left(\frac{P}{\sqrt{2}\,\pi K}\right)^{2/3}. \tag{5.16}$$

Relations (5.14) and (5.16) can be obtained, except for the numerical factor, from dimensional analysis (cf. (5.6) and (5.8)).

If a disk-shaped crack is kept open by equal and opposite forces $P$ whose points of application are $2L$ apart along the common line of action, then the radius of a mobile-equilibrium crack $R$ is determined from the equation

$$\frac{P}{KL^{3/2}} = \pi \sqrt{2} \left(\frac{L}{R}\right)^{-3/2} \left(1 + \frac{L^2}{R^2}\right)^2 \left(1 + \frac{2-\nu}{1-\nu} \frac{L^2}{R^2}\right)^{-1} \tag{5.17}$$

The above solutions were obtained in [56, 57]; the interpretation of the relations obtained is quite similar to the corresponding cases for a straight crack.

### 3. *The Extension of Isolated Cracks Under Proportional Loading; Stability of Isolated Cracks*

The problem of this section is a special case of problem A. A complete investigation is carried out for symmetrical loading and initial cracks, straight and disk-shaped cracks being considered simultaneously. An example of a problem concerning the growth of an unsymmetrical initial crack is given, which illustrates the general procedure of solving this problem. Under proportional loading the tensile stresses at the place of the crack, but in a *continuous* body subjected to the same load, are proportional to the loading parameter $\lambda$; hence $p(x) = \lambda f(x)$ and $p(r) = \lambda f(r)$ in the cases of straight and disk-shaped cracks, respectively. Introducing the dimensionless variable $\xi$ equal to $x/l$ and $r/R$ in these cases, respectively, one obtains relations (5.3) and (5.12) in the form

$$\frac{\sqrt{2}\,\lambda}{K} = \varphi(c), \tag{5.18}$$

where $\varphi(c)$ is defined respectively by

$$\varphi(c) = \left[\sqrt{c}\int_0^1 \frac{f(c\xi)d\xi}{\sqrt{1-\xi^2}}\right]^{-1}, \quad \varphi(c) = \left[\sqrt{c}\int_0^1 \frac{f(c\xi)\xi d\xi}{\sqrt{1-\xi^2}}\right]^{-1} \tag{5.19}$$

and $c$ denotes, respectively the half-length $l/2$ or the radius $R$.

Thus the relation of the crack length to the parameter $\lambda$ of proportional loading is completely determined by the length of the initial crack and by the function $\varphi(c)$, corresponding to a given load distribution. Certain properties of the function $\varphi(c)$ can be obtained under the most general assumptions. Omitting the case of a crack maintained by concentrated forces applied at its surface, let us suppose that the crack is kept open by any loads, in particular, by concentrated loads applied inside the body and perhaps by distributed loads applied at the crack surface. In this case the functions $p(x)$, $p(r)$, and, consequently, $f(c\xi)$ are obviously bounded. For small $c$ we obtain from (5.19), respectively:

$$\varphi(c) = \frac{2}{\pi f(0)\sqrt{c}} + \dots, \qquad \varphi(c) = \frac{1}{f(0)\sqrt{c}} + \dots . \tag{5.20}$$

Suppose that the tensile loads applied to the body on each side of the crack are bounded and, for definiteness, equal to $\lambda P$. Then the following relations are valid:

$$\begin{aligned} &\int_{-\infty}^{\infty} p(x)dx = \lambda P, &&\int_0^{\infty} f(c\xi)d\xi = \frac{P}{2c}, \\ &\int_0^{\infty} p(r)rdr = \frac{\lambda P}{2\pi} &&\int_0^{\infty} f(c\xi)\xi d\xi = \frac{P}{2\pi c^2}. \end{aligned} \tag{5.21}$$

Eqs. (5.21) and (5.19) yield asymptotic representations for the functions $\varphi(c)$ when $c \to \infty$:

$$\varphi(c) = \frac{2\sqrt{c}}{P} + \dots, \qquad \varphi(c) = \frac{2\pi c^{3/2}}{P} + \dots. \tag{5.22}$$

Thus, under the assumptions made, $\varphi(c)$ tends to infinity when $c \to 0$ and $c \to \infty$. Owing to the boundedness of $f(c\xi)$, the integrals in expressions (5.19) do not become infinite at any $c$, therefore $\varphi(c)$ vanishes nowhere and, consequently, has at least one positive minimum, one falling branch, and one rising branch. If the forces applied to the body on either side of the crack are not bounded, then the function $\varphi(c)$ may not have rising branches and, consequently, minima. This happens in particular in case of a uniform tensile stress field when $p = \lambda p_0$ and

$$\varphi(c) = \frac{2}{\pi p_0 \sqrt{c}}, \qquad \varphi(c) = \frac{1}{p_0 \sqrt{c}} \tag{5.23}$$

for a straight and axisymmetrical crack, respectively.

By definition, an equilibrium crack is stable if no (sufficiently small) change in its contour produces forces which tend to move the crack further away from the disturbed state of equilibrium. It is evident that immobile-equilibrium cracks are always stable. For stability of a mobile-equilibrium crack it is necessary that its size should grow with an increase of the loading parameter $\lambda$. Suppose indeed that the corresponding size of a mobile-equilibrium crack $c$ grows with increasing load. If the crack size is diminished without changing the load ($\lambda = \text{const}$), the crack extension force will be greater than it was in equilibrium. Therefore the equilibrium is disturbed, and the crack tends to widen under the action of the excess force. Conversely, if the crack size is slightly increased compared with its equilibrium size, then the equilibrium is disturbed in the opposite direction, and the crack tends to close, if it is reversible.* If near a given equilibrium state the equilibrium crack size $c$ diminishes with an increase of $\lambda$, then it is obvious that its small change under a constant load will produce forces favouring further departure from the equilibrium state. The corresponding equilibrium state will be unstable. Hence the equilibrium state of a crack is stable, if for given $c$ and $\lambda$ the following condition is satisfied:

$$\frac{dc}{d\lambda} > 0. \tag{5.24}$$

* If the crack is irreversible, then with an increase in its size no reverse closing takes place, but no further expansion of the crack takes place either. Equilibrium is attained in this case because of diminution of forces of cohesion acting in the edge region of the crack.

Differentiating (5.18) with respect to $\lambda$, we find

$$\frac{dc}{d\lambda} = \frac{\sqrt{2}}{K\varphi'(c)}. \tag{5.25}$$

Thus the condition for stability of the state of mobile equilibrium is

$$\varphi'(c) > 0, \tag{5.26}$$

and only those states of mobile equilibrium are stable which correspond to rising portions of the curve $\varphi(c)$.

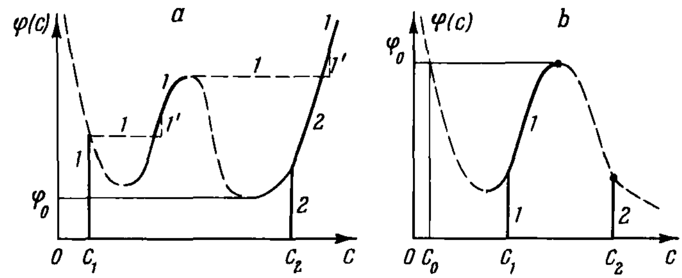


FIG. 21.

Now we have everything that is necessary for the complete investigation of the extension of an isolated symmetrical crack under proportional loading. Let a function $\varphi(c)$, such as shown in the graph of Fig. 21, correspond to a given system of loads applied to the body and consider first the case when $\varphi(c) \to \infty$ as $c \to \infty$ (Fig. 21a). Such a case occurs in particular when the loads applied on both sides of the crack are bounded. Let the dimension of the initial crack $2c_1$ correspond to an unstable brach of $\varphi(c)$. Then the crack length remains constant with increase of $\lambda$, until $\lambda$ reaches the magnitude, for which the initial crack of size $2c_1$ becomes one of mobile equilibrium. Since the mobile equilibrium is unstable, the crack begins to expand under constant load, until it reaches the nearest stable mobile-equilibrium state. With further increase of $\lambda$ the crack size grows continuously, until the load corresponding to a maximum of $\varphi(c)$ is reached, then changes again in a stepwise manner when the transition to another stable branch takes place, after which it grows continuously with increasing $\lambda$. The path of the point representing the change of the crack is indicated by the number 1 in Fig. 21a.* Let now the size of an initial crack $2c_2$ correspond to a stable

* Owing to dynamic effects that occur in this transition, the crack may overexpand a little beyond the size of the stable mobile-equilibrium crack corresponding to the given load (apparently that happened in the experiments described in paper [52]). In this case, a further increase in the load leaves the length unchanged up to reaching mobile equilibrium, after which the crack starts to lengthen further. Naturally, the purely static theory considered here cannot describe these dynamic effects; the corresponding parts of the graph in Fig. 21a are dotted and designated by the number 1'.

branch of $\varphi(c)$. The crack size now remains unchanged up to the load at which it reaches mobile equilibrium, after which it increases continuously. The path of the representative point is indicated by the number 2 in Fig. 21a. In the case considered, no fracture of the body occurs for any values of the parameter $\lambda$. If $\lambda$ is less than its critical value (corresponding to the lowest of the minima of $\varphi(c)$), then great as the size of the initial crack may be, it does not expand under a given load. The size of the mobile-equilibrium crack corresponding to this critical value of $\lambda$ is finite.

This means in particular: if a crack is kept open by forces applied inside the body and perhaps by loads distributed over the crack surface, and if the forces applied on each side of the crack are bounded, then there exists a critical value of the parameter $\lambda$; for all values of $\lambda$ greater than the critical one there exists at least one stable and one unstable state of mobile equilibrium.

Let us now turn to the case when $\varphi(c) \to 0$ as $c \to \infty$ (Fig. 21b). If the size of an initial crack $2c_1$ corresponds to a stable branch of $\varphi(c)$, then the crack does not expand until a load is reached at which its state becomes a mobile equilibrium. After that, the crack grows continuously with increasing $\lambda$, until a value of $\lambda$ is reached that corresponds to a maximum. If this $\lambda$-value is exceeded, the solution of the problem does not exist any longer, and fracture of the body occurs. The path of the representative point is indicated by the number 1 in Fig. 21b. If the size of an initial crack $2c_2$ corresponds to the right-hand unstable branch of $\varphi(c)$, then no expansion of the initial crack occurs with increasing $\lambda$, until a value of $\lambda$ is reached for which the state of the initial crack becomes a mobile equilibrium. The slightest exceeding of this value of $\lambda$ causes complete fracture of the body. If the size of an initial crack $2c_3$ corresponds to the left-hand unstable branch of the curve $\varphi(c)$, then for $c_3 < c_0$ the crack develops in the same manner as in case 2; for $c_3 > c_0$ the development of the crack is similar to case 1 in Fig. 21a before reaching a maximum, after which the body fractures.

The investigation of other forms of the curve $\varphi(c)$ can easily be carried out by combining the cases considered. We see that the knowledge of the function $\varphi(c)$ makes it possible to describe completely the behavior of a symmetrical isolated crack in an infinite body under proportional loading. In the case of reversible cracks, a change in the crack size can be traced by means of the graph of $\varphi(c)$ also for a non-monotonous variation in the load. It is of interest to note that in this case a decrease in the load produces a stepwise diminution of the crack size, but this happens, in general, when critical equilibrium states are passed that are different from those corresponding to an increase in the load.

Recently, L. M. Kachanov [84a] carried out an investigation generalizing the previous treatments so as to cover the case of the time-dependent modulus of cohesion. This investigation is of basic importance in connection with the problems of so-called "stress rupture".

The analysis carried out in the present section is based on [59].

Consider now the solution of a problem concerning the extension of an unsymmetrical initial crack in one simple case. Let a straight initial crack with the end coordinates $x = -a_0$ and $x = b_0$ be given in an infinite unloaded body (for definiteness assume $b_0 < a_0$) and let equal and opposite concentrated forces $P$ be applied at opposite points of the crack surfaces, say, at $x = 0$. The magnitude of the force $P$ plays the role of the loading parameter. According to (5.1), the values of the tensile-stress intensity factors $N_0$ at $x = -a$ and $x = b$ are, respectively,

$$N_a = \frac{P}{\pi\sqrt{b+a}}\sqrt{\frac{b}{a}}, \qquad N_b = \frac{P}{\pi\sqrt{b+a}}\sqrt{\frac{a}{b}}. \tag{5.27}$$

When $P < P_1$, where

$$\frac{P_1^2}{K^2} = \frac{(b_0 + a_0)b_0}{a_0}, \tag{5.28}$$

both factors $N_a$ and $N_b$ are less than $K/\pi$ so that the crack expands neither to the right nor to the left. At $P = P_1$ the factor $N_b$ becomes equal to $K/\pi$,

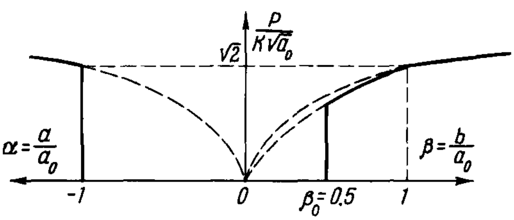


FIG. 22.

mobile equilibrium is reached and the end $b$ begins to move to the right. The advance depends on the magnitude of the applied force according to the relation

$$\frac{P^2}{K^2} = \frac{b(a_0 + b)}{a_0}. \tag{5.29}$$

As long as $P < P_2$, where

$$\frac{P_2^2}{K^2} = 2a_0, \tag{5.30}$$

we have $N_a < K/\pi$, and the left end does not move. At $P = P_2$, we have $b = -a_0$, a symmetrical crack in mobile equilibrium, and at $P > P_2$ the development of the crack continues according to (5.8). The development of the initial crack with changing $P$ is plotted in Fig. 22.

### 4. *Cracks Extending to the Surface of the Body*

If a crack extends to the surface of the body, it becomes difficult to obtain effective analytical solutions. Mapping of the corresponding region on a half-plane cannot be carried out by means of rational functions, and Muskhelishvili's method does not make it possible to obtain solution in finite form. Therefore it is necessary to resort to numerical methods in analysing such problems.

A number of numerical solutions have been derived up to now; the mobile-equilibrium states are unstable in all analysed cases.

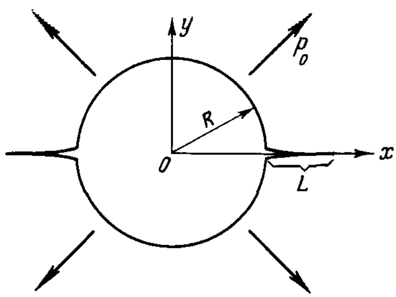


FIG. 23.

O. L. Bowie [22] treated the problem of a system of $k$ symmetrically located cracks of equal length extending to the free surface of a circular cut in an infinite body (Fig. 23). The body is stretched at infinity by the all-round stress $p_0$. Bowie employed Muskhelishvili's method for calculating stresses and strains. To obtain the solution in effective form, the author used a polynomial approximation to the analytical function mapping the exterior of the circle with adjacent cuts on the exterior of the unit circle. For the determination of the dimensions of mobile-equilibrium cracks Bowie used directly Griffith's energy method and computed the strain-energy release rate. Numerical calculations were made for cases of one crack and two diametrically opposite cracks. To obtain sufficient accuracy of calculations it proved necessary to retain about thirty terms in the polynomial representation of the mapping function. The numerical results for the cases $k = 1$ and $k = 2$ obtained by Bowie are shown in Fig. 24. It follows from these computations that at $L/R > 1$ the tensile stress for two cracks with a circular cavity is very close to the tensile stress for one crack of length

$2(L + R)$, so that the influence of the cavity proper is almost unnoticeable. Furthermore, in the case of small crack lengths the conditions of mobile equilibrium are obviously determined by the tensile stresses directly at the

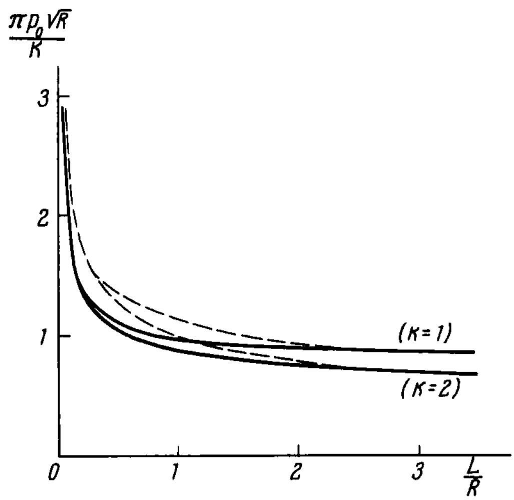


FIG. 24. ———uniaxial tension, - - - all-round tension.

surface of the circular cavity. As is known, in case of uniaxial extension the highest tensile stress at the boundary of the cavity is equal to $3p_0$ and in case of all-round extension $2p_0$. Thus the ratio of equilibrium loads in these cases should approach 2/3, and this is found in agreement with Bowie's calculations.

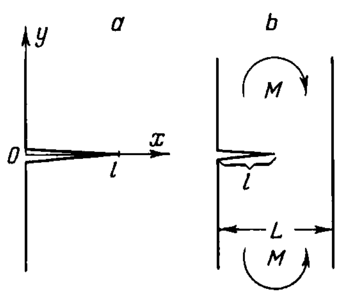


FIG. 25.

The problem of a straight crack ending on a straight free boundary of the half-space (Fig. 25) was treated independently by L. A. Wigglesworth [85] and G. R. Irwin [51] using different methods.

Wigglesworth [85] investigated the case of an arbitrary distribution of normal and shearing stresses over the faces of the crack. For a symmetrical distribution of stresses he reduced the problem to an integral equation for the complex displacement $w(x) = u(x) + iv(x)$ of points of the crack surface:

$$\int_0^l L(x,t)w(t)dt = -\frac{4(1-\nu^2)}{E}\int_0^x p(x)dx. \tag{5.31}$$

Here $L(x,t)$ is a singular integral operator and $p(x) = \sigma(x) + i\tau(x)$; $\sigma(x)$ is the distribution of normal stresses; $\tau(x)$ is the distribution of shearing stresses. Equation (5.31) is solved in the paper by an integral-transform method. Detailed calculations are made for the case when the surfaces of the crack and boundary are free of stresses, the tensile stress $p_0$ being applied at infinity parallel to the boundary of the half-space.

For stresses near the crack end the author obtains in this special case the following relations:

$$\begin{aligned} \sigma_x + \sigma_y &= 1.586 \sqrt{\frac{l}{s}}\, p_0 \sin\frac{\phi}{q}, \\ \sigma_x - \sigma_y + 2i\sigma_{xy} &= -\,0.793 \sqrt{\frac{l}{s}}\, p_0 \sin\phi \exp\left(\frac{3i\phi}{q}\right), \end{aligned} \tag{5.32}$$

hence we find at the prolongation of the crack ($\Phi = \pi$)

$$\sigma_x = \sigma_y = 0.793\, p_0 \sqrt{\frac{l}{s}}, \qquad \sigma_{xy} = 0, \tag{5.33}$$

which together with (4.6) gives the expression for the length of the mobile-equilibrium crack in the form

$$l = \frac{K^2}{\pi^2 (0.793)^2 p_0{}^2} \approx 1.61 \frac{K^2}{p_0{}^2}. \tag{5.34}$$

Irwin [51] investigated only the last special case. He represented the unknown solution as the sum of three fields. The first field corresponds to a crack ($-\,l \leqslant x \leqslant l,\ y = 0$) in an infinite body subjected to constant tensile stress $p_0$ at infinity, the second field corresponds to the same crack under normal stresses $Q(x)$ symmetrical with respect to the $x$ and $y$ axes and applied at the crack surface, the third field corresponds to a half-space $x \geqslant 0$ without crack, at the boundary of which ($x = 0$) the distribution of normal stresses $P(y)$, symmetrical with respect to the $x$ axis, is given. Satisfying the boundary conditions at the free boundary and the crack surface, Irwin obtained for $P(y)$ and $Q(x)$ the system of integral equations

$$\begin{aligned} &\int_0^l Q(x)\, \frac{2\sqrt{l^2 - x^2}\, y}{\pi(y^2 + x^2)\sqrt{y^2 + l^2}} \left[\frac{2y^2}{y^2 + x^2} + \frac{y^2}{y^2 + l^2} - 2\right] dx + \\ &\qquad + p_0 \left[\frac{2y}{\sqrt{y^2 + l^2}} - \frac{y^3}{(y^2 + l^2)^{3/2}} - 1\right] = P(y), \\ &\qquad -\,4 \int_0^\infty P(y)\, \frac{x y^2}{\pi (x^2 + y^2)^2}\, dy = Q(x), \end{aligned} \tag{5.35}$$

which he solved by the method of successive approximations. The first approximation yields a relation for the length of the mobile-equilibrium crack $l$:

$$l = \frac{2K^2}{\pi^2\, 1.095^2 p_0{}^2} = 1.69\, \frac{K^2}{p_0{}^2}, \tag{5.36}$$

which differs, as is seen, insignificantly from the more exact relation (5.34).

H. F. Bueckner [50] treated a problem of one straight crack reaching the boundary of a circular cavity in an infinite body. No stress is applied at infinity and at the boundary of the cavity, the surface of the crack is free of shearing stresses, normal stresses are applied symmetrically and vary according to a given law $-p(x)$. Such a form of the problem arises in the analysis of rupture of rotating disks. Like Wigglesworth [85], Bueckner proceeds independently from a singular integral equation for the lateral displacements of points of the crack surface. He considers a one-parameter family of particular solutions of this equation, corresponding to certain special distributions $p_n(x)$. In the general case it is recommended to represent $p(x)$ as a linear combination of $p_n(x)$:

$$p(x) = \sum_{n=0}^{n=m} \alpha_n p_n(x); \tag{5.37}$$

the coefficients $\alpha_n$ are determined by the least-square method or by collocation. The factor of stress intensity at the crack end $N_0$ is expressed in terms of the coefficients $\alpha_n$.

If the length of the crack is far less than the radius of the circular cavity, then we have in the limit the previous particular case of a straight boundary. As it follows from Bueckner's calculations in this particular case when $p = p_0 = \text{Const}$, the expression for the length of a mobile-equilibrium crack is

$$l = \frac{2K^2}{\pi^2\, 1.13^2 p_0{}^2} = 0.159\, \frac{p_0{}^2}{K^2}, \tag{5.38}$$

which is in good agreement with (5.34) and (5.36).

In [50] Bueckner also treated a problem of a crack reaching the surface of an infinitely long strip of finite width under an arbitrary load, symmetrical with respect to the line of the crack (Fig. 25b). He showed that it is possible to replace with a high degree of accuracy the integral equation occurring in this case by one with a degenerated kernel. The numerical solution obtained by Bueckner in the special case when the load is produced by couples $M$, applied on both sides of the crack at infinity, gives the relation between the length of a mobile-equilibrium crack and the load; it is represented by the curve in Fig. 26.

As has already been pointed out, in all cases discussed in the present section mobile-equilibrium cracks are unstable. Thus, when loads increase, extension of an initial crack does not take place until it reaches mobile equilibrium, after which the body fractures. In these problems the load at which an initial crack reaches mobile equilibrium coincides with the breaking load, which is in general not true.

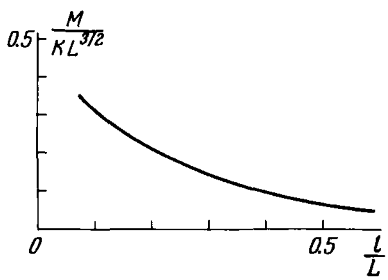


FIG. 26.

In the paper by D. H. Winne and B. M. Wundt [32] some of the solutions presented in this section were employed for the analysis of fracture of rotating notched disks, and of notched beams in bending. The experiments conducted by Winne and Wundt, analysed on the basis of these calculations, revealed close coincidence of the values of surface-energy density $T$ (or, which amounts to the same, of the moduli of cohesion $K$) determined from the angular speed, at which fracture of rotating notched disks occurs, and from the loads at which fracture of notched beams in bending occurs. This confirms that the quantities $T$ and $K$ are characteristics of the material and do not depend on the nature of the state of stress.

## 5. *Cracks near Boundaries of a Body; Systems of Cracks*

Crack development in bounded bodies possesses some characteristic peculiarities. Difficulties of a mathematical character do not allow us to

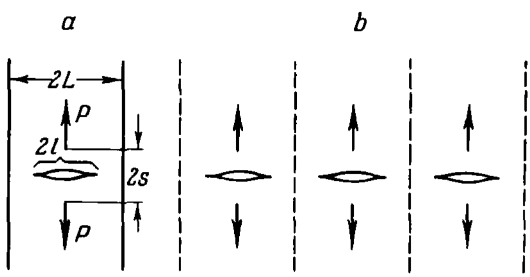


FIG. 27.

carry out here as complete an investigation as in the case of isolated cracks. However, the qualitative features and some of the quantitative characteristics of this phenomenon can readily be elucidated in connection with the simplest problems that yield to analytical solution. Let us examine first of all the problem of a straight crack in a strip of finite width (Fig. 27a). The crack is assumed to be symmetrical with respect to the middle line of the strip, and the direction of its propagation is normal to the free boundary. The load keeping the crack open is considered symmetrical with respect to the line of the crack and the middle line of the strip.

In solving the problem we use the method of successive approximations developed by D. I. Sherman [86] and S. G. Mikhlin [87]. As the first approximation we take the solution of a problem in the theory of elasticity for the exterior of a periodical system of cuts (Fig. 27b). Denoting again by $p(t)$ the distribution of tensile stresses, which would be at the place of the cracks in a continuous body under the same loads, we obtain the equation determining the half-length of a mobile-equilibrium crack $l$ in the form

$$\int_{-m}^{m} p[t_0(t)] \sqrt{\frac{m+t}{m-t}}\,dt = K\sqrt{\frac{\pi m}{2L}}, \tag{5.39}$$

where $t = \sin(\pi t_0/2L)$, $m = \sin(\pi l/2L)$. In the particular case represented in Fig. 27, when the crack is maintained by equal and opposite concentrated forces $P$ with points of application $2s$ apart along their common line of action, (5.39) becomes

$$\frac{P}{K\sqrt{L}} = \frac{\sqrt{8(\alpha^2+1)\sin(\pi l/L)}}{\pi\cosh\sigma\left[1-\nu+(1+\nu)\dfrac{\sigma(2\alpha^2+1)\cosh\sigma}{\alpha(\alpha^2+1)m,}\right]}, \tag{5.40}$$

where $\alpha = \sinh\sigma/m$, $\sigma = \pi s/2L$. When $s = 0$ (concentrated forces applied at the crack surface), (5.40) reduces to

$$\frac{P}{K\sqrt{L}} = \sqrt{\frac{2}{\pi}\sin\frac{\pi l}{L}}. \tag{5.41}$$

Let us also quote the relation between the size of a mobile-equilibrium crack and the load for the case of a uniform tensile stress at infinity, $P/2L$,

$$\frac{P}{K\sqrt{L}} = \sqrt{\frac{2}{\pi}\cot\frac{\pi l}{2L}}. \tag{5.42}$$

Relation (5.40) for various $\sigma$ is presented in Fig. 28. The solid and dotted lines denote, as usual, stable and unstable branches. As is seen, for $\sigma \geqslant \sigma_c \approx 0.5$ there are no stable branches, hence for distances between points of application of forces exceeding $2L/\pi \approx 0.64\,L$ mobile-equilibrium cracks are always unstable. Quite similarly to the analysis in Section V.3 (extension of an isolated crack under proportional loading) the graph in Fig. 28 makes it possible to describe completely the extension of any symmetrical initial crack when the load increases.

The present analysis is based on papers [58, 88]. The solution of the corresponding problem in the theory of elasticity for the case $s = 0$ was obtained by Irwin [45]. The problem of a periodical system of cracks under uniform loading at infinity was solved by Westergaard [13] and independently by W. T. Koiter [89].

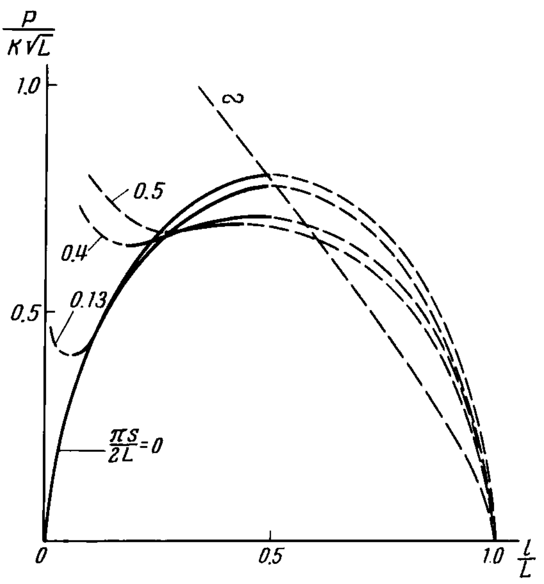


FIG. 28.

In the first approximation only the shearing stresses vanish at the lines of symmetry (shown by the dotted lines in Fig. 27b, which correspond to the boundaries of the strip); the normal stresses are different from zero. To obtain the second approximation, the first approximation is added to the solution for an uncracked strip, at the boundaries of which the normal stresses are given; their distribution is chosen in such a manner as to compensate the normal stresses at the boundary obtained in the first approximation. Now the boundary condition is no longer satisfied at the crack surface. To obtain the third approximation, the second approximation is added to the solution for the exterior of a periodical system of cuts, at the surface of which the distribution of normal stresses is equal to the difference between the given stresses and those obtained in the second approximation, and so on.

Special estimates obtained in [88] show that for stable mobile-equilibrium states the considerations of the second and subsequent approximations leads to corrections of the order of 2.5–3 per cent in the above relations. This permits us to confine ourselves to the first approximation.

In addition to these problems (the periodical system of cracks and the system of radial cracks ending in a circular cavity), several other problems of systems of cracks have been treated; they deal with straight cracks located along one straight line. Mathematical methods developed by Muskhelishvili [90, 18], D. I. Sherman [91], and Westergaard [13] permit

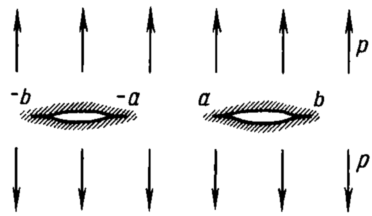


FIG. 29.

the reduction of any such problem to quadratures. Let us here consider the simplest example: it is the problem of the extension of two collinear straight cracks of the same length in an infinite body, stretched by a uniform stress $p$ at infinity (Fig. 29). This problem was treated by Willmore [21]; it also occurs in a paper by Winne and Wundt [32] (the authors refer to a private communication by Irwin). According to the solution presented in [21], the sizes of the cracks remain unchanged at $p < p_1$, where

$$p_1 = \sqrt{\frac{2}{b}}\,\frac{K}{\pi}\left\{\frac{K'(\alpha)\sqrt{\alpha(1-\alpha^2)}}{E'(\alpha) - \alpha^2 K'(\alpha)}\right\}, \qquad \alpha = \frac{a}{b} < 1. \tag{5.43}$$

Here $K'$, $E'$ are standard notations of elliptic integrals.

At $p = p_1$ the cracks attain an unstable state of mobile equilibrium, after which the inside edges of the cracks join and form a crack of length $2b$. The further extension of the crack depends on whether the bracketed expression in (5.43) is greater or less than unity. If it is less than unity, which happens for $\alpha < 0.027$, the size of the crack resulting from the joining of the inside edges is less than the size of the mobile-equilibrium crack corresponding to the load $p_1$. In this case the crack remains unchanged up to the load $p_2 = \sqrt{2}\,K/\pi\sqrt{b}$, after which the body fractures. If it is greater than unity, complete fracture of the body occurs immediately upon reaching the load $p_1$. Assuming $b - a = 2l$ and making $b \to \infty$ in (5.43), we obtain in the limit (5.6), as expected. The solution given in [32] leads to the same qualitative results. However, it cannot be accepted as correct because it is based on the erroneous expressions of the stress intensity factors given in [47].

The case of two identical cracks maintained open by concentrated forces applied at their surface was treated in [88]. A complete investigation of the general case of symmetrical loading for a system of two cracks can be carried out quite similarly, with expressions for the stress intensity factors at the crack ends $x = a$ and $x = b$

$$N_b = \frac{\sqrt{2}}{\pi \sqrt{b(b^2 - a^2)}} \left[ \int_a^b p(t) t \sqrt{\frac{t^2 - a^2}{b^2 - t^2}} dt + C \right]$$

$$N_a = \frac{\sqrt{2}}{\pi \sqrt{a(b^2 - a^2)}} \left[ \int_a^b p(t) t \sqrt{\frac{b^2 - t^2}{t^2 - a^2}} dt - C \right]$$

$$C = \frac{b}{K'\left(\frac{a}{b}\right)} \int_a^b \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} \int_a^b \frac{p(t_0) \sqrt{(b^2 - t_0^2)(t_0^2 - a^2)}\, dt_0}{t_0 + t}$$

(5.44)

As is seen from these examples, collinear cracks "weaken" each other and reduce their stability. Ya. B. Zeldovitch noticed that in the case

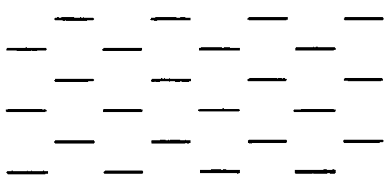

FIG. 30.

of a "chess-board" pattern of cracks (Fig. 30) the inverse phenomenon occurs. As the calculations show, even for uniform normal loads at the crack surfaces, mobile-equilibrium cracks may become stable for a certain mutual position.

We consider briefly the so-called "size effect" in the brittle fracture of bounded bodies. Take similarly shaped bodies, which differ only in the characteristic size $d$ and in the characteristic scale of the applied extensional load $S$ (it is supposed that macroscopic cracks present in the bodies are also geometrically similar). In brittle fracture, the value $S = S_0$ that corresponds to fracture depends only on the characteristic size of the body $d$ and the modulus of cohesion $K$. There is only one way to form a characteristic having the dimension $S$ from the quantities $K$ and $d$, and it is impossible to make any dimensionless combinations. Therefore the following simple relations govern the magnitude of the breaking load:

$$S_0 = \varepsilon_1 K d^{3/2}, \qquad S_0 = \varepsilon_2 K d^{1/2}, \qquad S_0 = \varepsilon_3 K d^{-1/2}, \tag{5.45}$$

where $S_0$ has the dimension of a force, of a force distributed along a line (as, for instance, a concentrated force in plane strain), and of a stress, respectively. The quantities $\varepsilon_i$ are constants for a given geometrical configuration of the body. About the fracture of geometrically similar bodies a great deal of experimental data is at present available, which permits clarification of the limits of applicability of the theory of brittle fracture. Detailed information on this topic can be found in a paper by B. M. Wundt [92], and some new results have been presented by S. Yusuff [93].

## 6. *Cracks in Rocks*

The investigation of crack extension in rock massifs is of great interest in theoretical geology. Cracks can form in rocks because of various causes of tectonic character, but also because of some artificial actions (mining excavations, hydraulic fracture of oil-bearing strata, etc.).

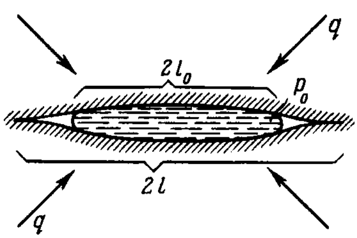


FIG. 31.

In connection with the theory of hydraulic fracture of an oil-bearing stratum a number of problems of the theory of cracks have been treated, among them *the problem of the vertical crack:* A crack in an infinite space subjected to all-round compressive pressure $q$ at infinity is maintained open by a flowing viscous fluid injected into it (Fig. 31). The main peculiarity of the problem is that the fluid does not fill the crack completely: there is always a free part of the crack on both sides of the wetted area. The pressure $p_0$ in the flowing fluid troughout the wetted area of the crack can be considered constant in first approximation. Indeed, at the end of the wetted area an abrupt narrowing of the crack takes place, and almost all of the pressure drop will occur there. The problem is called so, because the actual fissure, idealised by this problem, is located in a vertical plane, and $q$ represents the lateral pressure of the rocks. In comparison with the action of lateral rock and fluid pressures the action of the forces of cohesion may be neglected, as estimates show.* Condition (5.3) determining crack sizes becomes

$$\int_0^l \frac{p(x)dx}{\sqrt{l^2 - x^2}} = 0, \qquad p(x) = \begin{cases} p_0 - q, & 0 \leqslant x \leqslant l_0; \\ -q, & l_0 < x \leqslant l; \end{cases} \tag{5.46}$$

hence

$$l = l_0 \left[\sin \frac{\pi q}{2p_0}\right]^{-1}. \tag{5.47}$$

The expression for the maximum half-opening of the crack $v_0$ is

$$v_0 = \frac{8(1 - \nu^2)p_0 l_0}{\pi E} \ln \cot \frac{\pi q}{4p_0}. \tag{5.48}$$

As calculations show, for values $l_0/l$ close to unity which are usually encountered in practice, the opening of the crack is almost constant all along the wetted area of the crack; the crack closes rapidly along the free part. — This problem of the vertical crack was first stated and solved in a paper by Zheltov and Khristianovitch [38].

*The problem of the horizontal crack* [40] is stated as follows. In a heavy half-space at a certain depth $H$ a horizontal disk-shaped crack is formed by injecting viscous fluid as before; the surface of the crack is again divided into a wetted part ($0 \leqslant r \leqslant R_0$) and a free part ($R_0 < r \leqslant R$), and the fluid pressure $p$ in the wetted part may again be considered as constant. Forces of cohesion, as in the preceding case, are neglected. Under the assumption that the depth of the crack position $H$ is sufficiently great, the boundary condition at the boundary of the half-space need not be taken into account. The condition of finiteness of stresses at the crack contour yields in this case

$$\frac{p - \gamma H}{p} = \sqrt{1 - \left(\frac{R_0}{R}\right)^2}. \tag{5.49}$$

where $\gamma$ is the specific weight of the rock. For the volume of the injected fluid one obtains

$$V = \frac{4(1 - \nu^2)pR^3}{E} \varphi\left(\frac{R_0}{R}\right), \quad \varphi(z) = z^3 \left[\frac{2}{3} - \frac{z}{3} - \frac{z}{3(1 + \sqrt{1 - z^2})}\right]. \tag{5.50}$$

---

* The condition that forces of cohesion be negligibly small is $K/q\sqrt{l} \ll 1$. It is in general not satisfied in laboratory scaling.

In practice, $z = R_0/R$ is close to unity so that it is possible to use the asymptotic form of (5.50)

$$V = \frac{4(1-\nu^2)pR^3}{3E} \sqrt{2(1-z)}\,[1 + \sqrt{2(1-z)} - 3(1-z)]. \tag{5.51}$$

The maximum half-opening of the crack is determined by the formula

$$v_0 = \frac{8(1-\nu^2)pR_0}{\pi E} \arccos\left(\frac{R_0}{R}\right). \tag{5.52}$$

Thus, if the depth of the crack position, the fluid pressure, and the specific weight of the rock are known, $R_0/R$ can be found according to (5.49). Then the crack radius is obtained from (5.51) and a knowledge of the total volume of the injected fluid $V$, after which the determination of the remaining parameters does not encounter any difficulties.

In [40, 41] problems were also treated concerning horizontal cracks in a radially varying pressure field caused by the higher lying rocks. Under certain conditions a complete wetting of the crack surface (i.e. the absence of a free part) may in this case occur.

Yu. P. Zheltov [43] proposed an approximate method for solving the problem of the horizontal crack in a radially varying vertical pressure field. A comparison between the results obtained by this method and the exact solutions for certain cases showed quite satisfactory agreement.

By using the method of successive approximations Yu. A. Ustinov [94] estimated the influence of the free boundary in the problem of the horizontal crack. If the depth is larger than twice the crack radius, the influence of the free boundary is negligibly small.

The problem of a crack formed by driving a horizontal wedge of constant thickness into a heavy space was treated in [39].

The solution of the problem of the vertical crack was extended by Zheltov [42] to cover the case when the rock is permeable and the injected fluid flows through the rock.

## VI. Wedging; Dynamic Problems in the Theory of Cracks

### 1. *Wedging of an Infinite Body*

Wedging is formation of a crack in a solid by driving a rigid wedge into it. The most characteristic property of the wedging of a brittle body is that the wedge surface never comes in complete contact with the body: there is always a free portion in the front part of the wedge; ahead of the wedge a free crack forms, which closes at some distance from the edge of the wedge (Fig. 32).

It appears that the problem of wedging of an infinite body by a fixed wedge [39, 58, 95] is the simplest to formulate among problems of this kind; it yields to an effective exact solution by the methods of elasticity theory and gives a qualitative idea of wedging under more complex conditions.

Let a uniform, isotropic brittle body be wedged by a thin, symmetrical, perfectly rigid semi-infinite wedge with thickness $2h$ at infinity (Fig. 32). In front of the wedge a free crack forms, which closes smoothly at a certain

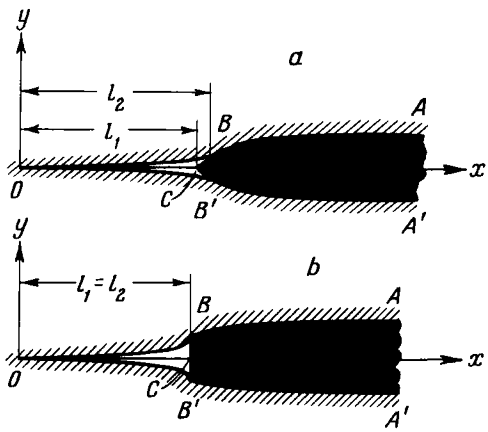


FIG. 32.

point $O$; the position of the point $O$ with respect to the front point of the wedge $C$ is not known beforehand and must be determined in the course of solving the problem. If the wedge has a rounded front part (Fig. 32a), the position of the points of departure of the crack surface from the wedge, $B$ and $B'$, is not prescribed and must also be determined in the course of solving the problem. If the wedge has a truncated front part (Fig. 32b) as e.g. in the case of a wedge of constant thickness, the position of the points of contact is quite definite; they coincide with the corners of the wedge front. It is evident that the stress at the points of departure is in this case infinite. We shall at first assume that there is no friction at the surface of contact between wedge and body.

The field of elastic stresses and strains satisfies the usual equations of static elasticity in the exterior of the crack. In view of the assumed slenderness of the wedge, the boundary conditions may be transferred from the crack surface proper to the $x$-axis. Without considering forces of cohesion, the boundary conditions are

$$\begin{aligned} &\sigma_{xy} = 0, \qquad \sigma_y = 0 \qquad (0 \leqslant x < l_2, \quad y = 0), \\ &v = \pm f(x - l_1), \qquad \sigma_{xy} = 0 \qquad (l_2 \leqslant x < \infty, \quad y = 0); \end{aligned} \tag{6.1}$$

here $\sigma_y$, $\sigma_{xy}$ are the stress-tensor components; $l_1$ and $l_2$ are the distances of the point $O$ from the edge of the wedge and from the points of departure $B$, $B'$; $f(t)$ defines the wedge surface in a system of coordinates with origin at the front point of the wedge; the positive and negative signs correspond to the upper and lower faces of the cut, respectively.

As is seen, the problem of wedging is a peculiar combination of the contact problem in the theory of elasticity [18, 72, 73] and the problem of the theory of cracks.

The position of the points of departure of the crack surface from the wedge in the case of a wedge with rounded edge, and the position of the point of closing with respect to the edge are determined from the following conditions:

1. *Stresses at the points of departure must be finite.* For the contact problem a similar condition was first suggested as a hypothesis by Muskhelishvili [96, 18] and independently by A. V. Bitsadze [97]; it was proved in [61].

2. *Stresses at the crack edge are finite or, which is the same, the opposite faces of a crack close smoothly at its end.* Since the intensity of forces of cohesion at the crack edge is maximal, stresses near the crack edge calculated without taking into account forces of cohesion must tend to infinity according to (4.7).

The problem of wedging is a mixed problem of the theory of elasticity. For its solution it is convenient to consider the singular integral equation for the compressive force acting on the face of the wedge, $\sigma_y = -\phi(x)$. If $\phi(x)$ is known, the determination of the elastic field obviously reduces to the solution of the first boundary-value problem in the theory of elasticity for the exterior of a semi-infinite straight-line cut, which can be found by Muskhelishvili's method ([18], § 95). This solution gives an expression for the lateral displacements at points of contact between wedge and crack surface:

$$v = \frac{4(1-\nu^2)}{\pi E}\int_{l_1}^{\infty}\phi(\sigma^2)\sigma\ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right|d\sigma, \tag{6.2}$$

where $\zeta = \sqrt{x}$, and the root may assume positive and negative values for displacements of the upper and lower face. The second condition (6.1) yields the fundamental integral equation of the problem:

$$\int_{l_1}^{\infty}\phi(\sigma^2)\sigma\ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right|d\sigma = \pm\frac{\pi E}{4(1-\nu^2)}f(\zeta^2-l_1), \tag{6.3}$$

which can be shown to be equivalent to the singular integral equation obtained from (6.3) by differentiation with respect to $\zeta$:

$$\int\limits_{|\sigma|>l_1} \frac{\phi(\sigma^2)\sigma d\sigma}{\sigma-\zeta} = \pm \frac{\pi E}{2(1-\nu^2)} \zeta f'(\zeta^2 - l_1), \tag{6.4}$$

and the condition

$$\phi(x) = \frac{Eh}{2\pi(1-\nu^2)x} + O\left(\frac{1}{x^2}\right) \quad (x \to \infty), \tag{6.5}$$

where $h = f(\infty)$. By using the methods for singular integral equations developed in the monograph by Muskhelishvili ([19], Chapter 5) the solution of equation (6.4) can be found in the form:

$$\phi(x) = \frac{1}{\pi\sqrt{x(x-l_2)}} \left[A - \frac{E}{2(1-\nu^2)} \int\limits_{l_2}^{\infty} \frac{f'(t-l_1)\sqrt{t(t-l_2)}\,dt}{t-x}\right], \tag{6.6}$$

where $A$ is an indefinite constant.

The integral in (6.6) does exist in view of the finiteness of $f(\infty) = h$, and it tends to zero as $x \to \infty$; this together with (6.5) determines the value of the constant $A$:

$$A = \frac{Eh}{2(1-\nu^2)}. \tag{6.7}$$

For finiteness of stress at the points of departure $x = l_2$ in case of a wedge with rounded edge, it is necessary and sufficient that the bracketed expression in (6.6) vanishes at $x = l_2$. This gives one equation for the determination of $l_1$ and $l_2$:

$$h = \int\limits_{l_2}^{\infty} f'(t-l_1)\sqrt{\frac{t}{t-l_2}}\,dt. \tag{6.8}$$

Now the following expression for the tensile stresses at the prolongation of the cut results from the solution:

$$\sigma_y = \frac{E}{2\pi(1-\nu^2)\sqrt{(l_2-x)(-x)}} \left[h - \int\limits_{l_2}^{\infty} \frac{f'(t-l_1\sqrt{t(t-l_2)}dt}{t-x}\right]. \tag{6.9}$$

Together with (4.7) it leads to

$$h - \int_{l_1}^{\infty} f'(t - l_1) \sqrt{\frac{t - l_2}{t}}\, dt = \frac{2K\sqrt{l_2}(1 - \nu^2)}{E} \tag{6.10}$$

Relations (6.8) and (6.10) are finite equations which determine the unknown constants $l_1$ and $l_2$.

In the particular case of constant wedge thickness $f(t) \equiv h$, condition (6.8), which is no longer valid, is replaced by the relation $l_1 = l_2$, and (6.10) gives the following expression for the length of a free crack in front of a "square" wedge:

$$l_1 = l_2 = \frac{E^2 h^2}{4(1 - \nu^2)^2 K^2}. \tag{6.11}$$

In [95] other special forms of the wedge are also treated such as a wedge rounded-off with a small radius of curvature and a wedge rounded-off according to a power law. Investigation of the first example shows that

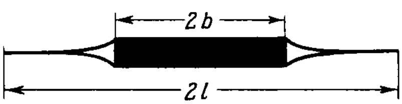


FIG. 33.

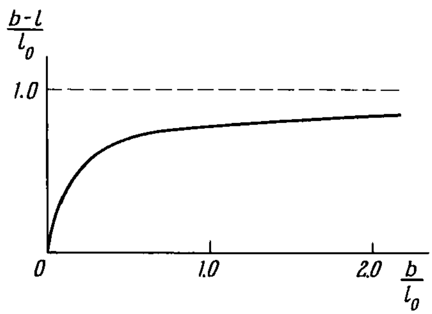


FIG. 34.

roundness affects slightly the length of the free crack in front of the wedge. In [95] also a case when Coulomb friction acts on the faces of the wedge is treated.

In [84] wedging of an anisotropic body by a semi-infinite rigid wedge is studied.

I. A. Markuzon [98] treated a problem of wedging an infinite body by a wedge of finite length $2b$ (Fig. 33). In case of constant thickness of the wedge $2h$, the relation between crack length $2l$ and wedge length $2b$, other things being equal, is as represented in Fig. 34 ($l_0$ is the length of a free crack for an infinite wedge defined by (6.11)).

In [98] the influence of a uniform compressive or tensile stress at infinity on the length of a free crack, when the wedge is of finite length, was also investigated.

Relation (6.11) can be used for the experimental determination of the modulus of cohesion $K$. For that purpose a wedge is driven into a plate of the testing material, the wedge being substantially more rigid than the plate. The length $L$ of the resulting free crack is measured. The modulus of cohesion can then be found according to the formula

$$K = \frac{Eh}{2(1-\nu^2)\sqrt{L}}. \tag{6.12}$$

The wedge must be sufficiently long in order to eliminate the influence of the plate boundary, and it should be driven in, until the distance between the wedge end and the crack end stops varying with further displacement of the wedge. The plate must be wide and sufficiently thick so that the state of stress essentially corresponds to plane strain. To insure a straight-line form of the crack, it is necessary to compress the specimen in the direction of crack propagation. This is recommended by Benbow and Roesler [9]. (It can be shown that (6.11) and (6.12) remain unchanged in this case.)

## 2. *Wedging of a Strip*

In strict formulation, problems concerning the wedging of bounded bodies are very difficult to solve. Up to now there are but a few approximate solutions, based on the application of the approximations of simple beam theory.

The first of these solutions was obtained by I. V. Obreimov [8]; as a matter of fact, this work was the first investigation in which wedging was considered. In connection with his experiments on the splitting of mica, Obreimov examined the case when a strip being torn off has small thickness and only one-point contact with the wedging body (Fig. 35). In order to establish a relation between the surface tension of mica and the parameters of the crack shape, Obreimov applied to this problem the methods of strength of materials, considering a shaving as a thin beam. The theoretical part of the work of Obreimov is not free from shortcomings. Later, corrections were introduced into these calculations in the book by V. D. Kuznetsov [99] as well as by M. S. Metsik [10] and N. N. Davidenkov [12]. In addition Metsik improved the experimental procedure of [8]. Application of the approximations of thin-beam theory for the determination of the crack length is justifiable in some cases. However, these approximations cannot be applied to describe the form of the crack surface in the immediate vicinity of its edge, even if the distribution of forces of cohesion in the edge region is explicitly considered, as it was done by Ya. I. Frenkel [5]. The fact is that the longitudinal dimension of the edge region cannot be assumed to be large compared to the shaving thickness; hence a shaving cannot be considered as a thin beam in the region where the forces of cohesion are acting.

To illustrate the approximate approach based on the methods of simple beam theory, we discuss the paper by Benbow and Roesler [9] in more detail. Note that in this work possibilities and limits of applicability of the above approach are most clearly pointed out.

The following statement of the problem is considered (Fig. 36). A strip of finite width $b$ is wedged symmetrically so that the crack passes along the

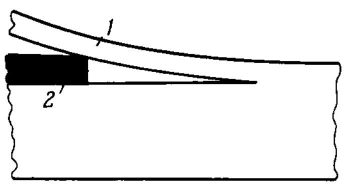


FIG. 35.
1. a Body being wedged.
1. a Wedge.

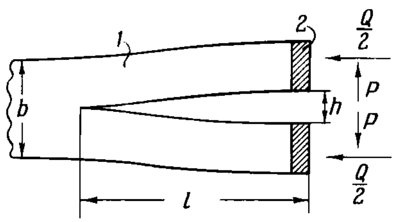


FIG. 36.
1. a Body being wedged.
2. Grips.

middle line of the strip. At the end of the strip, compressive forces $Q/2$ are applied to insure straight crack propagation; the wedging force $P$ produces a crack length $l$ and initial width $h$.

Having obtained an expression for the strain energy from dimensional considerations, the authors write the equilibrium condition for the crack in the form

$$\frac{T}{E} = \frac{h^2}{l} \phi\left(\frac{b}{l}\right), \tag{6.13}$$

so that for a given material the quantity $h^2/l$ is uniquely determined by the quantity $b/l$. The experiments made with specimens of two different plastics [9] give a conclusive proof of the existence of such a one-one relation.

For small $b/l$, i.e. for long cracks, it is possible to obtain an asymptotic form of relation (6.13) by considering both halves of the strip as thin beams fixed at the section corresponding to the crack end. The expression for the strain energy of the strip is in this case

$$U = 3h^2B/l^3, \tag{6.14}$$

where $B = EJ$ is the stiffness of the beam, $J = nb^3/96$, and $n$ is the transverse thickness of the beam. The surface energy of the crack is, evidently, $2Tnl$.

In the mobile-equilibrium state the variation of surface energy corresponding to a small variation of the crack length $\delta l$ is equal to the corresponding variation of strain energy of the strip. Hence it follows that

$$-\frac{\partial U}{\partial l} = 2Tn \qquad \text{or} \qquad \frac{T}{E} = \frac{3h^2b^3}{64l^4}. \tag{6.15}$$

By comparing the second formula (6.15) with (6.13), an asymptotic expression for $\Phi(b/l)$ can be found as $b/l \to 0$:

$$\phi = \frac{3}{64}\left(\frac{b}{l}\right)^3. \tag{6.16}$$

From (6.15) an expression for the length of an equilibrium crack is obtained:

$$l = \sqrt[4]{\frac{3h^2b^3E}{64T}} = \sqrt[4]{\frac{3h^2b^3E\pi}{64K^2(1-\nu^2)}}. \tag{6.17}$$

Thus in this case the length of the crack is proportional only to $\sqrt{h}$, whereas in the wedging of an infinite body by a semi-infinite wedge the length of the crack is proportional to $h^2$ (cf. (6.11)).

Relation (6.15) was used by Benbow and Roesler for the determination of the surface-energy density of the plastics investigated. The careful experimentation and the scrupulous evaluation of the sources of possible errors and of their magnitude are remarkable.

In the recent review of J. J. Gilman [11] a detailed summary and a bibliography of experimental investigations on wedging can be found.

### 3. *Dynamic Problems in the Theory of Cracks*

Considerable attention is nowadays given to questions of dynamics of cracks. A detailed consideration of these questions is beyond the scope of the present review; we shall confine ourselves here to a brief information about basic results achieved in theoretical investigations of dynamics of cracks.

In the paper by N. F. Mott [36] the crack-expansion process is treated in the case of an isolated straight crack in an infinite body subjected to a uniform field of tensile stresses $p_0$.* On the basis of dimensional analysis Mott obtained an expression for the kinetic energy of a body,

$$\mathscr{E} = k\rho l^2V^2p_0^2/E^2, \tag{6.18}$$

* Unlike [36], plane strain is here considered rather than the state of plane stress.

where $\rho$ is the density of the body, $l$ the half-length of the crack, $V$ the rate of crack expansion, and $k$ a dimensionless factor which Mott considered constant and left indefinite. Adding to the static-energy equation (2.1) the derivative with respect to $l$ of the kinetic energy (6.18) and assuming the remaining terms in (2.1) to be the same as in Griffith's static problem, Mott found the rate of crack expansion

$$V = \left[\frac{\pi(1-\nu^2)}{k}\right]^{1/2}\left(\frac{E}{\rho}\right)^{1/2}\left(1-\frac{l_*}{l}\right), \tag{6.19}$$

where $l_*$ is the half-length of the mobile-equilibrium crack defined by (5.4). Thus, as the crack propagates, its extension rate increases, approaching the limit

$$V_0 = \left[\frac{\pi(1-\nu^2)}{k}\right]^{1/2}\left(\frac{E}{\rho}\right)^{1/2}. \tag{6.20}$$

The ultimate rate constitutes, according to Mott, a certain part of the longitudinal wave propagation velocity. In this reasoning the use of the static expression for the decrease of the strain energy $W$ remains unfounded. Moreover, the quantity $k$ in (6.18) and (6.19) need not be constant in general; it may depend on $l/l^*$, $V/c_1$ and other dimensionless combinations.

E. Yoffe [100], using the exact formulation of dynamic elasticity theory, investigated the problem of a straight crack of constant length, moving with constant velocity in an infinite body stretched by uniform stress at infinity. Notwithstanding the somewhat artificial character of the problem, an important result was obtained in this paper, which has quite a general meaning: If the crack propagation rate becomes greater than a certain critical rate, the direction of crack propagation is no longer the direction of maximum tensile stress, and the crack begins to curve. The magnitude of the critical rate $V_1$ is about $0.4\,c_1$, where $c_1$ is the longitudinal wave propagation velocity in the given material (the ratio $V_1/c_1$ depends slightly on Poisson's ratio $\nu$ of the material).

D. K. Roberts and A. A. Wells [101] made an attempt to evaluate the constant $k$, which remained indefinite in [36]. Using the value of $k$ obtained, they found the ultimate crack expansion rate close to that found by Yoffe. However, their estimate, based as it is on the solution of a static problem, is too rough; and since the straight-line direction of crack propagation in [101] was assumed as certain, the close agreement between the critical rate found by Yoffe [100] and the ultimate rate obtained in [101] must be considered as incidental.

If the straight-line direction of crack propagation is somehow insured (for instance, by a large compression of the body in the direction of crack propagation or by the anisotropy of the material), then the maximum rate of crack propagation coincides with the velocity of propagation of Rayleigh surface waves in the given material, which is about $0.6\, c_1$.

The fact that the ultimate rate of crack propagation coincides with the Rayleigh velocity was first stated by A. N. Stroh [102]. The heuristic proof given in that paper amounts to the following. Stroh correctly notes that the ultimate rate of crack propagation does not depend on the surface energy of the body, and he assumes the surface energy to be zero. Proceeding from this, Stroh is led by energy considerations to the conclusion that the tensile stress near the crack end (on its prolongation) is equal to zero. Thus the crack may be thought of as a disturbance moving on a stress-free surface, which can propagate only with the Rayleigh velocity. In fact, from Stroh's reasoning it may only be concluded that the tensile stress at the very contour of the crack is equal to zero. From this fact, however, it does not follow that the rate of crack propagation is equal to the Rayleigh velocity, as can be seen from the following simple example. Take a body subjected to all-round compressive stress at infinity and wedged by a semi-infinite wedge as in Fig. 32, moving with infinitely small velocity. Forces of cohesion and, consequently, surface energy are assumed to be zero. In view of the infinitesimal velocity of the wedge, dynamic effects are insignificant, hence, according to Section III,2, the tensile stress at the crack end must vanish. At the same time, the rate of crack propagation is equal to the velocity of the wedge, i.e. it is also infinitesimal.

By arguments based on the analysis of exact solutions of the dynamic equations of elasticity, the conclusion that the ultimate rate of crack propagation is equal to Rayleigh velocity was drawn independently and simultaneously by several authors. I. W. Craggs [103] considered steady propagation of a semi-infinite straight crack with symmetrically distributed normal and shearing stresses applied on a part of the crack surface adjacent to the edge. In a paper by Dang Dinh An [104] a non-steady field of stresses and strains was investigated, acting in an infinite body with a semi-infinite crack, along the surface of which symmetrical concentrated forces normal to the crack surface begin to move suddenly away from the edge with constant velocity. Paper [95] examines the wedging of an infinite isotropic brittle body by a semi-infinite rigid wedge of arbitrary form, moving with constant velocity. In [84] a similar problem is treated for a case of an anisotropic body. B. R. Baker [105] considers a non-steady distribution of stresses and strains in a solid with a semi-infinite crack, at the surface of which constant normal stress is applied at the initial moment, after which the crack begins to expand with constant velocity.

From the various problems treated in these papers the following general result was obtained which led to our earlier conclusion: when the characteristic rate involved in the problem approaches the Rayleigh velocity, peculiar resonance phenomena arise. Note that the appearance of resonance when the Rayleigh velocity is approached is not specific for the problems of cracks: the investigation of the problem of a punch moving along the boundary of a half-space, carried out by L. A. Galin [72] and J. R. M. Radok [106], reveals [95] that the same resonance phenomena occur, when the velocity of the punch approaches the Rayleigh velocity. It appears that the limiting character of the Rayleigh velocity is most directly illustrated by a problem of wedging. Obviously the maximum possible rate of crack propagation can be reached in wedging a body by a moving wedge. The analysis of this problem shows [95] that with increasing velocity of the wedge the length of the free crack in front of the wedge decreases and tends to zero when the Rayleigh velocity is approached. For larger wedge velocity a free crack does not form in front of the wedge. Hence the maximum rate with which a crack can expand is equal to Rayleigh velocity.

K. B. Broberg [107, 108] treated the problem of a uniformly expanding crack of finite length in an infinite body subjected to a uniform tensile stress field. The solution obtained by Broberg is an asymptotic representation for great values of time of the solution of the problem treated by Mott [36] and Roberts and Wells [101]. However, unlike [101], Broberg's solution was obtained on the basis of the exact methods of the dynamic theory of elasticity. Independently of [102–104, 57, 95, 105] and in full accord with the results of these investigations, Broberg obtained that the rate of crack expansion in his problem, equal to the ultimate rate of crack expansion in the problem considered in [36, 101], coincides with the Rayleigh velocity.

Note the papers by B. A. Bilby and R. Bullough [109] and F. A. McClintock and S. P. Sukhatme [110] which treat uniformly moving cracks of finite and infinite length, respectively, at the surface of which symmetrical shearing stresses parallel to the crack edge were applied. Instead of plane strain we have in this problem what is often called *anti-plane strain*: one displacement component, parallel to the crack edge, is different from zero. The investigation of such cracks reduces to the solution of a single wave equation (reducing to Laplace's equation for equilibrium cracks). Cracks under anti-plane strain conditions are of considerable interest, being the simplest model for which an effective solution is possible for many problems, which are intractable for cracks under plane-strain conditions because of the great mathematical difficulties.

An analysis of the dynamics of crack propagation on the basis of the approximations of the simple beam theory was carried out by J. J. Gilman [11] and J. C. Suits [111].

ACKNOWLEDGEMENT

The author is very grateful to Prof. Ya. B. Zeldovitch and Prof. Yu. N. Rabotnov (USSR Academy of Sciences) and Dr. S. S. Grigorian for the invariable interest and attention given to his work on cracks and for a number of valuable advices. He recalls with appreciation the valuable discussions with Prof. S. A. Khristianovitch (USSR Academy of Sciences). The author considers it his pleasant duty to express his sincere thanks to Prof. G. Kuerti (USA) and Prof. G. G. Chernyi for the amiable assistance in writing this review. Credit is also given to I. A. Markuzon who assisted the author in compiling the bibliography.

## *References*

1. Inglis, C. E., Stresses in a plate due to the presence of cracks and sharp corners, *Trans. Inst. Nav. Arch.* **55**, 219–230 (1913).
2. Muskhelishvili, N. I., Sur l'intégration de l'équation biharmonique, *Izvestiya Ross. Akad. nauk* **13**, 6 ser., 663–686 (1919).
3. Griffith, A. A., The phenomenon of rupture and flow in solids, *Phil. Trans. Roy. Soc.* **A 221**, 163–198 (1920).
4. Griffith, A. A., The theory of rupture, *Proc. 1st Intern. Congr. Appl. Mech., Delft*, pp. 55–63 (1924).
5. Frenkel, Ya. I., The theory of reversible and irreversible cracks in solids, *Zhurn. tekhn. fiz.* **22**, 1857–1866 (1952) (in Russian).
6. Smekal, A., Technische Festigkeit und molekulare Festigkeit, *Naturwiss.* **10**, 799–804 (1922).
7. Wolf, K., Zur Bruchtheorie von A. Griffith, *Zeitschr. ang. Math. Mech.* **3**, 107–112 (1923).
8. Obreimov, I. V., The splitting strength of mica, *Proc. Roy. Soc.* **A 127**, 290–297 (1930).
9. Benbow, J. J., and Roesler, F. C., Experiments on controlled fractures, *Proc. Phys. Soc.* **B 70**, 201–211 (1957).
10. Metsik, M. S., A photoelastic measurement of the splitting work of mica crystals, *Izvestiya VUZ, ser. fiz.* No. 2, 58–63 (1958) (in Russian).
11. Gilman, J. J., Cleavage, ductility and tenacity in crystals, *in* "Fracture" (B. L. Averbach et al., eds.), pp. 193–221. Wiley, New York, 1959.
12. Davidenkov, N. N., On the surface energy of mica, *Prikladna mekhanika* **6**, No. 2, 138–142 (1960) (in Ukrainian).
13. Westergaard, H. M., Bearing pressures and cracks, *J. Appl. Mech.* **6**, No. 2, A-49–A-53 (1939).
14. Sneddon, I. N., The distribution of stress in the neighborhood of a crack in an elastic solid, *Proc. Roy. Soc.* **A 187**, 229–260 (1946).
15. Sneddon, I. N., "Fourier Transforms". McGraw Hill, New York, 1951.
16. Sneddon, I. N., and Elliot, H. A., The opening of a Griffith crack under internal pressure, *Quart. Appl. Math.* **4**, 262–267 (1946).
17. Williams, M. L., On the stress distribution at the base of a stationary crack, *J. Appl. Mech.* **24**, 109–114 (1957).
18. Muskhelishvili, N. I., "Some Basic Problems of the Mathematical Theory of Elasticity", 4th ed. Izd. AN SSSR, M.-L., 1954 (Transl. from 3rd ed., Nordhoof, Holland, 1953).
19. Muskhelishvili, N. I., "Singular Integral Equations". GTTI, M.-L., 1946 (Transl. Nordhoof, Holland, 1953).

20. SACK, R. A., Extension of Griffith theory of rupture to three dimensions, *Proc. Phys. Soc.* **58**, 729–736 (1946).
21. WILLMORE, T. J., The distribution of stress in the neighborhood of a crack, *Quart. Journ. Mech. Appl. Math.* **2**, 53–64 (1949).
22. BOWIE, O. L., Analysis of an infinite plate containing radial cracks originating at the boundary of an internal circular hole, *J. Math. and Phys.* **25**, 60–71 (1956).
23. IRWIN, G. R., Fracture dynamics, *in* "Fracturing of Metals", pp. 147–166. ASM, Cleveland, Ohio, 1948.
24. OROWAN, E. O., Fundamentals of brittle behavior of metals, *in* "Fatigue and Fracture of Metals" (W. M. Murray, ed.), pp. 139–167. Wiley, New York, 1950.
25. IRWIN, G. R., and KIES, J. A., Fracturing and fracture dynamics, *Weld. Journ. Res. Suppl.* **31**, 95s–100s (1952).
26. IRWIN, G. R., and KIES, J. A., Critical energy rate analysis of fracture strength, *Weld. Journ. Res. Suppl.* **33**, 193s–198s (1954).
27. OROWAN, E. O., Energy criteria of fracture, *Weld. Journ. Res. Suppl.* **34**, 157s–160s (1955).
28. FELBECK, D. K., and OROWAN, E. O., Experiments on brittle fracture of steel plates, *Weld. Journ. Res. Suppl.* **34**, 570s–575s (1955).
29. WELLS, A. A., The mechanics of notch brittle fracture, *Weld. Research* **7**, 34r–56r (1953).
30. WELLS, A. A., The brittle fracture strength of welded steel plates, *Quart. Trans. Inst. Nav. Arch.* **98**, 296–311 (1956).
31. WELLS, A. A., Strain energy release rates for fractures caused by wedge action, *NRL Rept.* No. 4705 (1956).
32. WINNE, D. H., and WUNDT, B. M., Application of the Griffith-Irwin theory of crack propagation to the bursting behavior of disks including analytical and experimental studies, *Trans. ASME* **80**, 1643–1658 (1958).
33. BUECKNER, H. F., The propagation of cracks and the energy of elastic deformation, *Trans. ASME* **80**, 1225–1230 (1958).
34. REBINDER, P. A., Physico-chemical study of deformation processes in solids, *in* "Yubileinyi Sbornik, Posviashchennyi XXX-letiu Velikoi Oktyabr'skoi Revolutsii", pp. 533–561. Izd. AN SSSR, M., 1947 (in Russian).
35. ELLIOT, H. A., An analysis of the conditions for rupture due to Griffith cracks, *Proc. Phys. Soc.* **59**, 208–223 (1946).
36. MOTT, N. F., Fracture of metals; theoretical consideration, *Engineering* **165**, 16–18 (1948).
37. RZHANITSYN, A. R., On the fracture process of the material in tension, *in* "Issledovaniya po Voprosam Stroitel'noi Mekhaniki i Teorii Plastichnosti" (A. R. Rzhanitsyn, ed.), pp. 66–83. Stroiizdat, M., 1956 (in Russian).
38. ZHELTOV, YU. P., and KHRISTIANOVITCH, S. A., On the mechanism of hydraulic fracture of an oil-bearing stratum, *Izvestiya AN SSSR, OTN*, No. 5, 3–41 (1955) (in Russian).
39. BARENBLATT, G. I., and KHRISTIANOVITCH, S. A., On roof stoping in mine excavations, *Izvestiya AN SSSR, OTN*, No. 11, 73–86 (1955) (in Russian).
40. BARENBLATT, G. I., On some problems of the theory of elasticity arising in investigating the mechanism of hydraulic fracture of an oil-bearing stratum, *Prikl. matem. i mekhan.* **20**, 475–486 (1956) (in Russian).
41. BARENBLATT, G. I., On the formation of horizontal cracks in hydraulic fracture of an oil-bearing stratum, *Izvestiya AN SSSR, OTN*, No. 9, 101–105 (1956) (in Russian).
42. ZHELTOV, YU. P., An approximate evaluation of the size of a crack forming in hydraulic fracture of a stratum, *Izvestiya AN SSSR, OTN*, No. 3, 180–182 (1957) (in Russian).

43. ZHELTOV, YU. P., On the formation of vertical cracks in a stratum by means of a filtrating fluid, *Izvestiya AN SSSR, OTN*, No. 8, 56–62 (1957) (in Russian).
44. WESTERGAARD, H. M., Stresses at a crack, size of the crack and the bending of reinforced concrete, *J. Americ. Concrete Inst.* **5**, No. 2, 93–102 (1933).
45. IRWIN, G. R., Analysis of stresses and strains near the end of a crack traversing a plate, *J. Appl. Mech.* **24**, 361–364 (1957).
46. IRWIN, G. R., Relation of stresses near a crack to the crack extension force, *Proc. 9th Intern. Congr. Appl. Mech., Brussels* pp. 245–251 (1957).
47. IRWIN, G. R., Fracture, *in* "Handbuch der Physik", B. VI (S. Flügge, ed.), pp. 551–590. Springer, Berlin, 1958.
48. IRWIN, G. R., KIES, J. A., and SMITH, H. L., Fracture strength relative to onset and arrest of crack propagation, *Proc. Americ. Soc. Test. Mater.* **58**, 640–657 (1958/1959).
49. KIES, J. A., SMITH, H. L., IRWIN, G. R., La mécanique des ruptures et son application aux travaux de l'ingénieur, *Mém. scient. rev. métallurgie* **57**, No. 2, 101–117 (1960).
50. BUECKNER, H. F., Some stress singularities and their computation by means of integral equations, *in* "Boundary Problems in Differential Equations" (R. E. Langer, ed.), pp. 215–230. Univ. Wisconsin Press, 1960.
51. IRWIN, G. R., The crack-extension-force for a crack at a free surface boundary, *NRL Rept.* No. 5120 (1958).
52. ROMUALDI, J. P., and SANDERS, P. H., Fracture arrest by riveted stiffeners, *Proc. 4th Midwest. Conf. Solid Mech.*, Univ. Texas Press pp. 74–90 (1959/1960).
53. ROESLER, F. C., Brittle fracture near equilibrium, *Proc. Phys. Soc.* **B 69**, 981–992 (1956).
54. BENBOW, J. J., Cone cracks in fused silica, *Proc. Phys. Soc.* **B 75**, 697–699 (1960).
55. DROZDOVSKY, B. A., and FRIDMAN, YA. B., "The Influence of Cracks on the Mechanical Properties of Structural Steels". Metallurgizdat, M., 1960 (in Russian).
56. BARENBLATT, G. I., On the equilibrium cracks due to brittle fracture, *Doklady AN SSSR* **127**, 47–50 (1959) (in Russian).
57. BARENBLATT, G. I., On the equilibrium cracks due to brittle fracture. Fundamentals and hypotheses. Axisymmetrical cracks, *Prikl. matem. i mekhan.* **23**, 434–444 (1959) (in Russian).
58. BARENBLATT, G. I., On the equilibrium cracks due to brittle fracture. Straight-line cracks in flat plates, *Prikl. matem. i mekhan.* **23**, 706–721 (1959) (in Russian).
59. BARENBLATT, G. I., On the equilibrium cracks due to brittle fracture. Stability of isolated cracks. Relation to energy theories, *Prikl. matem. i mekhan.* **23**, 893–900 (1959) (in Russian).
60. BARENBLATT, G. I., On conditions of finiteness in mechanics of continua. Static problems in the theory of elasticity, *Prikl. matem. i mekhan.* **24**, 316–322 (1960) (in Russian).
61. BARENBLATT, G. I., On some basic ideas of the theory of equilibrium cracks forming during brittle fracture, *in* "Problems of Continuum Mechanics" (M. A. Lavrent'ev, ed.), pp. 21–38. SIAM, Philadelphia, 1961.
62. ALEXANDROV, A. P., ZHURKOV, S. P., "The Phenomenon of Brittle Fracture". GTTI, M., 1933 (in Russian).
63. GARBER, R. I., and GUINDIN, I. A., Strength physics of crystalline bodies, *Uspekhi fizicheskikh nauk* **70**, 57–110 (1960) (in Russian).
64. DERYAGIN, B. V., ABRIKOSOVA, I. I., and LIFSHITZ, E. M., Molecular attraction of condensed bodies, *Uspekhi fizicheskikh nauk* **64**, 493–528 (1958) (in Russian).
65. DERJAGUIN, B. V., and ABRIKOSSOVA, I. I., Direct measurements of molecular attraction of solids, *J. Phys. Chem. Solids* **5**, No. 1/2, 1–10 (1958).

66. JONGH, J. G. V., DE, "Measurements of Retarded van der Waals-Forces". Thesis, Abels, Utrecht, 1958.
67. BORN, M. and KUN HUANG, "Dynamical Theory of Crystal Lattices". Clarendon Press, Oxford, 1954.
68. LEIBFRIED, G., Gittertheorie der mechanischen und thermischen Eigenschaften der Krystalle, *in* "Handbuch der Physik", B. VII, I (S. Flügge, ed.), p. 104–324. Springer, Berlin, 1955.
69. LEONOV, M. YA., and PANASYUK, V. V., Development of the finest cracks in a solid, *Prikladna mekhanika* **5**, 391–401 (1959) (in Ukrainian).
70. PANASYUK, V. V., Determination of stresses and strains near the finest crack, *Nauchn. zapiski In-ta mashinovedenia i avtomatiki AN USSR* **7**, 114–127 (1960) (in Russian).
71. LANDAU, L. D., and LIFSHITZ, E. M., "Theory of Elasticity". Pergamon Press, London, 1960.
72. GALIN, L. A., "Contact Problems of the Theory of Elasticity". GITTL, M., 1953 (in Russian).
73. SHTAERMAN, I. YA., "Contact Problem of the Theory of Elasticity". GITTL, M., 1949 (in Russian).
74. PARKER, E. R., "Brittle Behavior of Engineering Structures". Wiley, New York, 1957.
75. DAVIDENKOV, N. N., "Impact Problem in Metallography". Izd. AN SSSR, M.-L., 1938 (in Russian).
76. PASHKOV, P. O., "Fracture of Metals". Sudpromgiz, L., 1960 (in Russian).
77. POTAK, YA. M., "Brittle Fracture of Steel and Steel Parts". Oborongiz, M., 1955 (in Russian).
78. AVERBACH, B. L., and oth. (eds.), "Fracture". Wiley, New York, 1959.
79. SEDOV, L. I., "Similarity and Dimensional Methods in Mechanics", 4th ed. GITTL, M., 1957 (Transl. from 4th ed., Acad. Press, New York, 1959).
80. NEUBER, H., "Kerbspannungslehre. Grundlagen für genaue Spannungsrechnung". Springer, Berlin, 1937.
81. FRANKLAND, I. M., Triaxial tension at the head of a rapidly running crack in a plate, *Paper ASME*, No. APM-11 (1959).
82. MASUBUCHI, K., Dislocation and strain energy release during crack propagation in residual stress field, *Proc. 8th Japan Nath. Congr. Appl. Mech.* pp. 147–150 (1958/1959).
83. STROH, A. N., Dislocations and cracks in anisotropic elasticity, *Phil. Mag., VIII ser.* **3**, 625–646 (1958).
84. BARENBLATT, G. I., and CHEREPANOV, G. P., On the equilibrium and propagation of cracks in an anisotropic medium, *Prikl. matem. i mekhan.* **25**, 46–55 (1961) (in Russian).
84a. KACHANOV, L. M., On kinetics of crack growth, *Prikl. matem. i mekhan.* **25**, 498–502 (1961) (in Russian).
85. WIGGLESWORTH, L. A., Stress distribution in a notched plate, *Mathematika* **4**, 76–96 (1957).
86. SHERMAN, D. I., A method of solving the static plane problem of the theory of elasticity for multiconnected regions, *Trudy Seismologicheskogo Instituta Akademii Nauk SSSR*, No. 54 (1935) (in Russian).
87. MIKHLIN, S. G., The plane problem of the theory of elasticity, *Trudy Seismologicheskogo Instituta Akademii Nauk SSSR*, No. 65 (1935) (in Russian).
88. BARENBLATT, G. I., and CHEREPANOV, G. P., On the influence of body boundaries on the development of cracks of brittle fracture, *Izvestiya AN SSSR, OTN, ser. mekh. i mash.*, No. 3, 79–88 (1960) (in Russian). Correction. The same journal, No. 1 (1962).

89. KOITER, W. T., An infinite row of collinear cracks in an infinite elastic sheet, *Ingenieur-Archiv* **28**, 168–172 (1959).
90. MUSKHELISHVILI, N. I., Basic boundary-value problems in the theory of elasticity for a plane with straight-line cuts, *Soobschenia AN Gruz. SSR* **3**, 103–110 (1942) (in Russian).
91. SHERMAN, D. I., A mixed problem in the theory of potential and in the theory of elasticity for a plane with a finite number of straight-line cuts, *Doklady AN SSSR* **27**, 330–334 (1940) (in Russian).
92. WUNDT, B. M., A unified interpretation of room-temperature strength of notched specimens as influenced by their size, *Paper ASME*, No. 59-MET-9 (1959).
93. YUSUFF, S., Fracture phenomena in metal plates, *Paper presented at the Xth Intern. Congr. Appl. Mech., Stresa* (1960).
94. USTINOV, YU. A., On the influence of the free boundary of a half-space on the crack propagation, *Izvestia AN SSSR, OTN, ser. mekh. i mash.*, No. 4, 181–183 (1959) (in Russian).
95. BARENBLATT, G. I., and CHEREPANOV, G. P., On the wedging of brittle bodies, *Prikl. matem. i mekhan.* **24**, 667–682 (1960) (in Russian).
96. MUSKHELISHVILI, N. I., "Some Basic Problems of the Mathematical Theory of Elasticity", 2nd ed. Izd. AN SSSR, M.-L., 1935 (in Russian).
97. BITSADZE, A. V., On local deformations of elastic bodies in compression, *Soobschenia AN Gruz. SSR* **3**, 419–424 (1942) (in Russian).
98. MARKUZON, I. A., On the wedging of a brittle body by a wedge of finite length, *Prikl. matem. i mekhan.* **25**, 356–361 (1961) (in Russian).
99. KUZNETSOV, V. D., "Surface Energy of Solids". GITTL, M., 1954 (Transl., H. M. Stat. Office, London, 1957).
100. YOFFE, E., The moving Griffith crack, *Phil. Mag., VII ser.* **42**, 739–750 (1951).
101. ROBERTS, D. K., and WELLS, A. A., The velocity of brittle fractures, *Engineering* **178**, 820–821 (1954).
102. STROH, A. N., A theory of the fracture of metals, *Advances in Physics* **6**, 418–465 (1957).
103. CRAGGS, I. W., On the propagation of a crack in an elastic-brittle material, *J. Mech. Phys. Solids* **8**, 66–75 (1960).
104. DANG DINH AN, Elastic waves by a force moving along a crack, *J. Math. and Phys.* **38**, 246–256 (1960).
105. BAKER, B. R., Dynamic stresses created by a moving crack, *Paper presented at the Xth Intern. Congr. Appl. Mech., Stresa* (1960).
106. RADOK, J. R. M., On the solutions of problems of dynamic plane elasticity, *Quart. Appl. Math.* **14**, 289–298 (1956).
107. BROBERG, K. B., The propagation of a brittle crack, *Paper presented at the Xth Intern. Congr. Appl. Mech., Stresa* (1960).
108. BROBERG, K. B., The propagation of a brittle crack, *Arkiv för Fysik* **18**, 159–129 (1960).
109. BILBY, B. A., and BULLOUGH, R., The formation of twins by a moving crack, *Phil. Mag., VII ser.* **45**, 631–646 (1954).
110. McCLINTOCK, F. A., and SUKHATME, S. P., Travelling cracks in elastic materials under longitudinal shear, *J. Mech. Phys. Solids* **8**, 187–193 (1960).
111. SUITS, J. C., Cleavage, ductility and tenacity in crystals. Discussion, *in* "Fracture" (B. L. Anderson, and oth., eds.), pp. 223–224. Wiley, New York, 1959.