Example: Lamé Problem.

As an example of boundary value problems, consider a spherical cavity in a large body, remote hydrostatic tension. The symmetry of the problem makes the use of a spherical coordinate system convenient.

a. List nonzero quantities.

- the radial displacement u,
- the radial stress σ_r, two equal hoop stresses $\sigma_\theta = \sigma_\phi$,
- the radial strain ε_r, two equal hoop strains $\varepsilon_\theta = \varepsilon_\phi$.
- They are all functions of *r*.

b. List equations. Use the basic equation sheet and simplify taking into account the symmetry of the problem:

- Equilibrium equation: $$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0.$$
- Deformation geometry: $$\varepsilon_r = \frac{du}{dr}; \qquad \varepsilon_\theta = \frac{u}{r}.$$
- Material law: $$\varepsilon_r = \frac{1}{E}\left(\sigma_r - 2\nu\sigma_\theta\right); \quad \varepsilon_\theta = \frac{1}{E}\left[\left(1-\nu\right)\sigma_\theta - \nu\sigma_r\right].$$

c. Reduce to a single ODE. The above are a set of 5 equations for 5 functions of *r*. You can follow a number of approaches to solve them. We'll take the approach below. We want to obtain a single equation in the radial stress, σ_r. From the equilibrium equation, we express σ_θ in terms of σ_r:

$$\sigma_\theta = \sigma_r + \frac{r}{2}\frac{d\sigma_r}{dr}.$$

Then we use the material law to express both strains in terms of σ_r:

$$\varepsilon_r = \frac{1}{E}\left[(1-2\nu)\sigma_r - \nu r \frac{d\sigma_r}{dr}\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[(1-2\nu)\sigma_r + (1-\nu)\frac{r}{2}\frac{d\sigma_r}{dr}\right]$$

We can eliminate *u* from the two equations for deformation geometry. This results in an equation in terms of the two strains:

$$\varepsilon_r = d(r\varepsilon_\theta)/dr\,.$$

Express this equation in terms of the radial stress, and we get

$$\frac{d^2\sigma_r}{dr^2} + \frac{4}{r}\frac{d\sigma_r}{dr} = 0\,.$$

d. Solve the ODE. This is an **equidimensional** equation. The solution is of the form $\sigma_r = r^m$. Substitute $\sigma_r = r^m$ into the ODE, and we find two roots: $m = 0$ and $m = -3$. Consequently, the full solution is

$$\sigma_r = A + \frac{B}{r^3}\,,$$

where *A* and *B* are constants to be determined by the boundary conditions. The hoop stress is given by

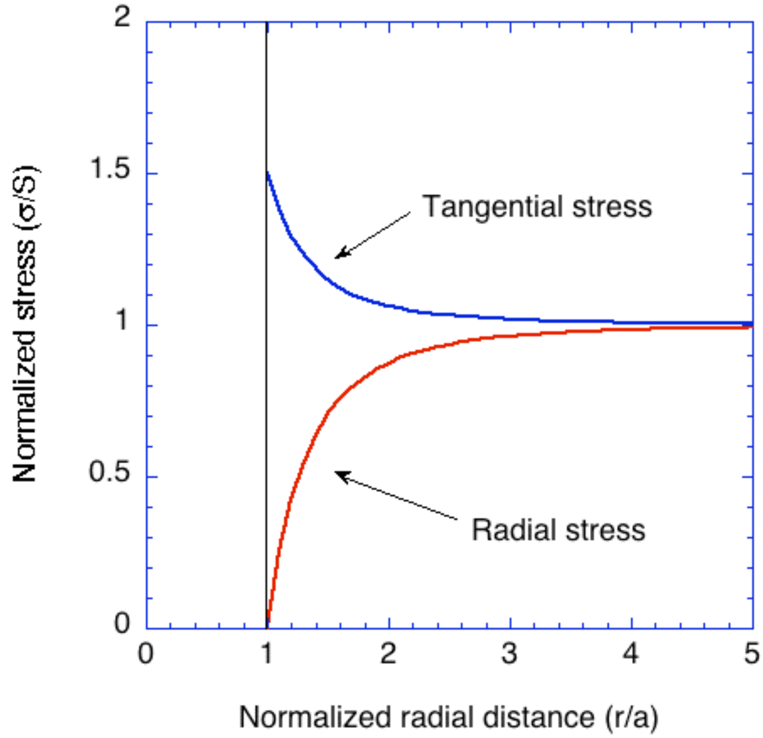
$$\sigma_\theta = A - \frac{B}{2r^3}\,.$$

The boundary conditions are

- Prescribed remote stress: $\sigma_r = S$ as $r = \infty$.
- Traction-free at the cavity surface: $\sigma_r = 0$ as $r = a$

Upon determining the two constants *A* and *B*, we obtain the stress distribution

$$\sigma_r = S\left[1 - \left(\frac{a}{r}\right)^3\right],\ \sigma_\theta = S\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right].$$



Stress concentration factor. Note that the tangential stress is nonzero near the cavity surface, where it stress reaches a maximum. The stress concentration factor is the ratio of the maximum stress over the applied stress. In this case, the stress concentration factor is 3/2.

A note on compatibility

The six independent components of strain at each point are completely determined by the displacement field ***u*** (u, v, w). As a result it is not possible to choose the six strain components independently – they have to satisfy a number of equations that arise when the displacement components are eliminated from the equations defining the strain components. In other words, one can ask the following question: If I choose six strain components (or six functions corresponding to these components), does there exist a corresponding displacement field. The answer to this question is simple: if I choose the functions arbitrarily, there does not generally exist a corresponding displacement field.

The conditions that need to be satisfied by the strain components in order to get a corresponding displacement field can be found by eliminating the displacement components u, v, w from the definitions of the strains. From the definitions of the various strain components, we get:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2}, \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial y \partial x^2},$$

from which we find that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

Two more relations of the same kind can be found by cyclical permutation of the letters x, y and z. From the derivatives

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z}, \qquad \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z},$$
$$\frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^3 w}{\partial x \partial y}, \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y},$$

it follows further that

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right).$$

Two more equations of this kind can be derived through cyclical permutation of the coordinates. These equations are known as the compatibility equations or the conditions of compatibility because they tell you under what conditions a real displacement field exists. One can show that for simply connected regions (i.e., materials without holes), the compatibility equations are necessary and sufficient to assure that the displacements exist and are single-valued. If the region is multiply connected, additional conditions need to be imposed. It should be emphasized that these equations are completely superfluous when you treat the displacements as variables in the problem; you only need them if you want to use the strain components instead.

Principal Stress

Imagine a material particle in a state of stress. The state of stress is fixed, but we can represent the material particle in many ways by cutting cubes in different orientations. For any given state of stress, it is always possible to cut a cube in a suitable orientation, such that the stress components on all the cube faces are normal to the faces, and there are no shear stresses on the cube faces. These cube faces are called the principal planes, the normal vectors to these faces the principal directions, and the stresses on them the principal stresses.

Examples

- Uniaxial tension
- Equal biaxial tension
- Hydrostatic pressure
- Pure shear is the same state of stress as the combination of pulling and pressing in 45°.

Given a stress state, how to find the principal stresses? When a plane is the principal plane, the traction on the plane is normal to the plane, namely, the traction vector $\boldsymbol{t}$ must be in the same direction as the unit normal vector $\boldsymbol{n}$. Let the magnitude of the traction be σ. On the principal plane, the traction vector is in the direction of the normal vector, $\mathbf{t} = \sigma\mathbf{n}$. Write in the matrix notion, and we have

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

Here both $\boldsymbol{t}$ and $\boldsymbol{n}$ are vectors, but σ is a scalar representing the magnitude of the principal stress.

Recall that the traction vector is the stress matrix times the normal vector, so that

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

This is an eigenvalue problem. When the stress state is known, i.e., the stress matrix is given, solve the above eigenvalue problem to determine the eigenvalue σ and the eigenvector ***n***. The eigenvalue σ is the principal stress, and the eigenvector ***n*** is the principal direction.

Linear algebra of eigenvalues. Because the stress tensor is a 3 by 3 symmetric matrix, you can always find three real eigenvalues, i.e., principal stresses, $\sigma_a, \sigma_b, \sigma_c$. We distinguish three cases:

(1) If the three principal stresses are unequal, the three principal directions are orthogonal (e.g., pure shear state).

(2) If two principal stresses are equal, but the third is different, the two equal principal stresses can be in any directions in a plane, and the third principal direction is normal to the plane (e.g., pure tensile state).

(3) If all the three principal stresses are equal, any direction is a principal direction. This stress state is called a hydrostatic state.

Maximum normal stress. Why do we care about the maximum normal stress? Chalk is made of a brittle material: it break by tensile stress, not by shear stress. When chalk is under bending, the tensile stress is along the axial direction of the chalk, so that the chalk breaks on a plane normal to the axial direction. When chalk is under torsion, the maximum tensile stress is 45° from the axial direction, so that it breaks in a direction 45° from the axial direction. (The fracture surface of the chalk under torsion is not a plane, because of some 3D effects.) We care about the principal stress because brittle materials fail by tensile stress, and we want to find the maximum tensile stress.

Let's order the three principal stresses as $\sigma_a \le \sigma_b \le \sigma_c$. This ordering takes into consideration the signs: a compressive stress (negative) is smaller than a tensile stress (positive). On an arbitrary plane, the traction vector may be decomposed into two components: one component normal the plane (the normal stress), and the other component parallel to the plane (the shear stress). Obviously, when you look at a plane with a different normal vector, you find different normal and shear stresses. You will be delighted by the following theorem:

Of all planes, the principal plane corresponding to σ_c has the maximum normal stress

Maximum shear stress. Why do we care about the maximum shear stress? Most metals are ductile materials: they fail by plastic yielding. When a material is under a complex stress state, it is known empirically that yielding first occurs on a plane with maximum shear stress. To find the maximum shear stress and the particular plane, you are helped by the following theorem:

The maximum shear stress is $\tau_{\max} = (\sigma_c - \sigma_a)/2$. The maximum shear stress $\tau_{\max}$ acts on a plane with the normal vector 45° from the principal directions n_a and n_c.

A proof of the above theorems is outlined below:

Consider a system of coordinates that coincide with three orthogonal directions of the principal stresses, $\sigma_a, \sigma_b, \sigma_c$. Then consider an arbitrary plane whose unit normal vector has components n_1, n_2, n_3 in this coordinate system. The components of the stress tensor in this coordinate system is

$$\begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_b & 0 \\ 0 & 0 & \sigma_c \end{bmatrix}.$$

Thus, on the plane with unit vector (n_1, n_2, n_3), the traction vector is $(\sigma_a n_1, \sigma_b n_2, \sigma_c n_3)$. The normal stress on the plane is

$$\sigma_n = \sigma_a n_1^2 + \sigma_b n_2^2 + \sigma_c n_3^2 .$$

We need to maximize σ_n under the constraint that $n_1^2 + n_2^2 + n_3^2 = 1$.

The shear stress on the plane τ is given by

$$\tau^2 = (\sigma_a n_1)^2 + (\sigma_b n_2)^2 + (\sigma_c n_3)^2 - (\sigma_a n_1^2 + \sigma_b n_2^2 + \sigma_c n_3^2)^2 .$$

We need to maximize τ under the constraint that $n_1^2 + n_2^2 + n_3^2 = 1$.

Coordinate transformations and tensors

1. The direction-cosine matrix relating two bases.

In 3D-space, let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be an orthonormal basis, namely,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} .$$

The base vectors are ordered to follow the right-hand rule.

Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be a new orthonormal basis, namely,

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta} .$$

Let the angle between the two vectors $\mathbf{e}_i$ and $\mathbf{e}'_\alpha$ be $\theta_{i\alpha}$. Denote the **direction cosine** of the two vectors by

$$l_{i\alpha} = \mathbf{e}_i \cdot \mathbf{e}'_\alpha = \cos\theta_{i\alpha} .$$

We follow the convention that the first index of $l_{i\alpha}$ refers to a coordinate in the old basis, and the second to a coordinate in the new basis. For the two bases, there are a total of 9 direction cosines. We can list $l_{i\alpha}$ as a 3 by 3 matrix. By our convention, the rows refer to the old basis, and the columns to the new basis.

Note that $l_{i\alpha}$ is the component of the vector $\mathbf{e}'_\alpha$ projected on the vector $\mathbf{e}_i$. We can express each new base vector as a linear combination of the three old base vectors:

$$\mathbf{e}'_\alpha = l_{1\alpha}\mathbf{e}_1 + l_{2\alpha}\mathbf{e}_2 + l_{3\alpha}\mathbf{e}_3 .$$

If you are tired of writing sums like this, you abbreviate it as

$$\mathbf{e}'_\alpha = l_{i\alpha}\mathbf{e}_i ,$$

with the convention that a repeated index implies summation over 1,2,3. Because the sum is the same whatever the repeated index is named, such an index is called a dummy index.

Similarly, we can express the old basis as a linear combination of the new basis:

$$\mathbf{e}_i = l_{i1}\mathbf{e}'_1 + l_{i2}\mathbf{e}'_2 + l_{i3}\mathbf{e}'_3 .$$

Using the summation convention, we write more concisely as

$$\mathbf{e}_i = l_{i\alpha}\mathbf{e}'_\alpha .$$

2. Transformation of components of a vector due to change of basis.

Let **f** be a vector. It is a linear combination of the base vectors:

$$\mathbf{f} = f_i\mathbf{e}_i ,$$

where f_1, f_2, f_3 are the components of the vector, and are commonly written as a column. Consider the vector pointing from Cambridge to Boston. When the basis is changed, the vector between Cambridge and Boston remains unchanged, but the components of the vector do change. Let f'_1, f'_2, f'_3 be the components of the vector **f** in the new basis, namely,

$$\mathbf{f} = f'_\alpha \mathbf{e}'_\alpha$$

Recall the transformation between the two sets of basis, $\mathbf{e}_i = l_{i\alpha}\mathbf{e}'_\alpha$, we write that

$$\mathbf{f} = f_i\mathbf{e}_i = f_i l_{i\alpha}\mathbf{e}'_\alpha$$

A comparison between the two expressions gives that

$$f'_\alpha = f_i l_{i\alpha} .$$

Thus, the component column in the new basis is the *transpose* of the direction-cosine matrix times the component column in the old basis.

Similarly, we can show that

$$f_i = l_{i\alpha} f'_\alpha$$

The component column in the old basis is the direction-cosine matrix times the component column in the old basis.

3. Transformation of stress components due to change of basis.

The **stress tensor, σ,** describes the *state of stress* suffered by a material particle. Represent the material particle by a cube. The stress components are the force per unit area on 6 faces of the cube. The stress tensor is represented by a 3 by 3 symmetric matrix. The state of stress of a material particle is a physical object, and is independent of your choice of the basis (i.e., how you cut a cube to represent the particle). However, the **components of the stress tensor** do depend on your choice of the basis. How do we transform the stress components when the basis is changed?

Consider the stress state of a material particle, and the traction vector on a given plane. In the old basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, denote the components of the stress state by σ_{ij}, the components of the unit vector normal to the plane by n_j, and the components of the traction vector on the plane by t_i. Using the summation convention, we write the traction-stress equations as

$$t_i = \sigma_{ij} n_j .$$

Recall that we obtained this relation by the balance of forces on a tetrahedron. In the language of linear algebra, we call the stress as a linear operator that maps the unit normal vector of a plane to the traction vector acting on the plane.

Similarly, in the new basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, denote the components of the stress state by $\sigma'_{\alpha\beta}$, the components of the unit vector normal to the plane by n'_β, and the components of the traction vector on the plane by t'_α. Force balance requires that

$$t'_\alpha = \sigma'_{\alpha\beta} n'_\beta . \qquad \text{(a)}$$

We now examine the relations between the components in the old basis and those in the new basis. The traction is a vector, so that its components transform as $t'_\alpha = l_{i\alpha} t_i$. Insert $t_i = \sigma_{ij} n_j$ into the above, and we obtain that $t'_\alpha = l_{i\alpha} \sigma_{ij} n_j$. The unit normal vector transforms as $n_j = l_{j\beta} n'_\beta$. Consequently, we obtain that

$$t'_\alpha = l_{i\alpha} \sigma_{ij} l_{j\beta} n'_\beta . \qquad \text{(b)}$$

Equations (a) and (b) are valid for any choice of the plane. Consequently, we must require that

$$\sigma'_{\alpha\beta} = l_{i\alpha} \sigma_{ij} l_{j\beta} .$$

Thus, the stress-component matrix in the new basis is the product of three matrices: the transpose of the direction-cosine matrix, the stress-component matrix in the old basis, and the direction-cosine matrix.

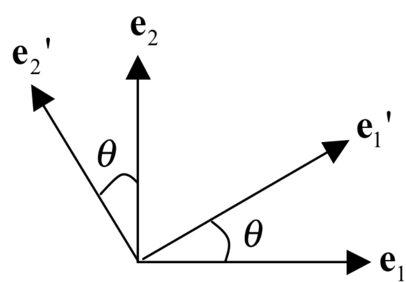
A special case: the new basis and the old basis differ by an angle θ around the axis $\mathbf{e}_3$

The sign convention for θ follows the right-hand rule. The direction cosines are

$$\mathbf{e}_1 \cdot \mathbf{e}'_1 = \cos\theta, \mathbf{e}_1 \cdot \mathbf{e}'_2 = -\sin\theta, \mathbf{e}_1 \cdot \mathbf{e}'_3 = 0,$$
$$\mathbf{e}_2 \cdot \mathbf{e}'_1 = \sin\theta, \mathbf{e}_2 \cdot \mathbf{e}'_2 = \cos\theta, \mathbf{e}_2 \cdot \mathbf{e}'_3 = 0,$$
$$\mathbf{e}_3 \cdot \mathbf{e}'_1 = 0, \mathbf{e}_3 \cdot \mathbf{e}'_2 = 0, \mathbf{e}_3 \cdot \mathbf{e}'_3 = 1.$$

Consequently, the matrix of the direction cosines is

$$[l_{i\alpha}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The components of a vector transform as

$$\begin{bmatrix} f'_1 \\ f'_2 \\ f'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Thus,

$$f'_1 = f_1 \cos\theta + f_2 \sin\theta$$
$$f'_2 = -f_1 \sin\theta + f_2 \cos\theta$$
$$f'_3 = f_3$$

The components of a stress state transform as

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$\sigma'_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta$$
$$\sigma'_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta - \sigma_{12}\sin 2\theta$$
$$\sigma'_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta$$
$$\sigma'_{13} = \sigma_{13}\cos\theta + \sigma_{23}\sin\theta$$
$$\sigma'_{23} = -\sigma_{13}\sin\theta + \sigma_{23}\cos\theta$$
$$\sigma'_{33} = \sigma_{33}$$

State of plane stress.

An even more special case is that the stress components out of the plane are absent, namely, $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$.

$$\sigma'_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta$$
$$\sigma'_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta - \sigma_{12}\sin 2\theta$$
$$\sigma'_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta$$

In Beer's Section 7.4, these equations are represented by graphically (i.e., Mohr's circle). In this case, $\mathbf{e}_3$ is one principal direction, and the principal stress in this direction is zero. To find the other two principal directions, we set $\sigma'_{12} = 0$, so that the two principal directions are at the angle θ_p from the $\mathbf{e}_1$ - and $\mathbf{e}_2$ -directions. This angle is given by

$$\tan 2\theta_p = \frac{2\tau_{12}}{\sigma_{11} - \sigma_{22}}.$$

The two principal stresses are given by $\frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$.

We need to order these two principal stresses and the zero principal stress in the $\mathbf{e}_3$ direction. The maximum shear stress is the half of the difference between the maximum principal stress and the minimum principal stress.

4. Scalars, vectors, and tensors.

When the basis is changed, a scalar (e.g., temperature, energy, and mass) does not change, the components of a vector transform as

$$f'_\alpha = f_i l_{i\alpha},$$

and the components of a tensor transform as

$$\sigma'_{\alpha\beta} = \sigma_{ij} l_{i\alpha} l_{j\beta}.$$

This transformation defines the second-rank tensor. By analogy, a vector is a first-rank tensor, and a scalar is a zeroth-rank tensor. We can also similarly define tensors of higher ranks.

Multilinear algebra. The rule of the above transformation is based on only one fact: the stress is a linear map from one vector to another vector. You can generate a new tensor from a linear map from one tensor to another tensor. You can also generate a tensor by a bilinear form, e.g., a bilinear map from two vectors to a scalar. Of course, a multilinear map of several tensors to a tensor is yet another tensor. Consequently, all tensors follow a similar rule under a change of basis. We write this rule again for a third-rank tensor:

$$g'_{\alpha\beta\gamma} = g_{ijk} l_{i\alpha} l_{j\beta} l_{k\gamma}.$$

Invariants of a tensor. A scalar is invariant under any change of basis. When the basis changes, the components of a vector change, but the length of the vector is invariant. Let $\mathbf{f}$ be a vector, and f_i be the components of the vector for a given basis. The length of the vector is the square root of

$$f_i f_i.$$

The index i is dummy. Thus, this combination of the components of a vector is a scalar, which is invariant under any change of basis. For a vector, there is only one independent invariant. Any other invariant of the vector is a function of the length of the vector.

This observation can be extended to high-order tensors. By definition, **an invariant of a tensor is a scalar formed by a combination of the components of the tensor.** For example, for a symmetric second-rank tensor σ_{ij}, we can form three independent invariants:

$$\sigma_{ii}, \quad \sigma_{ij}\sigma_{ij}, \quad \sigma_{ij}\sigma_{jk}\sigma_{ki}.$$

In each case, all indices are dummy, resulting in a scalar. Any other invariant of the tensor is a function of the above three invariants.

Exercise. For a nonsymmetric second-rank tensor, give all the independent invariants. Write each invariant using the summation convention, and then explicitly in all its terms.

Exercise. Give all the independent invariants of a third-rank tensor.

The state of strain of a material particle is a second-rank tensor.

In the old basis, the coordinates of a material particle are (x_1, x_2, x_3), and the components of the displacement field are $u_i(x_1, x_2, x_3, t)$. In the new basis, the coordinates of the same material particle are (x'_1, x'_2, x'_3), and the components of the displacement field are $u'_\alpha(x'_1, x'_2, x'_3, t)$. Recall that $u'_\alpha = l_{i\alpha} u_i$ and $x_j = l_{j\beta} x'_\beta$, so that

$$\frac{\partial u'_\alpha}{\partial x'_\beta} = l_{i\alpha} \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial x'_\beta} = l_{i\alpha} l_{j\beta} \frac{\partial u_i}{\partial x_j}.$$

Thus, the gradients of the displacement field, $\partial u_i / \partial x_j$, form the components of a second-rank tensor. Consequently, the state of strain, being the symmetric part of the displacement gradient, is a second-rank tensor. When the basis is changed, the components of the strain state transform as

$$\varepsilon'_{\alpha\beta} = \varepsilon_{ij} l_{i\alpha} l_{j\beta}.$$