$$(\nabla\wedge \underset{\sim}{W})_{ij} = \varepsilon_{ipq} W_{jq,p} = \varepsilon_{ipq}\varepsilon_{jqk} v_{k,p}$$

$$= -\varepsilon_{qip}\varepsilon_{qjk} v_{k,p} = -(\delta_{ij}\delta_{pk} - \delta_{ik}\delta_{pj}) v_{k,p}$$

$$= -\underbrace{\delta_{ij} v_{k,k}}_{\text{zero by (3)}} + v_{i,j} = v_{i,j}$$

whence $\nabla\wedge \underset{\sim}{W} = \nabla \underset{\sim}{v}$, so that

$$\nabla\wedge \underset{\sim}{W} = \nabla \underset{\sim}{v} = \underline{I} \text{ on } D.$$

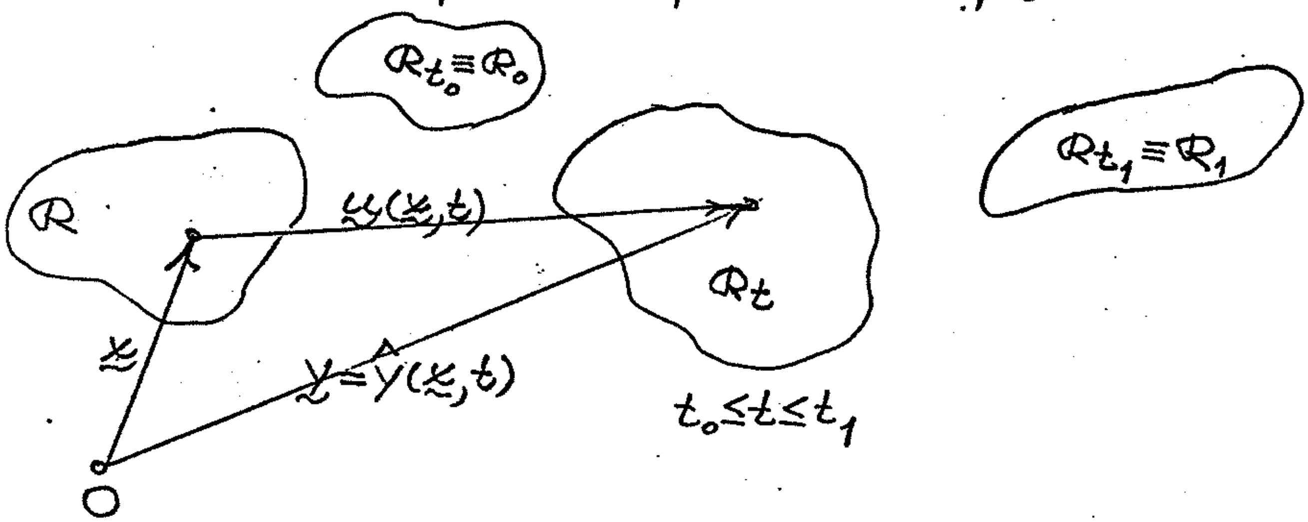
2. Kinematics of motions and deformations of a continuous medium

Objective: Study of purely geom. aspects of motion of an arb. continuous medium.

Introduction. A motion of a "body" (continuous medium) relative to a given $E_3 = E$ is specifie[d] if we know the position in the reference space [of] all material points (particles) of body at every instant during the motion.

A motion of a body (cont. medium) $\mathcal{B}$ relative to E is fully specified if we know the position in E of each material point of $\mathcal{B}$ throughout some interval of time (duration of motion).

Let $\mathcal{R} \subseteq E$ be a closed region occupied by $\mathcal{B}$ in some specified "reference configuration".



$\underset{\sim}{x}$ ... position vector in $\mathcal{R}$ of generic mat. point of $\mathcal{B}$

$\hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$ ... position vector at time t of particle with position vector $\underset{\sim}{x}$ in $\mathcal{R}$.

$\underset{\sim}{u}(\underset{\sim}{x},t)$ ... displacement vector (explain)

$$\left.\begin{aligned} \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x},t) \ \forall\ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T},\ \mathcal{T}=[t_0,t_1] \\ \hat{\underset{\sim}{y}}(\mathcal{R},t) = \mathcal{R}_t\ \ (t_0 \le t \le t_1),\ \mathcal{R}_{t_0}=\mathcal{R}_0,\ \mathcal{R}_{t_1}=\mathcal{R}_1 \end{aligned}\right\} ($$

Note that $\mathcal{R}$ may, but need not, coincide with $\mathcal{R}_t$ for some $t \in \mathcal{T}$. In particular, the reference config. may, but need not, coincide with the initial configuration.

Let $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ be a coord. frame for

$x_i = \underset{\sim}{x} \cdot \underset{\sim}{e}_i$ ... material (Lagrangian) coords. in X of $\mathcal{B}$

$y_i = \hat{\underset{\sim}{y}}(\underset{\sim}{x}, t) \cdot \underset{\sim}{e}_i$ ... spatial (Eulerian) coords. in X of $\mathcal{B}$

Comment on misnomers (Euler, D'Alembert)

<u>Definition</u>. Let $\mathcal{R} \subset E$ be the closed region occupied by a body $\mathcal{B}$ in a reference configuration. A <u>motion</u> $\hat{\underset{\sim}{y}}(\cdot, t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ of $\mathcal{B}$ relative to E is a one-parameter family of mappings of $\mathcal{R}$ into E, depending on the time as parameter and specified by (2.1). The motion is <u>admissible</u> (regular) if:

(i) $\hat{y} \in C^2(\mathcal{R} \times \mathcal{T})$

(ii) the mapping $\hat{\underset{\sim}{y}}(\cdot, t)$ is one-to-one on $\mathcal{R}$ $\forall t \in$

(iii) $J \equiv \det \nabla \underset{\sim}{y} = \det [y_{i,j}] > 0$ on $\mathcal{R} \times \mathcal{T}$

Implications, discussion

(a) (i), (ii) $\Rightarrow \exists$ inverse mapping $\hat{\underset{\sim}{y}}^{-1}(\cdot,t) \equiv \hat{\underset{\sim}{x}}(\cdot,t)$ defined on $\mathcal{R}_t$ $\forall\, t \in \mathcal{T} \ni$

$$\left.\begin{aligned} \underset{\sim}{x} &= \hat{\underset{\sim}{x}}(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t) \ \forall\ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}, \\ \underset{\sim}{y} &= \hat{\underset{\sim}{y}}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t) \ \forall\ \underset{\sim}{y} \in \mathcal{R}_t \quad (t_0 \le t \le t_1). \end{aligned}\right\} \quad (2.2)$$

Further, in view of (iii) & inverse function thm (see Calculus books),

$$\hat{\underset{\sim}{x}}(\cdot,t) \in \mathcal{C}^2(\mathcal{R}_t) \ \forall\ t \in \mathcal{T}$$

(b) (i), (ii) $\Rightarrow \mathcal{R}_t$ is a closed region in E $\forall\, t \in \mathcal{T}$. Regions, surfaces, curves $\subset \mathcal{R}$ are mapped onto such in $\mathcal{R}_t$ and viceversa $\forall\, t \in \mathcal{T}$. No tearing of $\mathcal{B}$. Also, two distinct material points cannot occupy same position at any one instant ("impenetrability")

(c) $\exists$ one-to-one correspondence between local directions in $\mathcal{R}$ and $\mathcal{R}_t$ $\forall\, t \in \mathcal{T}$. Explain by considering images of two arcs with common local tangent, etc.

(d) $\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$ $(t_0 \le t \le t_1)$ for fixed $\underset{\sim}{x} \in \mathcal{R}$ is the parametric eq. of path of material point with po $\underset{\sim}{x} \in \mathcal{R}$. (i) assures existence of velocity and acceleration fields:

$$\left.\begin{aligned}
&\underset{\sim}{v}(\underset{\sim}{x},t) = \frac{\partial}{\partial t}\hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \dot{\hat{\underset{\sim}{y}}}(\underset{\sim}{x},t), \\
&\underset{\sim}{a}(\underset{\sim}{x},t) = \frac{\partial^2}{\partial t^2}\hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \dot{\underset{\sim}{v}}(\underset{\sim}{x},t) \ \forall \ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}; \\
&\underset{\sim}{v}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t) = \overline{\underset{\sim}{v}}(\underset{\sim}{y},t) \equiv \underset{\sim}{q}(\underset{\sim}{y},t), \\
&\underset{\sim}{a}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t) = \overline{\underset{\sim}{a}}(\underset{\sim}{y},t) \ \forall \ \underset{\sim}{y} \in \mathcal{R}_t \ (t_0 \le t \le t_1)
\end{aligned}\right\} (2.$$

Interpret $\overline{\underset{\sim}{v}}(\underset{\sim}{y},t)$, $\overline{a}(\underset{\sim}{y},t)$. Note from (i) that

$$\underset{\sim}{v} \in \mathcal{C}^1(\mathcal{R}\times\mathcal{T}),\ \underset{\sim}{a} \in \mathcal{C}(\mathcal{R}\times\mathcal{T})$$

(e) Mention need for relaxation of regularity assumptions (singular problems) and occasional need for additional regularity requirements.

Deformation gradient, displacement gradient and Jacobian determinant

Consider an admissible motion $\hat{\underset{\sim}{y}}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le$ and define

$$\underset{\sim}{F} = \nabla\hat{\underset{\sim}{y}} = \underset{\sim}{1} + \nabla\underset{\sim}{u} \quad \text{or} \quad F_{ij} = \hat{y}_{i,j} = \delta_{ij} + u_{i,j} \text{ on } \mathcal{R}\times\mathcal{T} \quad (2$$

One calls $\underset{\sim}{F}(\cdot,t)$ & $\nabla\underset{\sim}{u}(\cdot,t)$ respectively the deformation grad. field & dispt. grad field at time t; both are two-tensor fields (explain). So,

$$J = \det[\hat{y}_{i,j}] = \det \nabla\hat{\underset{\sim}{y}} = \det \underset{\sim}{F} \text{ on } \mathcal{R}\times\mathcal{T}. \quad (2.5$$

$J(\cdot,t)$ is evidently a scalar field.

<u>Claim</u>. If $\hat{\underset{\sim}{x}}(\cdot,t) = \hat{\underset{\sim}{y}}^{-1}(\cdot,t)$ on $\mathcal{R}_t$ $(t_0 \le t \le t_1)$, one ha

$$\nabla\hat{\underset{\sim}{x}}(\underset{\sim}{y},t) = \underset{\sim}{F}^{-1}(\underset{\sim}{x},t) \;\forall\; (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T},\; \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) \quad (2.6)$$

To see this, recall from (2.2) that

$$y_i = \hat{y}_i(\underset{\sim}{x},t) = \hat{y}_i(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t) \;\forall\; \underset{\sim}{y}\in\mathcal{R}_t \;(t_0\le t\le t_1) \quad (\S)$$

Differentiate (§) w.r. y_j and use chain-rule to get

$$\delta_{ij} = \frac{\partial\hat{y}_i}{\partial x_k}\frac{\partial\hat{x}_k}{\partial y_j} \quad \text{or} \quad \underset{\sim}{F}(\underset{\sim}{x},t)\nabla\hat{\underset{\sim}{x}}(\underset{\sim}{y},t) = \underset{\sim}{1}, \quad (*)$$

which implies (2.6).

<u>Remark</u>. If in the def. of admiss. motion we replace (iii), i.e. $J \equiv \det \underset{\sim}{F} > 0$, by

$$\hat{\underset{\sim}{x}}(\cdot,t) \equiv \hat{\underset{\sim}{y}}^{-1}(\cdot,t) \in \mathcal{C}^1(\mathcal{R}_t) \quad (t_0 \le t \le t_1),$$

then (*) follows again & implies $J \equiv \det \underset{\sim}{F} \neq 0$ on $\mathcal{R} \times \mathcal{T}$. If further $\underset{\sim}{u}(\underset{\sim}{x},t_0) = \underset{\sim}{0}\ \forall\, \underset{\sim}{x} \in \mathcal{R}$, so that $\mathcal{R} = \mathcal{R}_0$; $\underset{\sim}{F}(\underset{\sim}{x},t_0) = \underset{\sim}{1}$, $J(\underset{\sim}{x},t_0) = 1$, whence by (i) $J > 0$ on $\mathcal{R} \times \mathcal{T}$. Finally, (2.6) in view of (i) and Exercise 4 now assure that

$$\hat{\underset{\sim}{x}}(\cdot,t) \in \mathcal{C}^2(\mathcal{R}_t) \quad (t_0 \le t \le t_1).$$

<u>Deformations of a body (continuous medium)</u>

Many aspects of kinematics of cont. media concern merely a comparison of an <u>instantaneous</u> configuration of $\mathcal{B}$ during a motion with a reference configuration of $\mathcal{B}$. In dealing with such issues one need consider merely the mapping

$$\hat{\underset{\sim}{y}}(\cdot,t_*) : \mathcal{R} \to \mathcal{R}_{t_*} = \mathcal{R}_* \quad \text{for fixed } t_* \in \mathcal{T}$$

This leads us to introduce the following def.

<u>Definitions</u>. Let $R \subset E$ have the same meaning as before. A <u>deformation</u> $\hat{\mathbf{y}}: R \to R_*$ of $\mathcal{B}$ is a mapping of R into E, specified by

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \ \forall \mathbf{x} \in R, \ \hat{\mathbf{y}}(R) = R_*. \quad (2.7)$$

The deformation is <u>admissible</u> (regular), provided

(i) $\hat{\mathbf{y}} \in \mathcal{C}^2(R)$

(ii) the mapping $\hat{\mathbf{y}}$ is one-to-one on R.

(iii) $J = \det \nabla \mathbf{y} = \det [y_{i,j}] > 0$ on R

Note that an admiss. motion of $\mathcal{B}$ is a suitably regular one-parameter family of admiss. deformations of $\mathcal{B}$, depending on time as parameter.

(ii), (iii) $\Rightarrow \exists \ \hat{\mathbf{x}} = \hat{\mathbf{y}}^{-1} \in \mathcal{C}^2(R_*) \ni$

$$\mathbf{x} = \hat{\mathbf{x}}(\hat{\mathbf{y}}(\mathbf{x})) \ \forall \mathbf{x} \in R, \ \mathbf{y} = \hat{\mathbf{y}}(\hat{\mathbf{x}}(\mathbf{y})) \ \forall \mathbf{y} \in R_* \quad (2.8)$$

Define

$$\mathbf{F} = \nabla \hat{\mathbf{y}} = \mathbf{1} + \nabla \mathbf{u} \ \text{on } R \ \text{(def. grad field)} \quad (2.$$

Then

$$J = \det \mathbf{F} > 0 \ \text{on } R, \ \mathbf{F}^{-1}(\mathbf{x}) = \nabla \hat{\mathbf{x}}(\hat{\mathbf{y}}(\mathbf{x})) \ \forall \mathbf{x} \in R \quad (2$$

Cf. discussion of admiss. motions.

Definition

(a) An admissible deformation $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ is *locally volume preserving* (l.v.p) if for *every* compact region $\mathcal{R}' \subset \mathcal{R}$, the volume of $\mathcal{R}'$ equals the volume of $\mathcal{R}'_* = \hat{\underset{\sim}{y}}(\mathcal{R}')$. (explain)

(b) A deformation $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ is *distance-preserving*

$$|\hat{\underset{\sim}{y}}(\underset{\sim}{x}) - \hat{\underset{\sim}{y}}(\underset{\sim}{z})| = |\underset{\sim}{x} - \underset{\sim}{z}| \quad \forall\ (\underset{\sim}{x}, \underset{\sim}{z}) \in \mathcal{R} \times \mathcal{R}.$$

A deformation is *rigid* if distance preserving *and* admissible.

Remarks. Illustrate distinction in (b). Let $\mathfrak{X} \in \mathcal{F}$, $\mathcal{R}$ the sphere $|\underset{\sim}{x}| \le 1$ and let $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}$ be given by

$$y_1 = \hat{y}_1(\underset{\sim}{x}) = x_1, \quad y_2 = \hat{y}_2(\underset{\sim}{x}) = x_2, \quad y_3 = \hat{y}_3(\underset{\sim}{x}) = -x_3 \quad \forall\ \underset{\sim}{x} \in \mathcal{R},$$

where $y_i = \underset{\sim}{y} \cdot \underset{\sim}{e}_i$, $x_i = \underset{\sim}{x} \cdot \underset{\sim}{e}_i$ if $\mathfrak{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$. Clearly $\hat{\underset{\sim}{y}}$ is distance-preserving, but $J = \det \nabla \underset{\sim}{y} = -$ whence deformation is not rigid (reflection about the plane $x_3 = 0$).

Define notions of "incompressible body" and "rigid body".

__Lemma__: Let $\mathcal{R}$ be a compact region and assume

$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x})\ \forall\ \underset{\sim}{x} \in \mathcal{R}$, $\hat{\underset{\sim}{y}} \in \mathcal{C}^1(\mathcal{R})$, $\hat{\underset{\sim}{y}}$ is $(1,1)$ & $J = \det \nabla \hat{\underset{\sim}{y}} > 0$ on

Suppose $\mathcal{R}^* = \hat{\underset{\sim}{y}}(\mathcal{R})$ and $f \in \mathcal{C}(\mathcal{R}_*)$ is scalar-valued

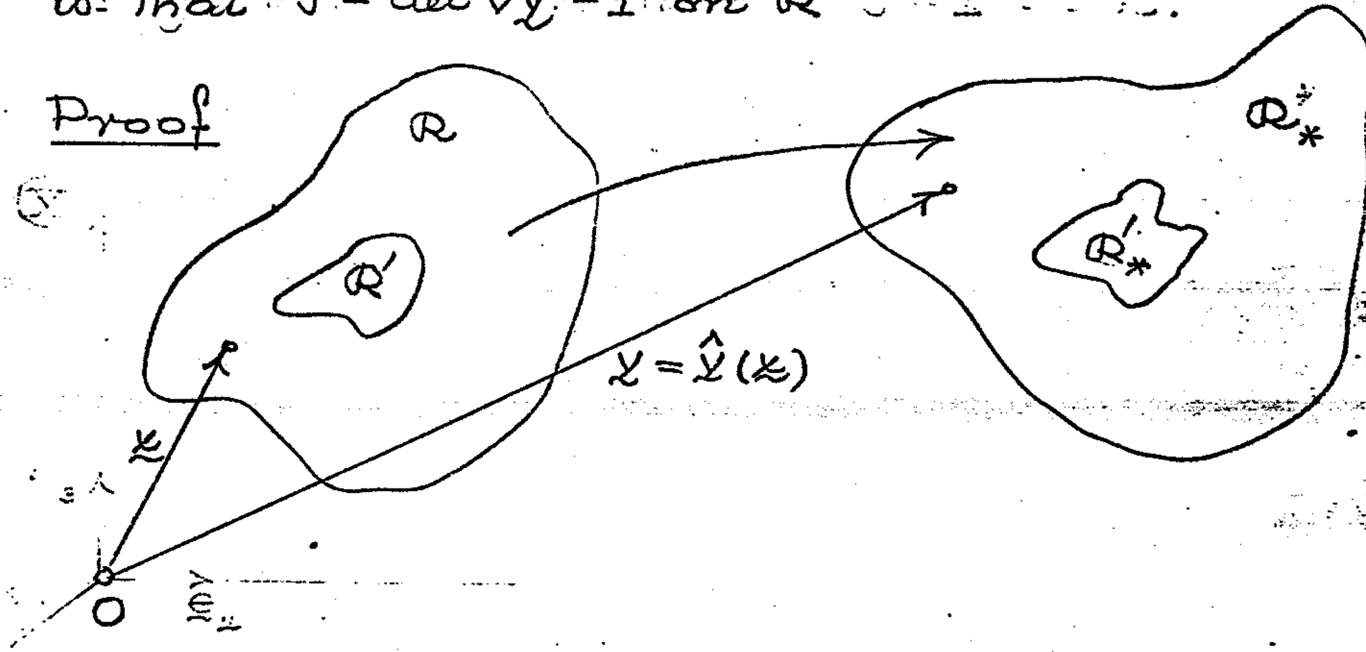
Then,

$$\int_{\mathcal{R}_*} f(\underset{\sim}{y})\, dV_{\underset{\sim}{y}} = \int_{\mathcal{R}} J(\underset{\sim}{x})\, f(\hat{\underset{\sim}{y}}(\underset{\sim}{x}))\, dV_{\underset{\sim}{x}}$$

For a proof see, e.g., Buck, Adv. Calculus, p.304 or Apostol, Math. Anal., p.270.

__Thm__. 2.1. A nec. and suff. cond. that an admissible $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ be locally volume preserving is that $J = \det \nabla \hat{\underset{\sim}{y}} = 1$ on $\mathcal{R}$

__Proof__



Let $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ be admissible.

If $\mathcal{R}' \subset \mathcal{R}$ is a compact region and $\mathcal{R}'_* = \hat{\underset{\sim}{y}}(\mathcal{R}')$ define

$$v(\mathcal{R}') = \int_{\mathcal{R}'} dV_{\underset{\sim}{x}} \ , \ v(\mathcal{R}'_*) = \int_{\mathcal{R}'_*} dV_{\underset{\sim}{y}} \quad \text{(volumes of } \mathcal{R}', \mathcal{R}'_*) \quad (1)$$

Then from the lemma, with $f(\underset{\sim}{y}) = 1 \ \forall \ \underset{\sim}{y} \in \mathcal{R}'_*$ one has

$$v(\mathcal{R}'_*) = \int_{\mathcal{R}'_*} dV_{\underset{\sim}{y}} = \int_{\mathcal{R}'} J(\underset{\sim}{x}) \, dV_{\underset{\sim}{x}} \quad (2)$$

Thus, $J = 1$ on $\mathcal{R} \Rightarrow v(\mathcal{R}'_*) = v(\mathcal{R}) \ \forall$ compact subregion $\mathcal{R}'$ of $\mathcal{R} \Rightarrow \hat{\underset{\sim}{y}}$ is l.v.p.

Conversely, suppose $\hat{\underset{\sim}{y}}$ is l.v.p.. Then, from (1),

$$v(\mathcal{R}'_*) - v(\mathcal{R}') = 0 \ \forall \text{ compact region } \mathcal{R}' \subset \mathcal{R} \quad (3)$$

Now define

$$g: \quad g(\underset{\sim}{x}) = J(\underset{\sim}{x}) - 1 \ \forall \ \underset{\sim}{x} \in \mathcal{R} \quad (4)$$

(1), (2), (3), (4) $\Rightarrow$

$$v(\mathcal{R}'_*) - v(\mathcal{R}) = \int_{\mathcal{R}'} [J(\underset{\sim}{x}) - 1] \, dV_{\underset{\sim}{x}} = \int_{\mathcal{R}'} g(\underset{\sim}{x}) \, dV_{\underset{\sim}{x}} = 0 \text{ for } \underline{\text{every}}$$

compact region $\mathcal{R}' \subset \mathcal{R}$. But $g \in \mathcal{C}(\mathcal{R})$. Therefore $g = 0$ on $\mathcal{R}$ (explain). qed.

Mention immediate counterpart of Thm 2.1 for admiss motions.

Distance preserving and rigid deformations

Motivate study of rigid deformations within the mechanics of deformable bodies.

Thm. 2.2.

(a) Let $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ be a deformation given by

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{c} \quad \forall\, \underset{\sim}{x} \in \mathcal{R} \quad (*)$$

where $\underset{\sim}{Q}$ is a const. orthogonal two-tensor & $\underset{\sim}{c}$ a const. vector. Then $\hat{\underset{\sim}{y}}$ is distance-preserving. Furth $\hat{\underset{\sim}{y}}$ is rigid if $\underset{\sim}{Q}$ is proper orthog., i.e. $\det \underset{\sim}{Q} = 1$.

(b) Conversely, if $\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_*$ is <u>any</u> deformation that preserves distance, then $\exists$ a unique orthogonal two-tensor $\underset{\sim}{Q}$ & a unique vector $\underset{\sim}{c} \ni (*)$ holds. Thus necessarily $\hat{\underset{\sim}{y}} \in C^{\infty}(\mathcal{R})$. Further, if $\hat{\underset{\sim}{y}}$ is <u>rigid</u>, i.e. distance-preserving and admissible, then $\underset{\sim}{Q}$ is <u>proper</u> orthogonal.

Emphasize weakness of hyp. in (b): no smoothness or invertibility assumptions are made! Continuity of $\hat{\underset{\sim}{y}}$ is immediate since $\hat{\underset{\sim}{y}}$ is distance preserving.

Proof.

Re (a). Suppose (*) holds with

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$$

Choose $\mathbf{x} \in \mathcal{R}$, $\mathbf{z} \in \mathcal{R}$ and note with the aid of (1.23) that

$$|\hat{\mathbf{y}}(\mathbf{x}) - \hat{\mathbf{y}}(\mathbf{z})|^2 = \{\mathbf{Q}(\mathbf{x}-\mathbf{z})\}\cdot\{\mathbf{Q}(\mathbf{x}-\mathbf{z})\} = \mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{z})\cdot(\mathbf{x}-\mathbf{z})$$

$$= (\mathbf{x}-\mathbf{z})\cdot(\mathbf{x}-\mathbf{z}) = |\mathbf{x}-\mathbf{z}|^2$$

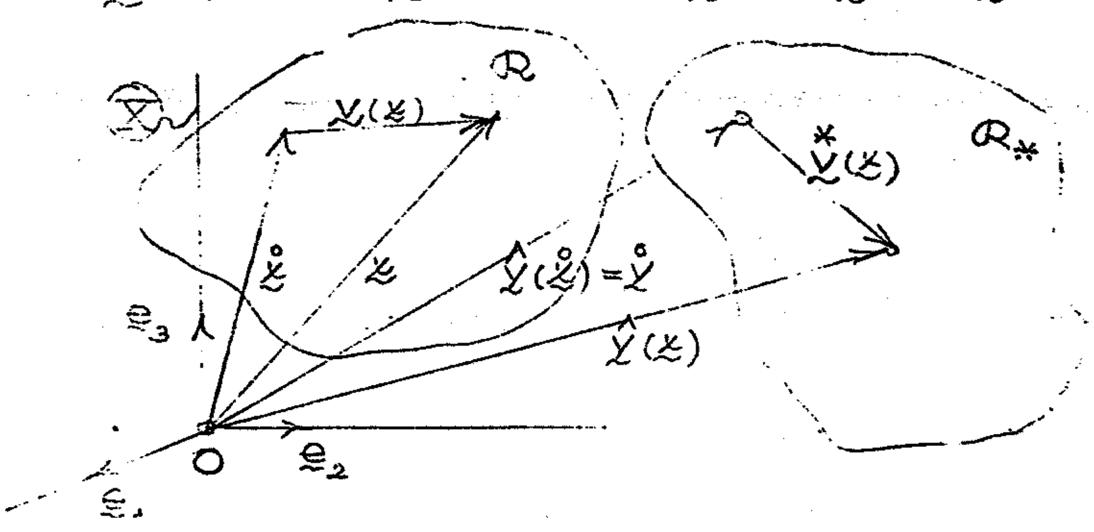
so that $\hat{\mathbf{y}}$ is distance-preserving. Also, clearly $\hat{\mathbf{y}}$ is admissible if $\det \mathbf{Q} = 1$ and thus rigid. (explain: $\hat{\mathbf{y}}$ now obeys (i), (ii), (iii) in def. of admiss. deform.

Re (b). Suppose

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) \ \forall\, \mathbf{x} \in \mathcal{R}, \ \hat{\mathbf{y}} \text{ preserves distances}$$

Choose and fix $\mathring{\mathbf{x}} \in \mathring{\mathcal{R}}$ (interior of $\mathcal{R}$) and set

$$\hat{\mathbf{y}}(\mathring{\mathbf{x}}) = \mathring{\mathbf{y}}, \quad \mathbf{v}(\mathbf{x}) = \mathbf{x} - \mathring{\mathbf{x}}, \quad \overset{*}{\mathbf{v}}(\mathbf{x}) = \hat{\mathbf{y}}(\mathbf{x}) - \mathring{\mathbf{y}} \ \forall\, \mathbf{x} \in \mathcal{R} \qquad (1)$$



Since $\hat{y}$ preserves distance, (1) $\Rightarrow$

$$|\underset{\sim}{v}(\underset{\sim}{x})| = |\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x})| \;,\; \underset{\sim}{v}^2(\underset{\sim}{x}) = \overset{*}{\underset{\sim}{v}}{}^2(\underset{\sim}{x}) \;\; \forall\, \underset{\sim}{x} \in \mathcal{R} \qquad (2)$$

$$\{\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}) - \overset{*}{\underset{\sim}{v}}(\underset{\sim}{z})\}^2 = \{\hat{\underset{\sim}{y}}(\underset{\sim}{x}) - \hat{\underset{\sim}{y}}(\underset{\sim}{z})\}^2 = (\underset{\sim}{x} - \underset{\sim}{z})^2 = \{\underset{\sim}{v}(\underset{\sim}{x}) - \underset{\sim}{v}(\underset{\sim}{z})\}^2 \; \forall\, (\underset{\sim}{x}, \underset{\sim}{z}) \in \mathcal{R}\times\mathcal{R}$$

Square out extreme members and use (2) to get

$$\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}) \cdot \overset{*}{\underset{\sim}{v}}(\underset{\sim}{z}) = \underset{\sim}{v}(\underset{\sim}{x}) \cdot \underset{\sim}{v}(\underset{\sim}{z}) \;\; \forall\; (\underset{\sim}{x}, \underset{\sim}{z}) \in \mathcal{R}\times\mathcal{R} \qquad (3)$$

Note that (2), (3) $\Rightarrow$ preservation of angles under $\hat{\underset{\sim}{y}}$.

Now choose $\mathfrak{X} = \{O, \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$. Since $\overset{\circ}{\underset{\sim}{x}} \in \overset{\circ}{\mathcal{R}}$ and $\overset{\circ}{\mathcal{R}}$ is open, $\exists\; \rho > 0 \ni$

$$\underset{\sim}{x}_i = \overset{\circ}{\underset{\sim}{x}} + \rho\, \underset{\sim}{e}_i \in \mathcal{R} \quad (i = 1, 2, 3). \qquad (4)$$

Define

$$\overset{*}{\underset{\sim}{e}}_i = \overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}_i) = \hat{\underset{\sim}{y}}(\underset{\sim}{x}_i) - \overset{\circ}{\underset{\sim}{y}}\,. \qquad (5)$$

(5), (3), (1), (4) $\Rightarrow$

$$\overset{*}{\underset{\sim}{e}}_i \cdot \overset{*}{\underset{\sim}{e}}_j = \overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}_i) \cdot \overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}_j) = \underset{\sim}{v}(\underset{\sim}{x}_i) \cdot \underset{\sim}{v}(\underset{\sim}{x}_j) = (\underset{\sim}{x}_i - \overset{\circ}{\underset{\sim}{x}}) \cdot (\underset{\sim}{x}_j - \overset{\circ}{\underset{\sim}{x}}) = \rho^2 \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j$$

so that

$$\overset{*}{\underset{\sim}{e}}_i \cdot \overset{*}{\underset{\sim}{e}}_j = \rho^2 \delta_{ij}\,. \qquad (6)$$

Thus $(\overset{*}{\underset{\sim}{e}}_1, \overset{*}{\underset{\sim}{e}}_2, \overset{*}{\underset{\sim}{e}}_3)$ constitute an orthog. base for E.

Hence, if $\underset{\sim}{w}$ is <u>any</u> vector in E, one has from (6)

$$\underset{\sim}{w} = w_j \overset{*}{\underset{\sim}{e}}_j \;,\quad w_i = \frac{1}{\rho^2}\, \underset{\sim}{w} \cdot \overset{*}{\underset{\sim}{e}}_i \qquad (7)$$

Apply (7) to $\underset{\sim}{w} = \overset{*}{\underset{\sim}{v}}(\underset{\sim}{x})\ \forall\, \underset{\sim}{x} \in \mathcal{R}$. Thus (7), (5), (3), (1), (4) ⇒

$$\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}) = \frac{1}{\rho^2}\{\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}) \cdot \overset{*}{\underset{\sim}{e}}_j\}\overset{*}{\underset{\sim}{e}}_j = \frac{1}{\rho^2}\{\overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}) \cdot \overset{*}{\underset{\sim}{v}}(\underset{\sim}{x}_j)\}\overset{*}{\underset{\sim}{e}}_j$$

$$= \frac{1}{\rho^2}\{\underset{\sim}{v}(\underset{\sim}{x}) \cdot \underset{\sim}{v}(\underset{\sim}{x}_j)\}\overset{*}{\underset{\sim}{e}}_j = \frac{1}{\rho^2}\{\underset{\sim}{v}(\underset{\sim}{x}) \cdot \rho\underset{\sim}{e}_j\}\overset{*}{\underset{\sim}{e}}_j \ \forall\, \underset{\sim}{x} \in \mathcal{R}$$

Therefore from (1),

$$\overset{*}{\underset{\sim}{y}}(\underset{\sim}{x}) = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) - \overset{\circ}{\underset{\sim}{y}} = \frac{1}{\rho}\{(\underset{\sim}{x} - \overset{\circ}{\underset{\sim}{x}}) \cdot \underset{\sim}{e}_j\}\overset{*}{\underset{\sim}{e}}_j \quad \forall\, \underset{\sim}{x} \in \mathcal{R} \text{ or}$$

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \frac{1}{\rho}(\underset{\sim}{x} \cdot \underset{\sim}{e}_j)\overset{*}{\underset{\sim}{e}}_j + \overset{\circ}{\underset{\sim}{y}} - \frac{1}{\rho}(\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{e}_j)\overset{*}{\underset{\sim}{e}}_j \ \forall\, \underset{\sim}{x} \in \mathcal{R} \qquad (8)$$

Now define

$$\underset{\sim}{L}(\underset{\sim}{x}) = \frac{1}{\rho}(\underset{\sim}{x} \cdot \underset{\sim}{e}_j)\overset{*}{\underset{\sim}{e}}_j \ \forall\, \underset{\sim}{x} \in E, \quad \underset{\sim}{c} = \overset{\circ}{\underset{\sim}{y}} - \frac{1}{\rho}(\overset{\circ}{\underset{\sim}{x}} \cdot \underset{\sim}{e}_j)\overset{*}{\underset{\sim}{e}}_j, \qquad (9)$$

so that (8) becomes

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{L}(\underset{\sim}{x}) + \underset{\sim}{c} \quad \forall\, \underset{\sim}{x} \in \mathcal{R}. \qquad (10)$$

Since $\underset{\sim}{L}$ is clearly a linear vector function,
Thm. 1.5 ⇒ ∃ a two-tensor $\underset{\sim}{Q} \ni \underset{\sim}{L}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x}\ \forall\, \underset{\sim}{x} \in E$. S

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{c} \quad \forall\, \underset{\sim}{x} \in \mathcal{R}. \qquad (11)$$

To see that $\underset{\sim}{Q}$ is <u>orthog.</u>, note from (11) & hyp.,

$$\{\hat{\underset{\sim}{y}}(\underset{\sim}{x})-\hat{\underset{\sim}{y}}(\mathring{\underset{\sim}{x}})\}^2 = \underset{\sim}{Q}(\underset{\sim}{x}-\mathring{\underset{\sim}{x}})\cdot\underset{\sim}{Q}(\underset{\sim}{x}-\mathring{\underset{\sim}{x}}) = \{\underset{\sim}{Q}^T\underset{\sim}{Q}(\underset{\sim}{x}-\mathring{\underset{\sim}{x}})\}\cdot(\underset{\sim}{x}-\mathring{\underset{\sim}{x}})$$

$$= (\underset{\sim}{x}-\mathring{\underset{\sim}{x}})^2 = (\underset{\sim}{x}-\mathring{\underset{\sim}{x}})\cdot(\underset{\sim}{x}-\mathring{\underset{\sim}{x}}) \;\forall\, \underset{\sim}{x}\in\mathcal{R}\,. \text{ or}$$

$$(\underset{\sim}{x}-\mathring{\underset{\sim}{x}})\cdot(\underset{\sim}{Q}^T\underset{\sim}{Q}-\underset{\sim}{1})(\underset{\sim}{x}-\mathring{\underset{\sim}{x}}) = 0 \;\underline{\underline{\forall}}\; \underset{\sim}{x}\in\mathcal{R}, \qquad (7)$$

which is easily seen to imply $\underset{\sim}{Q}^T\underset{\sim}{Q}=\underset{\sim}{1}$ (explain§)

To confirm <u>uniqueness</u> of $\underset{\sim}{Q}, \underset{\sim}{c}$ suppose $\exists\, \underset{\sim}{Q}', \underset{\sim}{c}' \ni$

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}'\underset{\sim}{x}+\underset{\sim}{c}' \;\forall\, \underset{\sim}{x}\in\mathcal{R}\,. \qquad (12)$$

But (11), (12) $\Rightarrow$

$$\underset{\sim}{Q}\,\underset{\sim}{x}+\underset{\sim}{c} = \underset{\sim}{Q}'\underset{\sim}{x}+\underset{\sim}{c}' \;\forall\, \underset{\sim}{x}\in\mathcal{R},\; \underset{\sim}{Q}\mathring{\underset{\sim}{x}}+\underset{\sim}{c} = \underset{\sim}{Q}'\mathring{\underset{\sim}{x}}+\underset{\sim}{c}' \Rightarrow$$

$$(\underset{\sim}{Q}-\underset{\sim}{Q}')(\underset{\sim}{x}-\mathring{\underset{\sim}{x}}) = \underset{\sim}{0} \;\underline{\underline{\forall}}\; \underset{\sim}{x}\in\mathcal{R} \Rightarrow \underset{\sim}{Q}=\underset{\sim}{Q}',\; \underset{\sim}{c}=\underset{\sim}{c}'.$$

Finally, if $\hat{\underset{\sim}{y}}$ is <u>rigid</u>, (11) $\Rightarrow J = \det\nabla\hat{\underset{\sim}{y}} = \det\underset{\sim}{Q} > 0$
$\Rightarrow \det\underset{\sim}{Q} = 1$; whence $\underset{\sim}{Q}$ is <u>proper</u> orthogonal. /
Mention simple proof if $\hat{\underset{\sim}{y}}\in\mathcal{C}^1(\mathcal{R})$ is <u>assumed</u>*.

<u>Consequences of Thm. 2.2</u>

(i) A deformation $\hat{\underset{\sim}{y}}:\mathcal{R}\to\mathcal{R}_*$ is rigid $\Leftrightarrow$

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x}+\underset{\sim}{c} \;\forall\, \underset{\sim}{x}\in\mathcal{R},\; \underset{\sim}{Q}\underset{\sim}{Q}^T=\underset{\sim}{1},\; \det\underset{\sim}{Q}=1.$$

(ii) if $\hat{\underset{\sim}{y}}:\mathcal{R}\to\mathcal{R}_*$ is rigid, it is the restriction to $\mathcal{R}$ of a unique rigid $\hat{\underset{\sim}{y}}: E\to E$ (explain). Remark <u>on "rigid extension" of a rigid body.</u>

Thus henceforth consider merely rigid deformats. on E.

(iii) Every rigid $\hat{\underset{\sim}{y}} : R \to R_*$ is locally volume preserving since $J = \det \underset{\sim}{F} = \det \underset{\sim}{Q} = 1$ (see Thm. 2.1).

Mention analogue of Thm. 2.2 for rigid motions. In particular: every rigid motion $\hat{\underset{\sim}{y}}(\cdot, t) : R \to R_t$ $(t_0 \le t \le t_1)$ admits the representation :

$$\left.\begin{aligned} &\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}, t) = \underset{\sim}{Q}(t)\underset{\sim}{x} + \underset{\sim}{c}(t) \ \forall \ (\underset{\sim}{x}, t) \in R \times \mathcal{T}, \ \underset{\sim}{Q} \in C^2(\mathcal{T}), \underset{\sim}{c} \in C^2(\mathcal{T}) \\ &\underset{\sim}{Q}(t)\underset{\sim}{Q}^T(t) = \underset{\sim}{1}, \ \det \underset{\sim}{Q}(t) = 1 \ \forall \ t \in \mathcal{T}, \ \mathcal{T} = [t_0, t_1] \end{aligned}\right\}($$

Special rigid deformations

Thm. 2.2 $\Rightarrow$ every rigid def. on E admits the representati

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{c} \ \forall \ \underset{\sim}{x} \in E, \ \underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{1}, \ \det \underset{\sim}{Q} = 1$$

(a) Translation : $\underset{\sim}{Q} = \underset{\sim}{1}$ ($\underset{\sim}{1}$ is proper orthogonal)

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{c} \ \forall \ \underset{\sim}{x} \in E \qquad (2.12)$$

(b) Rigid deformation about $\underset{\sim}{0}$: $\hat{\underset{\sim}{y}}(\underset{\sim}{0}) = \underset{\sim}{0}$

$$\underset{\sim}{c} = \underset{\sim}{0} \ ; \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{x} \ \forall \ \underset{\sim}{x} \in E \ (\underset{\sim}{Q} \text{ prop. orthog}) \quad (2.13$$

Note that

$$\underset{\sim}{z} = \underset{\sim}{Q}\underset{\sim}{x}, \ \underset{\sim}{y} = \underset{\sim}{z} + \underset{\sim}{c} \Rightarrow \underset{\sim}{y} = \underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{c}$$

Interpret kinematically.

Thm. 2.3. Let $\underset{\sim}{Q}$ be a proper orthog. two-tensor. Then

(a) $\lambda = 1$ is a p. value of $\underset{\sim}{Q}$, i.e. $\exists$ a vector $\underset{\sim}{m}$ $\ni$

$\underset{\sim}{Q}\underset{\sim}{m} = \underset{\sim}{m}$, $|\underset{\sim}{m}| = 1$;

(b) The p. axis belonging to this p. value is unique unless $\underset{\sim}{Q} = \underset{\sim}{1}$; i.e. if $\exists$ two non-parallel p. direction vectors belonging to $\lambda = 1$, then $\underset{\sim}{Q} = \underset{\sim}{1}$

Proof.

Re (a). By hyp.,

$$\underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{1}, \quad \det \underset{\sim}{Q} = 1 \qquad (1)$$

We know $\lambda = 1$ is a p. value of $\underset{\sim}{Q}$ if and only if

$$\det(\underset{\sim}{Q} - \underset{\sim}{1}) = 0.$$

But (1) $\Rightarrow$

$$\det(\underset{\sim}{Q} - \underset{\sim}{1}) = \det(\underset{\sim}{Q} - \underset{\sim}{Q}\underset{\sim}{Q}^T) = \det\{\underset{\sim}{Q}(\underset{\sim}{1} - \underset{\sim}{Q}^T)\} = \det(\underset{\sim}{1} - \underset{\sim}{Q}^T)$$
$$= \det(\underset{\sim}{1} - \underset{\sim}{Q})^T = \det(\underset{\sim}{1} - \underset{\sim}{Q}) = (-1)^3 \det(\underset{\sim}{Q} - 1)$$
$$= -\det(\underset{\sim}{Q} - \underset{\sim}{1}) = 0$$

Re (b). Suppose $\exists$ two linearly independent p. direction vectors of $\underset{\sim}{Q}$ both belonging to $\lambda = 1$. Thus,

$$\underset{\sim}{Q}\underset{\sim}{m} = \underset{\sim}{m}, \quad \underset{\sim}{Q}\underset{\sim}{n} = \underset{\sim}{n}, \quad |\underset{\sim}{m}| = |\underset{\sim}{n}| = 1, \quad \underset{\sim}{m} \neq \pm \underset{\sim}{n} \qquad (2)$$

Assume without loss of generality (explain) that

$$\underset{\sim}{m} \cdot \underset{\sim}{n} = 0 \qquad (3)$$

Define
$$\underset{\sim}{p} = \underset{\sim}{m} \wedge \underset{\sim}{n} \quad (4)$$
(3), (4) $\Rightarrow$ $(\underset{\sim}{m}, \underset{\sim}{n}, \underset{\sim}{p})$ form a proper orthonormal base, whence
$$\mathfrak{X} = \{0; \underset{\sim}{m}, \underset{\sim}{n}, \underset{\sim}{p}\} \in \mathcal{F} \quad (5)$$
Next, (1), (2), (5) $\Rightarrow$
$$\underset{\sim}{p} \cdot \underset{\sim}{m} = \underset{\sim}{p} \cdot \underset{\sim}{Q}^T \underset{\sim}{m} = \underset{\sim}{Q}\underset{\sim}{p} \cdot \underset{\sim}{m} = 0 \,, \quad \underset{\sim}{Q}\underset{\sim}{p} \cdot \underset{\sim}{n} = 0,$$
whence
$$\underset{\sim}{Q}\underset{\sim}{p} = \mu \underset{\sim}{p} \quad (6)$$
(2), (6) $\Rightarrow$ $\mathfrak{X}$ is p. for $\underset{\sim}{Q}$ and
$$[\overset{\mathfrak{X}}{Q_{ij}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}$$
But $\det \underset{\sim}{Q} = 1 \Rightarrow \mu = 1$ and thus $\underset{\sim}{Q} = \underset{\sim}{1}$. /

The preceding theorem on proper orthog. two-tensors enables one to infer the truth of the following result concerning rigid deformations.

<u>Euler's theorem</u>. Every rigid deformation $\hat{\underset{\sim}{y}} : E \to E$ that carries a point $\overset{\circ}{\underset{\sim}{x}}$ into itself carries an axis through $\overset{\circ}{\underset{\sim}{x}}$ into itself, i.e. is a rotation about a

fixed axis through $\mathring{\mathbf{x}}$. Further, this axis is unique unless $\hat{\mathbf{y}}$ is the identity mapping.

<u>Proof</u>. Take $\mathring{\mathbf{x}} = \mathbf{0}$ without loss of generality. Thus,

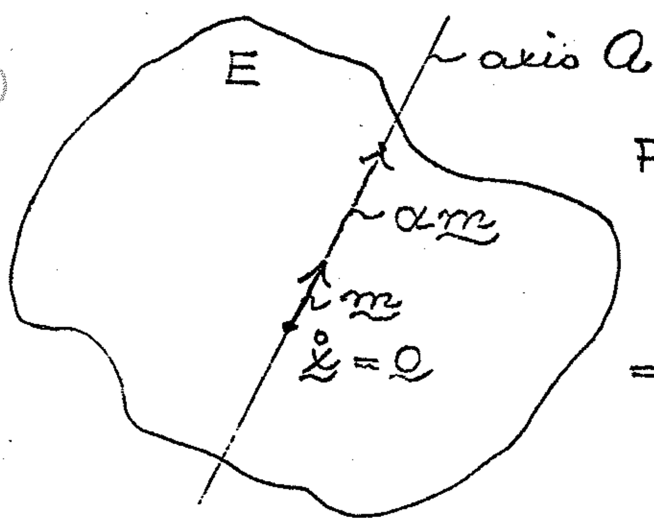
$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{Q}\mathbf{x} \text{ on } E, \quad \mathbf{Q} \text{ proper orthog.}$$

Let $\mathbf{Q} \neq \mathbf{1}$. By Thm. 2.3, $\exists$ a vector $\mathbf{m} \ni$

$$\mathbf{Q}\mathbf{m} = \mathbf{m}, \quad |\mathbf{m}| = 1.$$

Thus

$$\hat{\mathbf{y}}(\alpha\mathbf{m}) = \alpha\mathbf{m} \quad (-\infty < \alpha < \infty), \text{ whence}$$



$$\mathbf{x} = \alpha\mathbf{m} \Rightarrow \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x}.$$

Further,

$$\hat{\mathbf{y}}(\alpha\mathbf{k}) = \alpha\mathbf{k} \ \forall \alpha, \ |\mathbf{k}| = 1$$

$$\Rightarrow \mathbf{k} = \pm\mathbf{m} \text{ since } \mathbf{Q} \neq \mathbf{1},$$

in view of (b) in Thm 2.3 /

Illustrate content of Euler's theorem (eg. globe).

We turn now to an important result concerning rigid <u>motions</u> about a fixed point.

<u>Claim</u>. Let $\hat{\underset{\sim}{y}}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ be a rigid motion about $\underset{\sim}{x} = \underset{\sim}{0}$ and let $\mathcal{T} = [t_0, t_1]$. Then $\exists$ a unique vector-valued function $\underset{\sim}{\Omega} \in \mathcal{C}^1(\mathcal{T})$ ("angular velocity of the motion") $\ni$

$$\overline{\underset{\sim}{v}}(\underset{\sim}{y},t) = \underset{\sim}{\Omega}(t) \wedge \underset{\sim}{y} \quad \forall\ \underset{\sim}{y} \in \mathcal{R}_t \ (t_0 \le t \le t_1) \qquad (*)$$

<u>Proof</u>. Recall from (2.11) that here

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \underset{\sim}{Q}(t)\underset{\sim}{x} \ \text{ on } \mathcal{R}\times\mathcal{T} \qquad (1)$$

with

$$\underset{\sim}{Q} \in \mathcal{C}^2(\mathcal{T}),\ \underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{1},\ \det \underset{\sim}{Q} = 1 \text{ on } \mathcal{T} \qquad (2)$$

(1), (2) $\Rightarrow$

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \dot{\underset{\sim}{Q}}(t)\underset{\sim}{x} = \dot{\underset{\sim}{Q}}(t)\underset{\sim}{Q}^T(t)\hat{\underset{\sim}{y}}(\underset{\sim}{x},t) \ \text{ on } \mathcal{R}\times\mathcal{T} \qquad (3)$$

$$\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T + \underset{\sim}{Q}\dot{\underset{\sim}{Q}}^T = \underset{\sim}{0} \quad \text{or} \quad \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T = -(\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T)^T \text{ on } \mathcal{T} \ (4)$$

Hence $\dot{\underset{\sim}{Q}}(t)\underset{\sim}{Q}^T(t)$ is a skew two-tensor $\forall\ t \in \mathcal{T}$.

Let

$$\underset{\sim}{\Omega} = \operatorname{dual}(\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T) \text{ on } \mathcal{T} \ (5)$$

Then $\underset{\sim}{\Omega} \in \mathcal{C}^1(\mathcal{T})$ and by (1.37), (3),

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{\Omega}(t) \wedge \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) \ \forall\ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T},$$

which implies (*).

To infer the uniqueness of $\underset{\sim}{\Omega}$, suppose (*) holds also with $\underset{\sim}{\Omega}$ replaced by $\underset{\sim}{\Omega}'$. Then,

$$\{\underset{\sim}{\Omega}(t) - \underset{\sim}{\Omega}'(t)\} \wedge \underset{\sim}{y} = \underset{=}{0} \;\; \forall \;\; \underset{\sim}{y} \in R_t \quad (t_0 \leq t \leq t_1)$$

But this assures $\underset{\sim}{\Omega}' = \underset{\sim}{\Omega}$ on $\mathcal{T}$. /

Remark on "instantaneous axis of rotation".

Exercise 8.

Let $\hat{\underset{\sim}{y}} : E \to E$ be a right-handed rotation through an angle $\theta \in [0, 2\pi)$ about an axis throug $\underset{\sim}{x} = \underset{\sim}{0}$ with the unit direction vector $\underset{\sim}{n}$. Use elementary geometric considerations to confirm that $\hat{\underset{\sim}{y}}$ admits the representation (Rodrigues' formula):

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} \cos\theta + (1-\cos\theta)(\underset{\sim}{x}\cdot\underset{\sim}{n})\underset{\sim}{n} + \sin\theta \; \underset{\sim}{n} \wedge \underset{\sim}{x} \;\; \forall \underset{\sim}{x} \in E \quad (2.$$

Show that (2.15) is equivalent to

$$\underset{\sim}{y} = \underset{\sim}{Q}\underset{\sim}{x} \,, \quad Q_{ij} = \delta_{ij} \cos\theta + (1-\cos\theta) n_i n_j - \sin\theta \, \varepsilon_{ijk} n_k . \quad (2.1$$

Show that $\underset{\sim}{Q}$ is a proper orthogonal two-tensor by referring $\underset{\sim}{Q}$ to a frame $\mathfrak{X} = \{0; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{n}\} \in \mathcal{F}$.

Discussion of Exercise 8

Draw picture showing axis, $\mathbf{n}$, θ, $\underset{\sim}{x}$, $\hat{y}(\underset{\sim}{x})$, $\underset{\sim}{u}(\underset{\sim}{x})$.

~~It follows from requested reduction and (b) in Ex. 8 that every proper orthog. transformation $\underset{\sim}{Q}$ admits the representation (2.16)~~. Note that all possible positions of a rigid body with O fixed are describable in terms of the four parameters n_i, θ which are restricted by $n_i n_i = 1$. Thus only three degrees of freedom. Mention Eulerian angles.

~~Re (b). This result follows easily from (1.37) (relation between a skew tensor and its vector dual). Contrast to contorted derivations in books on rigid body dynamics~~

Homogeneous deformations (affine mappings)

Definition. $\hat{\underset{\sim}{y}}: \mathcal{R} \to \mathcal{R}_*$ is a (proper) homogeneous deformation of $\mathcal{R}$ {(proper) affine mapping of $\mathcal{R}$} if $\hat{\underset{\sim}{y}}$ admits the representation:

$H: \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{E}\,\underset{\sim}{x} + \underset{\sim}{c} \quad \forall\, \underset{\sim}{x} \in \mathcal{R}, \ \det \underset{\sim}{E} > 0, \ (*)$

where $\underset{\sim}{E}$ is a constant two-tensor and $\underset{\sim}{c}$ is a constant vector. One calls $\underset{\sim}{E}$ the tensor and $\underset{\sim}{c}$ the vector of the homog. deformation.

<u>Remark.</u> Motivate study of homog. deformations:

<u>Thm. 2.4</u>. Let $\hat{\underset{\sim}{y}}: \mathcal{R} \to \mathcal{R}_*$.

(a) A nec. & suff. cond. that $\hat{\underset{\sim}{y}}$ be a homog. def. on $\mathcal{R}$ is that it be an admiss. def. whose def. grad. field $\nabla\hat{\underset{\sim}{y}} = \underset{\sim}{E}$ is constant on $\mathcal{R}$.

(b) A nec. & suff. cond. that $\hat{\underset{\sim}{y}}$ be a <u>rigid</u> def. on $\mathcal{R}$ is that it be a homog. def. on $\mathcal{R}$ whose tensor $\underset{\sim}{E}$ is orthogonal.

(c) Every homog. def. on $\mathcal{R}$ is the restriction to $\mathcal{R}$ of a unique homog. def. on the entire space $E \equiv E_3$.

<u>Remarks</u>. Sketch trivial proof. In view of (c) we need from now on consider merely homog. deformations of E.

Let $\mathbb{H}$ be a homogeneous deformation of E, so that

$$\left.\begin{aligned} \mathbb{H}: \quad & \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{E}\,\underset{\sim}{x} + \underset{\sim}{c} \quad \forall\ \underset{\sim}{x} \in E,\ \det \underset{\sim}{E} > 0 \\ & \nabla \hat{\underset{\sim}{y}} = \underset{\sim}{E} = \underset{\sim}{1} + \nabla \underset{\sim}{u}\ ,\ J = \det \underset{\sim}{E} \end{aligned}\right\} (2.1$$

Thm. 2.5 (Inversion & composition of homog. deformations)

(a) Let $\mathbb{H}: \underset{\sim}{y} = \underset{\sim}{E}\,\underset{\sim}{x} + \underset{\sim}{c}$, $\det \underset{\sim}{E} > 0$. Then,

$$\left.\begin{aligned} & \mathbb{H}^{-1}: \underset{\sim}{x} = \overset{*}{\underset{\sim}{E}}\,\underset{\sim}{y} + \overset{*}{\underset{\sim}{c}} \quad \forall\ \underset{\sim}{y} \in E \\ & \overset{*}{\underset{\sim}{E}} = \underset{\sim}{E}^{-1},\ \ \overset{*}{\underset{\sim}{c}} = -\underset{\sim}{E}^{-1}\underset{\sim}{c},\ \det \overset{*}{\underset{\sim}{E}} = (\det \underset{\sim}{E})^{-1} > 0 \end{aligned}\right\} (2.18a)$$

and hence $\mathbb{H}^{-1}$ is a homog. deformation on E.

(b) Let

$\mathbb{H}': \underset{\sim}{y}' = \underset{\sim}{E}'\underset{\sim}{x} + \underset{\sim}{c}$, $\det \underset{\sim}{E}' > 0$; $\mathbb{H}'': \underset{\sim}{y} = \underset{\sim}{E}''\underset{\sim}{y}' + \underset{\sim}{c}''$, $\det \underset{\sim}{E}'' > 0$

Then,

$$\left.\begin{aligned} & \mathbb{H} = \mathbb{H}''\mathbb{H}': \ \underset{\sim}{y} = \underset{\sim}{E}\,\underset{\sim}{x} + \underset{\sim}{c} \\ & \underset{\sim}{E} = \underset{\sim}{E}''\underset{\sim}{E}',\ \underset{\sim}{c} = \underset{\sim}{E}''\underset{\sim}{c}' + \underset{\sim}{c}'',\ \det \underset{\sim}{E} = \det \underset{\sim}{E}' \det \underset{\sim}{E}'' > 0 \end{aligned}\right\} (2.18b)$$

and hence $\mathbb{H}''\mathbb{H}'$ is a homog. deformation on E

<u>Proof.</u> Immediate

Clearly, $\mathbb{H}''\mathbb{H}' \neq \mathbb{H}'\mathbb{H}''$

Let $\mathcal{A}$ be the collection of all affine mappings of E

Evidently A has the properties:

(i) $H' \in A,\ H'' \in A \Rightarrow H''H' \in A$

(ii) $H' \in A,\ H'' \in A,\ H''' \in A \Rightarrow H'''(H''H') = (H'''H'')H'$

(iii) $\exists\, H_o \in A \ni H \in A \Rightarrow H_o H = H$

(iv) $\forall\, H \in A\ \exists$ unique $H^{-1} \in A \ni H^{-1}H = H_o$

Thus A is a (non-commutative) group.

Thm. 2.6. Every homogeneous deformation maps:

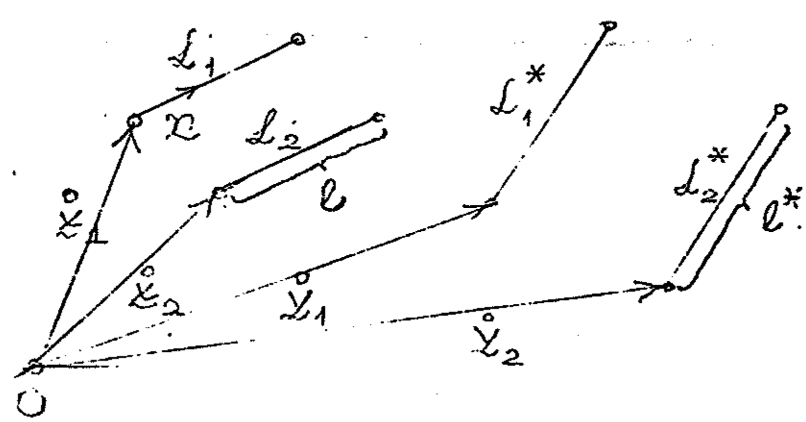
(a) Parallel straight-line segments of equal length on parallel straight-line segments of equal length;

(b) Parallel planes onto parallel planes;

(c) Quadric surfaces onto quadric surfaces (in particular ellipsoids onto ellipsoids).

Proof.

Re (a). Let H be given by (2.17). Choose $\overset{\circ}{x}_\alpha \in E$ $(\alpha = 1,2)$



Let $\underset{\sim}{n}$ be a unit-vector. Consider

$$\mathcal{L}_\alpha = \{\underset{\sim}{x} \mid \underset{\sim}{x} = \overset{\circ}{\underset{\sim}{x}}_\alpha + \eta \underset{\sim}{n}, \; 0 \le \eta \le \ell\} \quad (\alpha = 1,2) \quad (i)$$

$\mathcal{L}_\alpha$ $(\alpha = 1,2)$ is a straight-line segmt. par. $\underset{\sim}{n}$ of length ℓ.

Let $\mathcal{L}^*_\alpha = \hat{\underset{\sim}{y}}(\mathcal{L}_\alpha)$ (image of $\mathcal{L}_\alpha$ under H), $\overset{\circ}{\underset{\sim}{y}}_\alpha = \hat{\underset{\sim}{y}}(\overset{\circ}{\underset{\sim}{x}}_\alpha)$ $(\alpha = 1,2)$

Then $\underset{\sim}{x} \in \mathcal{L}_\alpha \Rightarrow \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{E}(\overset{\circ}{\underset{\sim}{x}}_\alpha + \eta \underset{\sim}{n}) + \underset{\sim}{c} = \overset{\circ}{\underset{\sim}{y}} + \eta \underset{\sim}{E}\underset{\sim}{n}$

$$\mathcal{L}^*_\alpha = \{\underset{\sim}{y} \mid \underset{\sim}{y} = \overset{\circ}{\underset{\sim}{y}}_\alpha + \eta \underset{\sim}{E}\underset{\sim}{n}, \; 0 \le \eta \le \ell\} \quad (\alpha = 1,2) \quad (ii)$$

Hence $\mathcal{L}^*_\alpha$ $(\alpha = 1,2)$ is a straight-line segment parallel to $\underset{\sim}{E}\underset{\sim}{n}$ of length $\ell^*_\alpha = |\underset{\sim}{E}\underset{\sim}{n}|\ell \equiv \ell^*$ $(\alpha = 1,2)$

<u>Exercise</u> 9. Prove parts (b) and (c) in Thm. 2.6.

<u>HINT</u>: Let $S = \{\underset{\sim}{x} \mid \underset{\sim}{x} \cdot \underset{\sim}{M}\underset{\sim}{x} + \underset{\sim}{m} \cdot \underset{\sim}{x} + k = 0, \; \underset{\sim}{M}^T = \underset{\sim}{M}\}$. Note S is a quadric surface if $\underset{\sim}{M} \neq \underset{\sim}{0}$ & a plane if $\underset{\sim}{M} = \underset{\sim}{0}, \underset{\sim}{m} \neq$

<u>Remarks.</u> Comment on ellipsoids $\rightarrow$ ellipsoids. Note Thm. 2.6 $\Rightarrow$ parallelepipeds $\rightarrow$ parallelepipeds.

<u>Transformation of straight-line segments under H</u>

Consider $H \in A$ and two points $\underset{\sim}{x} \in E$, $\underset{\sim}{z} \in E$. Set

$$\underset{\sim}{v} = \underset{\sim}{z} - \underset{\sim}{x}, \quad \overset{*}{\underset{\sim}{v}} = \hat{\underset{\sim}{y}}(\underset{\sim}{z}) - \hat{\underset{\sim}{y}}(\underset{\sim}{x}) \quad (\S)$$ (draw pictur

We call $\overset{*}{\underset{\sim}{v}}$ the "image of the vector $\underset{\sim}{v}$ under H"

From (δ) and (2.17),

$$\overset{*}{\underset{\sim}{v}} = \underset{\sim}{F}\,\underset{\sim}{v} \quad \text{or} \quad \overset{*}{v}_i = F_{ij} v_j \tag{2.18}$$

<u>Thm</u> 2.7. Let $\underset{\sim}{v}, \underset{\sim}{w}$ be vectors and $\overset{*}{\underset{\sim}{v}}, \overset{*}{\underset{\sim}{w}}$ their images under $\mathcal{H}$, and let $\underset{\sim}{F}$ be the tensor of $\mathcal{H}$. Then,

$$\left.\begin{aligned} \overset{*}{\underset{\sim}{v}} \cdot \overset{*}{\underset{\sim}{w}} - \underset{\sim}{v} \cdot \underset{\sim}{w} &= 2\,\underset{\sim}{v} \cdot \underset{\sim}{\varphi}\,\underset{\sim}{w} = 2\varphi_{ij} v_i w_j \\ \underset{\sim}{\varphi} = \underset{\sim}{\varphi}^T = \tfrac{1}{2}(\underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{1}) &\quad \text{or} \quad \varphi_{ij} = \tfrac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \end{aligned}\right\} \tag{2.19}$$

One calls the symmetric two-tensor $\underset{\sim}{\varphi}$ the <u>Lagrangian strain tensor</u> of $\mathcal{H}$.

<u>Proof</u>. By hyp., (2.18), (1.23),

$$\begin{aligned} \overset{*}{\underset{\sim}{v}} \cdot \overset{*}{\underset{\sim}{w}} - \underset{\sim}{v} \cdot \underset{\sim}{w} &= \underset{\sim}{F}\,\underset{\sim}{v} \cdot \underset{\sim}{F}\,\underset{\sim}{w} - \underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{v} \cdot \underset{\sim}{F}^T \underset{\sim}{F}\,\underset{\sim}{w} - \underset{\sim}{v} \cdot \underset{\sim}{w} \\ &= \underset{\sim}{v} \cdot (\underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{1})\,\underset{\sim}{w} = 2\,\underset{\sim}{v} \cdot \underset{\sim}{\varphi}\,\underset{\sim}{w} = 2\varphi_{ij} v_i w_j \end{aligned}$$

<u>Remark on notation</u>. From here on two-tensors will be denoted by Latin caps. <u>or</u> Greek l.c. letters.

If $\underset{\sim}{F}$ is the tensor of a homog. deformation, <u>define</u>

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}, \quad \underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T \tag{2.20}$$

One calls $\underset{\sim}{C}$ and $\underset{\sim}{G}$ the <u>right and left deformation tensors</u> of $\mathcal{H}$, respectively. ($\underset{\sim}{C}$ is also "Lagrangian deformation tensor"). $\underset{\sim}{C}$ and $\underset{\sim}{E}$ are both symm. pos. definite two-tensors which have the same scalar invariants and principal values, but in general not same p. axes.

(2.20), (2.19) $\Rightarrow$

$$\left.\begin{array}{l} \underset{\sim}{\varphi} = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{1}) \text{ or } \varphi_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) \\ \overset{*}{\underset{\sim}{v}} \cdot \overset{*}{\underset{\sim}{w}} = \underset{\sim}{v} \cdot \underset{\sim}{C}\underset{\sim}{w} = C_{ij} v_i w_j \end{array}\right\} \quad (2.21)$$

We shall make use of the $\underset{\sim}{G}$ tensor later on.

Geometric interpretation of Lagrangian strain compts.

As suggested by (2.19), $\underset{\sim}{\varphi}$ governs all length and angle changes under H. Note that $\underset{\sim}{\varphi} = \underset{\sim}{0}$ iff H is rigid!

Let $\underset{\sim}{v} \neq \underset{\sim}{0}$ be a vector with image $\overset{*}{\underset{\sim}{v}}$ under H.

Define

$$\left.\begin{array}{l} \lambda(\underset{\sim}{v}) = \dfrac{|\overset{*}{\underset{\sim}{v}}|}{|\underset{\sim}{v}|} \ \ldots \text{stretch-ratio of } \underset{\sim}{v} \\ \varepsilon(\underset{\sim}{v}) = \dfrac{|\overset{*}{\underset{\sim}{v}}| - |\underset{\sim}{v}|}{|\underset{\sim}{v}|} = \lambda(\underset{\sim}{v}) - 1 \ \ldots \text{extension ratio of } \underset{\sim}{v} \end{array}\right\} (2.2$$

(2.22), (2.18) $\Rightarrow$

$$0 < \lambda(\underset{\sim}{v}) < \infty,\ -1 < \varepsilon(\underset{\sim}{v}) < \infty,\ \lambda(\alpha\underset{\sim}{v}) = \lambda(\underset{\sim}{v}),\ \varepsilon(\alpha\underset{\sim}{v}) = \varepsilon(\underset{\sim}{v})\ \forall\, \alpha \neq$$

(2.2

$\lambda(\underset{\sim}{v}) > 1 \Leftrightarrow \varepsilon(\underset{\sim}{v}) > 0$ ... lengthening

$\lambda(\underset{\sim}{v}) < 1 \Leftrightarrow \varepsilon(\underset{\sim}{v}) < 0$ ... shortening

Thus all parallel (non-null) vectors have same stretch and extension ratios

Take $\underset{\sim}{v} = \underset{\sim}{w} \neq \underset{\sim}{0}$ in (2.19). Thus,

$$|\overset{*}{\underset{\sim}{v}}|^2 - |\underset{\sim}{v}|^2 = 2\underset{\sim}{v}\cdot\underset{\sim}{\varphi}\underset{\sim}{v}\ ,\quad \frac{|\overset{*}{\underset{\sim}{v}}|}{|\underset{\sim}{v}|} = \sqrt{1+(2\underset{\sim}{v}\cdot\underset{\sim}{\varphi}\underset{\sim}{v})/v^2} = \lambda(\underset{\sim}{v})$$

$$\varepsilon(\underset{\sim}{v}) = \lambda(\underset{\sim}{v}) - 1 = \sqrt{1+(2\underset{\sim}{v}\cdot\underset{\sim}{\varphi}\underset{\sim}{v})/v^2} - 1 \quad \forall\ \underset{\sim}{v} \neq \underset{\sim}{0} \qquad (2.24)$$

In particular, if $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathfrak{F}$, $\varphi_{ij} = \varphi_{ij}^{X}$,

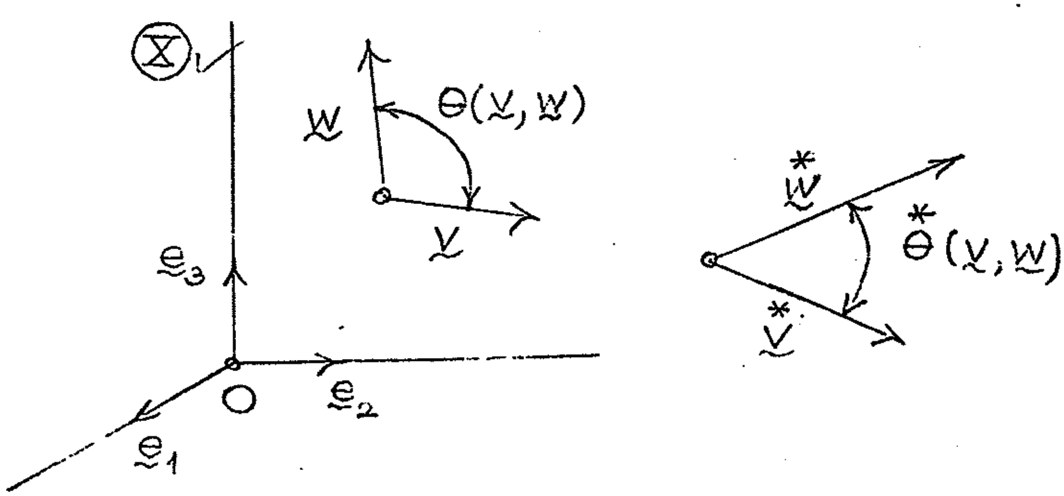
$$\varepsilon(\underset{\sim}{e}_i) = \sqrt{1 + 2\underset{\sim}{e}_i\cdot\underset{\sim}{\varphi}\underset{\sim}{e}_i} - 1 \quad (\text{no sum})$$

But (1.21') $\Rightarrow$ $\underset{\sim}{e}_i\cdot\underset{\sim}{\varphi}\underset{\sim}{e}_j = \varphi_{ij}$. Thus,

$$\varepsilon(\underset{\sim}{e}_i) = \varepsilon(\alpha\underset{\sim}{e}_i) = \sqrt{1+2\varphi_{ii}} - 1 \quad (\text{no sum})\ \forall\ \alpha \neq 0 \qquad (2.25)$$

This reveals geometric significance of $\varphi_{11}, \varphi_{22}, \varphi_{33}$.

Turn to geometric interpretation of φ_{ij} $(i \neq j)$. Consider



Assume $\underset{\sim}{v} \neq \underset{\sim}{0}$, $\underset{\sim}{w} \neq \underset{\sim}{0}$, $0 \leq \theta(\underset{\sim}{v},\underset{\sim}{w}) \leq \pi$, $0 \leq \overset{*}{\theta}(\underset{\sim}{v},\underset{\sim}{w}) \leq \pi$

From (2.19), (2.22) follows:

$$\overset{*}{\underset{\sim}{v}} \cdot \overset{*}{\underset{\sim}{w}} = |\overset{*}{\underset{\sim}{v}}||\overset{*}{w}| \cos(\overset{*}{\theta}(\underset{\sim}{v},\underset{\sim}{w})) = \underset{\sim}{v}\cdot\underset{\sim}{w} + 2\underset{\sim}{v}\cdot\underset{\sim}{\varphi}\underset{\sim}{w} \qquad (1)$$

$$|\overset{*}{\underset{\sim}{v}}| = |v|\,\lambda(\underset{\sim}{v}) = |\underset{\sim}{v}|[1+\varepsilon(\underset{\sim}{v})], \quad |\overset{*}{\underset{\sim}{w}}| = |\underset{\sim}{w}|[1+\varepsilon(\underset{\sim}{w})] \qquad (2)$$

(1), (2) $\Rightarrow$

$$\cos(\overset{*}{\theta}(v,\underset{\sim}{w})) = \frac{\underset{\sim}{v}\cdot\underset{\sim}{w} + 2\underset{\sim}{v}\cdot\underset{\sim}{\varphi}\underset{\sim}{w}}{|v||\underset{\sim}{w}|[1+\varepsilon(v)][1+\varepsilon(\underset{\sim}{w})]} \qquad (2.26)$$

For every $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ define $\mu_{ij} = \overset{X}{\mu_{ij}}$ through

$$\mu_{ij} = \mu_{ji} = \frac{\pi}{2} - \overset{*}{\theta}(\underset{\sim}{e}_i, \underset{\sim}{e}_j) \quad (i \neq j) \quad \text{"shears"} \qquad (2.27)$$

Describe geom. meaning of μ_{ij}. Note that $-\frac{\pi}{2} \le \mu_{ij} \le \frac{\pi}{2}$

Now take $\underset{\sim}{v} = \underset{\sim}{e}_i$, $\underset{\sim}{w} = \underset{\sim}{e}_j$ $(i \neq j)$, $(i,j$ fixed$)$. Then from (2.26), (2.27), (2.25),

$$\sin\mu_{ij} = \frac{2\varphi_{ij}}{\sqrt{1+2\varphi_{ii}}\sqrt{1+2\varphi_{jj}}} \quad (i \neq j), \text{(no sums)} \qquad (2.28)$$

Discuss (2.28).

Since $\underset{\sim}{\varphi}$ is a symm. two-tensor, $\exists X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$

$$[\overset{X}{\varphi_{ij}}] = \begin{bmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & \varphi_3 \end{bmatrix}, \quad \underset{\sim}{\varphi}\underset{\sim}{e}_i = \varphi_i \underset{\sim}{e}_i \quad \text{(no sums)} \qquad (2.29)$$

φ_i ... principal values of $\underset{\sim}{\varphi}$ (p. Lagrangian strains)

(2.27), (2.28), (2.29) $\Rightarrow$

Any triplet of mutually $\perp$ straight lines that is parallel to a p. frame of $\underset{\sim}{\varphi}$ has mutually $\perp$ images under H. There always exists at least one such triplet (family).

(2.21) $\Rightarrow$

A frame $\mathbb{X} \in \mathfrak{F}$ is p-for $\underset{\sim}{\varphi}$ if and only if it is p. for $\underset{\sim}{C}$

Let $\mathbb{X} \in \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be p. for $\underset{\sim}{\varphi}, \underset{\sim}{C}$. Let $\lambda_i^2 > 0$ be the p. value of the pos. def. symm. tensors $\underset{\sim}{C}, \underset{\sim}{\varphi}$. Then, by (2.20), (2.21),

$$[C_{ij}^{\mathbb{X}}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1+2\varphi_1 & 0 & 0 \\ 0 & 1+2\varphi_2 & 0 \\ 0 & 0 & 1+2\varphi_3 \end{bmatrix} \tag{2.30}$$

Define

$$\varepsilon_i = \varepsilon(\underset{\sim}{e}_i), \quad \lambda_i = \sqrt{\lambda_i^2} > 0 \tag{2.31}$$

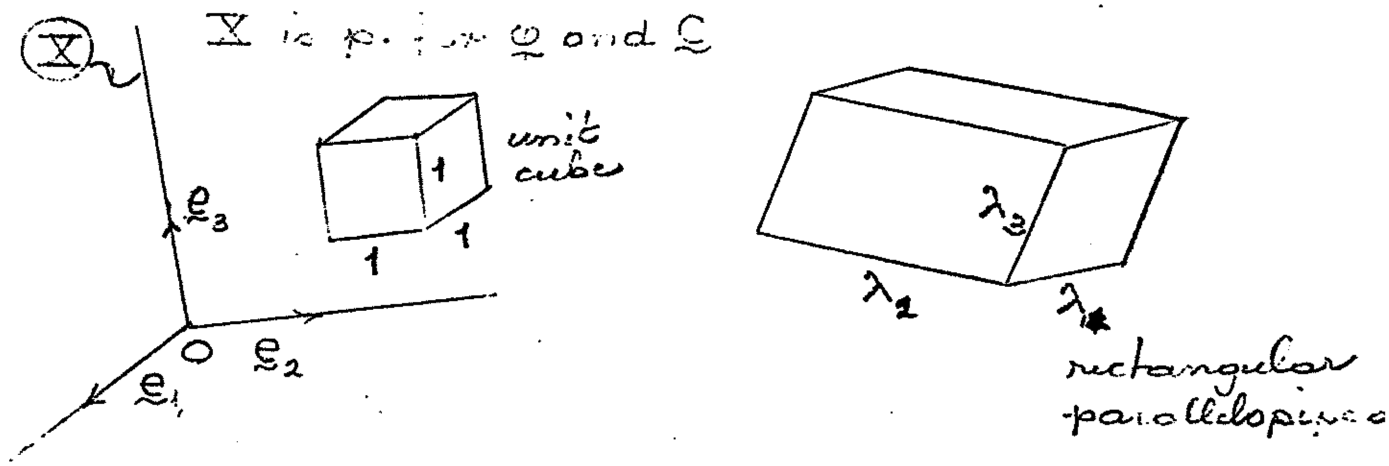
By (2.30), (2.31), (2.25), (2.22), one has

$$\lambda_i = \sqrt{1+2\varphi_i} = 1 + \varepsilon(\underset{\sim}{e}_i) = 1 + \varepsilon_i = \lambda(\underset{\sim}{e}_i) \tag{2.32}$$

This interprets the λ_i kinematically (draw picture, p.79)

ε_i ... princ. extension-ratios, λ_i ... princ. stretch-ratios.

Note: A unit cube with edges parallel to the axes of a p. frame for $\underset{\sim}{\varphi}$ (and $\underset{\sim}{C}$) $\rightarrow$ rectangular parallelepiped with side lengths $\rho_i = \lambda_i$.



With the aid of (2.32), (2.25), (2.29), (2.22) and Thm. 1.5 (extremum property of p. values) one finds easily

$$\left.\begin{aligned} &\varphi_1 = \max_{\underset{\sim}{X} \in \mathcal{F}} \varphi_{ii}\,, \quad \varphi_3 = \min_{\underset{\sim}{X} \in \mathcal{F}} \varphi_{ii} \quad (\text{no sum}) \quad (\text{if } \varphi_1 \geq \varphi_2 \geq \varphi_3) \\ &\varepsilon_1 = \max_{\underset{\sim}{v} \neq \underset{\sim}{0}} \varepsilon(\underset{\sim}{v})\,, \quad \varepsilon_3 = \min_{\underset{\sim}{v} \neq \underset{\sim}{0}} \varepsilon(\underset{\sim}{v})\,, \quad \lambda_1 = \max_{\underset{\sim}{v} \neq \underset{\sim}{0}} \lambda(\underset{\sim}{v})\,, \quad \lambda_3 = \min_{\underset{\sim}{v} \neq \underset{\sim}{0}} \lambda(\underset{\sim}{v}) \end{aligned}\right\} (2$$

<u>Thm</u>. 2.8 (Volume change under H). Let H be a homog. deformation with the tensor E. Let $\mathcal{R} \subset E$ be a bounded region with image $\mathcal{R}^*$ under H. Let $v(\mathcal{R})$ and $v(\mathcal{R}^*)$ be the volumes of $\mathcal{R}$ and $\mathcal{R}^*$, respectively. Then $v(\mathcal{R}^*)/v(\mathcal{R})$ is independent of $\mathcal{R}$ and

$$\left.\begin{aligned} \Delta &= \frac{v(\mathcal{R}^*)}{v(\mathcal{R})} = J = \det E \;\ldots \text{volume ratio of } H \\ \Theta &= \frac{v(\mathcal{R}^*) - v(\mathcal{R})}{v(\mathcal{R})} = \Delta - 1 = J - 1 \ldots \text{vol. dilatat. of } H \end{aligned}\right\} (2.$$

Further, if $\Phi_i = I_i(\underset{\sim}{\varphi})$ are the scalar invariants of the Lagrangian strain tensor $\underset{\sim}{\varphi}$ of H, one has

$$\Theta = \sqrt{1+2\Phi_1+4\Phi_2+8\Phi_3} - 1. \quad (*)$$

<u>Proof</u>. From Lemma preceding Thm. 2.1,

$$\vartheta(\mathcal{R}^*) = \int_{\mathcal{R}^*} dV_{\underset{\sim}{y}} = \int_{\mathcal{R}} J\, dV_{\underset{\sim}{x}} = J \int_{\mathcal{R}} dV_{\underset{\sim}{x}} = J\,\vartheta(\mathcal{R})$$

so that (2.34) is true. Next, to confirm (*) take R to unit cube with edges par. to p. axes of $\underset{\sim}{\varphi}$ (see picture on p.79). Then, by (2.34), (2.32)

$$\Theta = \vartheta(\mathcal{R}^*) - 1 = \lambda_1\lambda_2\lambda_3 - 1 = \sqrt{(1+2\varphi_1)(1+2\varphi_2)(1+2\varphi_3)} - 1 \quad (a)$$

But from (1.34),

$$\Phi_1 = \varphi_1+\varphi_2+\varphi_3\,,\ \Phi_2 = \varphi_1\varphi_2+\varphi_2\varphi_3+\varphi_3\varphi_1\,,\ \Phi_3 = \varphi_1\varphi_2\varphi_3 \quad (b)$$

(a), (b) and trivial algebra yield (*). qed.

<u>Def</u>. H is a <u>pure homogeneous deformation</u> if its vector is null and its tensor is symm., positive definite, i.e. if it is of the form

$$H: \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{E}\underset{\sim}{x} \ \forall \underset{\sim}{x} \in E\,,\ \underset{\sim}{E} = \underset{\sim}{E}^T,\ \underset{\sim}{E} \text{ pos. def.} \quad (2.35).$$

Thm. 2.9

(a) If H is pure and $\underset{\sim}{E}$ is the tensor of H, then

$$\underset{\sim}{L} = \frac{1}{2}(\underset{\sim}{E}^2 - \underset{\sim}{1}),\ \underset{\sim}{C} = \underset{\sim}{G} = \underset{\sim}{E}^2,\ \underset{\sim}{E} = \sqrt{\underset{\sim}{C}} = \sqrt{\underset{\sim}{G}} \qquad (2.36)$$

and $\underset{\sim}{E}, \underset{\sim}{L}, \underset{\sim}{C}, \underset{\sim}{G}$ have common princ. axes. Further, if $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ is p. frame for $\underset{\sim}{E}$, then

$$H: y_i = \lambda_i x_i \text{ (no sums)},\ \lambda_i = \lambda(\underset{\sim}{e}_i) > 0, \qquad (2.37)$$

where λ_i are the p. values of $\underset{\sim}{E}$, $x_i = \underset{\sim}{x} \cdot \underset{\sim}{e}_i$, $y_i = \underset{\sim}{y} \cdot \underset{\sim}{e}_i$.

(b) Conversely, if for some $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ a deformation H is given by (2.37), then H is pure homog. and X is a p. frame for the tensors $\underset{\sim}{E}$, $\underset{\sim}{L}, \underset{\sim}{C}, \underset{\sim}{G}$ associated with H, and λ_i are the p. stretch ratios of H

Proof

(a) (2.36) and the claim concerning the p. axes are immediate. Also, if λ_i^2 are the p. values of $\underset{\sim}{C}, \underset{\sim}{G}$, one has from (2.36),

$$[\overset{X}{C_{ij}}] = [\overset{X}{G_{ij}}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix},\ [\overset{X}{F_{ij}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (\lambda_i > 0),$$

But $H: \underset{\sim}{y} = \underset{\sim}{F}\underset{\sim}{x}\ \forall \underset{\sim}{x} \in E \Rightarrow y_i^{\overline{X}} = F_{ij}^{\overline{X}}\, x_j^{\overline{X}}$ now gives (2.37).

(b) Equally trivial.

<u>Remark</u>. Thus any pure homog. H is a superposition of three mutually orthog. stretches parallel to p. axes of the tensor of H. H carries the p. axes of $\underset{\sim}{F}$ into themselves. Also, it maps the unit sphere $x_i x_i = 1$ onto the ellipsoid
$$\frac{y_1^2}{\lambda_1^2} + \frac{y_2^2}{\lambda_2^2} + \frac{y_3^2}{\lambda_3^2} = 1. \quad (y_i \text{ coords. in p. frame})$$

<u>Thm</u>. 2.10. (Resolution of arbitrary homog. deformat.) Let
$$H: \underset{\sim}{y} = \underset{\sim}{F}\underset{\sim}{x} + \underset{\sim}{c}\ \ \forall \underset{\sim}{x} \in E,\ \det \underset{\sim}{F} > 0$$
and let
$$\underset{\sim}{F} = \underset{\sim}{Q}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{Q},\ \underset{\sim}{U} = \sqrt{\underset{\sim}{F}^T\underset{\sim}{F}} = \sqrt{\underset{\sim}{C}},\ \underset{\sim}{V} = \sqrt{\underset{\sim}{F}\underset{\sim}{F}^T} = \sqrt{\underset{\sim}{G}},\ \underset{\sim}{Q} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}.$$

(a) Suppose
$$\left.\begin{aligned} &H': \underset{\sim}{x} \to \underset{\sim}{y}': \underset{\sim}{y}' = \underset{\sim}{U}\underset{\sim}{x} \quad (\text{pure homog. def.}) \\ &H'': \underset{\sim}{y}' \to \underset{\sim}{y}'': \underset{\sim}{y}'' = \underset{\sim}{Q}\underset{\sim}{y}' \quad (\text{rigid def. about } O)\ , \\ &H''': \underset{\sim}{y}'' \to \underset{\sim}{y}: \underset{\sim}{y} = \underset{\sim}{y}'' + \underset{\sim}{c} \quad (\text{translation}) \end{aligned}\right\}(2.38c$$

Then,

$$\underset{\sim}{H} = \underset{\sim}{H}''' \underset{\sim}{H}'' \underset{\sim}{H}' \qquad (*)$$

Conversely, $(*) \Rightarrow$ (2.38a) if H' is a pure homog. def., H'' a rigid def. about O, and H''' is a translation.

(b) Suppose

$$\left.\begin{array}{lll} H': \underset{\sim}{x} \to \underset{\sim}{y}': & \underset{\sim}{y}' = \underset{\sim}{Q}\,\underset{\sim}{x} & \text{(rigid def. about O)} \\ H'': \underset{\sim}{y}' \to \underset{\sim}{y}'': & \underset{\sim}{y}'' = \underset{\sim}{V}\,\underset{\sim}{y}' & \text{(pure homog. def.)} \\ H''': \underset{\sim}{y}'' \to \underset{\sim}{y}: & \underset{\sim}{y} = \underset{\sim}{y}'' + \underset{\sim}{c} & \text{(translation)} \end{array}\right\} (2.38b)$$

Then $(*)$ holds again. Conversely, $(*) \Rightarrow$ (2.38b) if H' is a rigid def. about O, H'' is pure, and H''' a trans

Proof. By substitution and uniqueness of right & left polar decompositions.

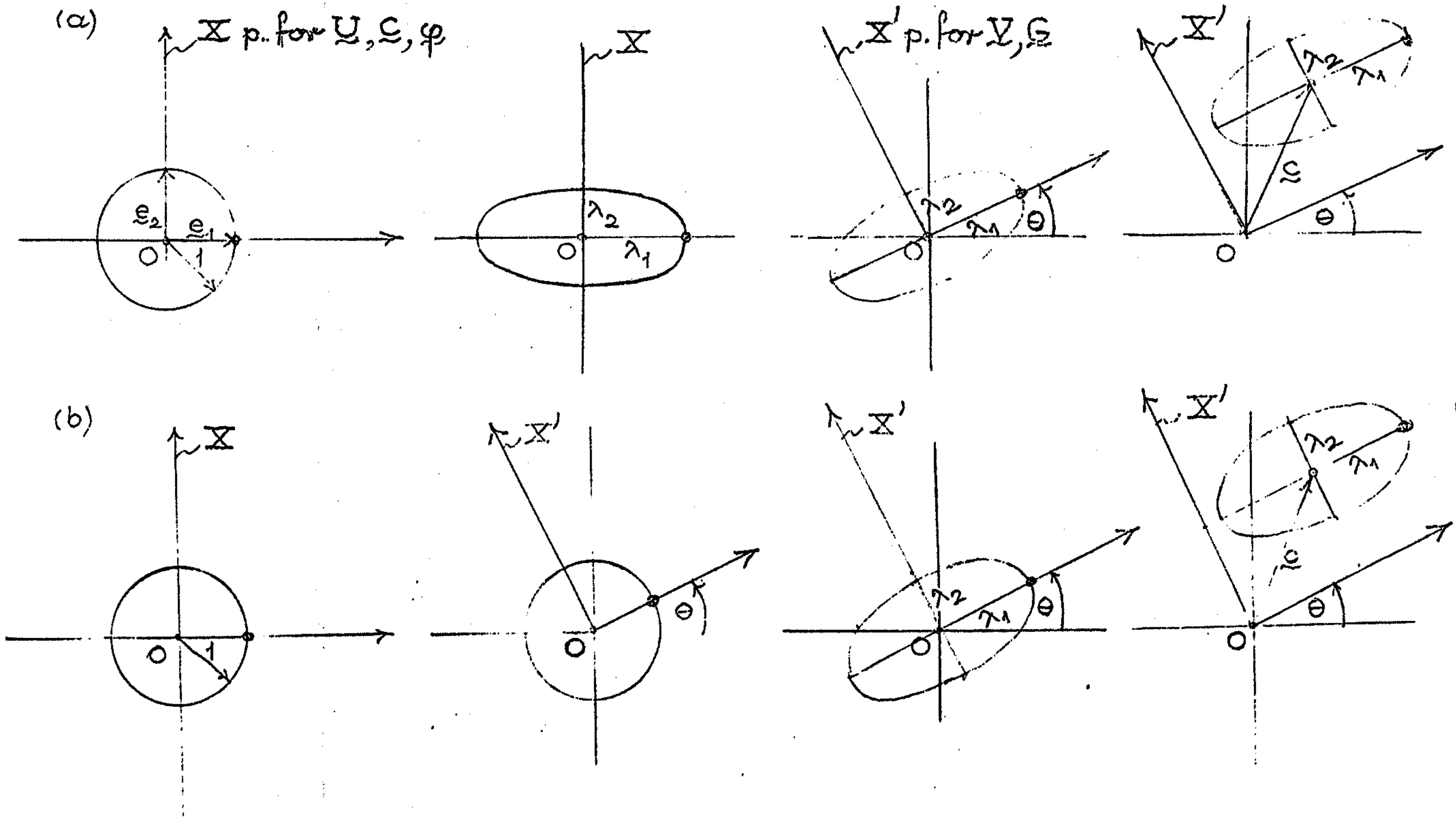
Remark. H' in (a) and H have the same Lagrangian strain tensor. To see this note by means of (2.21),

$$\underset{\sim}{L} = \frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{1}) = \frac{1}{2}\{(\underset{\sim}{Q}\underset{\sim}{U})^T(\underset{\sim}{Q}\underset{\sim}{U}) - \underset{\sim}{1}\} = \frac{1}{2}(\underset{\sim}{U}^T\underset{\sim}{Q}^T\underset{\sim}{Q}\underset{\sim}{U} - \underset{\sim}{1})$$

$$= \frac{1}{2}(\underset{\sim}{U}^T\underset{\sim}{U} - \underset{\sim}{1}) = \underset{\sim}{L}' \checkmark$$

Note, however, that $\underset{\sim}{L} \neq \underset{\sim}{L}'$ in the resolution (b)!

Schematic two-dimensional illustration of the two resolutions of $\mathbf{H}$



See Exercise 6, (b) for relation between p. axes of $\underset{\sim}{U}$ & $\underset{\sim}{V}$.

Consider

$$H: \underset{\sim}{y} = \underset{\sim}{F}\underset{\sim}{x} + \underset{\sim}{c} = \underset{\sim}{x} + \underbrace{(\underset{\sim}{F} - \underset{\sim}{1})\underset{\sim}{x}}_{\underset{\sim}{u}(\underset{\sim}{x})} + \underset{\sim}{c} \ \forall \underset{\sim}{x} \in E, \ \det \underset{\sim}{F} > 0 \quad (*)$$

Set

$$\underset{\sim}{\beta} = \nabla \underset{\sim}{u} = \underset{\sim}{F} - \underset{\sim}{1} \tag{2.39}$$

$\underset{\sim}{\beta}$ ... displacement-gradient tensor of H

For future convenience introduce the notation:

$$\left.\begin{aligned} \underset{\sim}{\gamma} &= \operatorname{sym} \underset{\sim}{\beta} = \tfrac{1}{2}(\underset{\sim}{\beta} + \underset{\sim}{\beta}^T) \quad \text{or} \quad \gamma_{ij} = \tfrac{1}{2}(\beta_{ij} + \beta_{ji}) = \beta_{(ij)} \\ \underset{\sim}{\omega} &= \operatorname{skw} \underset{\sim}{\beta} = \tfrac{1}{2}(\underset{\sim}{\beta} - \underset{\sim}{\beta}^T) \quad \text{or} \quad \omega_{ij} = \tfrac{1}{2}(\beta_{ij} - \beta_{ji}) = \beta_{[ij]} \\ \underset{\sim}{w} &= \operatorname{dual} \underset{\sim}{\omega} \quad \text{or} \quad w_i = \tfrac{1}{2}\varepsilon_{ijk}\omega_{kj}, \quad \vartheta = \operatorname{tr} \underset{\sim}{\gamma} = \gamma_{ii} = \beta_{ii} \end{aligned}\right\} (2.40)$$

We shall not attempt to interpret $\underset{\sim}{\gamma}, \underset{\sim}{\omega}, \underset{\sim}{w}, \vartheta$ kinematically, although it is possible to do so (see Novozhilov, 1961, Ch. 1). As will emerge presently, these quantities assume a simple approximate kinematic significance if $|\underset{\sim}{\beta}| \equiv |\nabla \underset{\sim}{u}| \ll 1$.

<u>Infinitesimal homogeneous deformations</u>

With a view to linearized kinematics for solid continua we now linearize the theory of homog. deformations on the assumption that

$$|\nabla \underset{\sim}{u}| = |\underset{\sim}{\beta}| = \sqrt{\beta_{ij}\beta_{ij}} \ll 1 \qquad (2.41)$$

This leads to the study of "infinites. homog deformations" (inf. affine mappings). Write (*) as

$$H: \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = (\underset{\sim}{1}+\underset{\sim}{\beta})\underset{\sim}{x}+\underset{\sim}{c} \quad \forall \underset{\sim}{x}\in E, \ \underset{\sim}{F} = \underset{\sim}{1}+\underset{\sim}{\beta}, \ \det \underset{\sim}{F}>0 \quad (2.42)$$

Recall from (2.19) that $\underset{\sim}{\varphi} = \frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F}-\underset{\sim}{1})$. Thus & by (2.40)

$$\underset{\sim}{\varphi} = \tfrac{1}{2}[(\underset{\sim}{1}+\underset{\sim}{\beta})^T(\underset{\sim}{1}+\underset{\sim}{\beta})-\underset{\sim}{1}] = \tfrac{1}{2}(\underset{\sim}{\beta}+\underset{\sim}{\beta}^T+\underset{\sim}{\beta}^T\underset{\sim}{\beta}) \sim \tfrac{1}{2}(\underset{\sim}{\beta}+\underset{\sim}{\beta}^T) = \underset{\sim}{\gamma} \quad (\S$$

To give a precise meaning to (§) consider a one-parame family of homogeneous deformations ∋

$$H_\eta: \underset{\sim}{y} = [\underset{\sim}{\beta}(\eta)+\underset{\sim}{1}]\underset{\sim}{x}+\underset{\sim}{c}, \ \det[\underset{\sim}{\beta}(\eta)+\underset{\sim}{1}]>0 \qquad (0<\eta<\eta_0)$$

$$\nabla \underset{\sim}{u}(\eta) \equiv \underset{\sim}{\beta}(\eta) = O(\eta) \ \text{as} \ \eta\to 0$$

Then,

$$\underset{\sim}{\varphi}(\eta) = \tfrac{1}{2}[\underset{\sim}{\beta}(\eta)+\underset{\sim}{\beta}^T(\eta)]+O(\eta^2) \ \text{as} \ \eta\to 0$$

Anyway, from above and since $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$, $\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T$, one has now

$$\left.\begin{array}{l} \underset{\sim}{\varphi} \sim \operatorname{sym} \underset{\sim}{\beta} = \underset{\sim}{\gamma} \quad \text{or} \quad \varphi_{ij} \sim \frac{1}{2}(\beta_{ij} + \beta_{ji}) = \gamma_{ij} \\ \underset{\sim}{C} \sim \underset{\sim}{G} \sim \underset{\sim}{1} + 2\underset{\sim}{\gamma} \end{array}\right\} \quad (2.43)$$

One calls $\underset{\sim}{\gamma}$ the <u>infinitesimal strain-tensor</u> of H in the present context.

Recall geom. meaning of φ_{ij} from (2.25), (2.27), (2.28) and note, upon linearization

$$\varepsilon(\underset{\sim}{e}_i) \sim \varphi_{ii} \sim \gamma_{ii} \;(\text{no sums}) \quad \tfrac{1}{2}\mu_{ij} \sim \varphi_{ij} \sim \gamma_{ij} \;(i \neq j) \quad (2.44)$$

Discuss (2.44). Next, from (2.34) the volume dilatation assoc. with H obeys

$$\Theta = \frac{v(R_*) - v(R)}{v(R)} = J - 1 = \det(\underset{\sim}{1} + \underset{\sim}{\beta}) - 1 = \begin{vmatrix} 1+\beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & 1+\beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & 1+\beta_{33} \end{vmatrix} - 1$$

Thus, using (2.40), one has

$$\Theta \sim \operatorname{tr} \underset{\sim}{\beta} = \beta_{ii} = \gamma_{ii} = \operatorname{tr} \underset{\sim}{\gamma} = I_1(\underset{\sim}{\gamma}) \quad (2.45)$$

<u>Composition of infinitesimal homog. deformations</u>

Consider two homog. deformations:

$$H': \ \underset{\sim}{y}' = \underset{\sim}{E}'\underset{\sim}{x} + \underset{\sim}{c}', \ \underset{\sim}{\beta}' = \underset{\sim}{E}' - \underset{\sim}{1} \ ; \ H'': \ \underset{\sim}{y} = \underset{\sim}{E}''\underset{\sim}{y}' + \underset{\sim}{c}'', \ \underset{\sim}{\beta}'' = \underset{\sim}{E}'' - \underset{\sim}{1}$$

$$H = H''H': \ \underset{\sim}{y} = \underset{\sim}{E}\underset{\sim}{x} + \underset{\sim}{c}, \ \underset{\sim}{\beta} = \underset{\sim}{E} - \underset{\sim}{1}$$

$$\underset{\sim}{E} = \underset{\sim}{E}''\underset{\sim}{E}', \ \underset{\sim}{c} = \underset{\sim}{E}''\underset{\sim}{c}' + \underset{\sim}{c}'' \quad \text{whence}$$

$$\underset{\sim}{E} = \underset{\sim}{1} + \underset{\sim}{\beta} = (\underset{\sim}{1} + \underset{\sim}{\beta}'')(\underset{\sim}{1} + \underset{\sim}{\beta}'), \quad \underset{\sim}{c} = (\underset{\sim}{1} + \underset{\sim}{\beta}'')\underset{\sim}{c}' + \underset{\sim}{c}''$$

Hence upon linearization with respect to $\underset{\sim}{\beta}', \underset{\sim}{\beta}''$,

$$H = H''H': \quad \underset{\sim}{\beta} \sim \underset{\sim}{\beta}' + \underset{\sim}{\beta}'', \ \underset{\sim}{c} \sim \underset{\sim}{c}' + \underset{\sim}{c}'' \tag{2.46}$$

In this sense the product of two inf. homog. deformations is commutative and this composition is additive as far as the displacements-gradient tensors and the translation vectors are concerned (superposition princ.)

<u>Infinitesimal rigid homogeneous deformations</u>

Recall that H is rigid if and only if $\underset{\sim}{E}\underset{\sim}{E}^T = \underset{\sim}{1}$ or, equivalently if and only if $\underset{\sim}{\varphi} = \underset{\sim}{0}$. But $\underset{\sim}{\varphi} \sim \underset{\sim}{\gamma}$ by (2.43). This motivates the following definition.

<u>Definition</u>. A homog. deformat. is <u>infinitesimally rigid</u>

$$\underset{\sim}{\gamma} = \operatorname{sym} \underset{\sim}{\beta} = \underset{\sim}{0} \quad \text{or} \quad \underset{\sim}{\beta} = -\underset{\sim}{\beta}^T \quad \text{or} \quad \underset{\sim}{\omega} = \operatorname{skw} \underset{\sim}{\beta} = \underset{\sim}{\beta} \tag{2.47}$$

Thus an <u>infinitesimally rigid H is of the form</u>

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{\omega}\,\underset{\sim}{x} + \underset{\sim}{c} \;\; \forall\, \underset{\sim}{x} \in E \;, \;\; \underset{\sim}{\omega} = -\underset{\sim}{\omega}^T \qquad (2.48)$$

$\underset{\sim}{\omega}$ ... <u>infinitesimal rotation tensor</u> of H

In view of (2.40) one may write (2.48) as

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{w} \wedge \underset{\sim}{x} + \underset{\sim}{c} \;\; \forall\, \underset{\sim}{x} \in E \;, \;\; \underset{\sim}{w} = \text{dual}\; \underset{\sim}{\omega} \qquad (2.49)$$

To obtain a <u>geom. interpretation</u> of $\underset{\sim}{w}$ recall from (2.15) in Exercise 8 that a right-handed rotation throug $\theta \in [0, 2\pi)$ about an axis through $\underset{\sim}{O}$ with the unit direction vector $\underset{\sim}{n}$ is characterized by

$$H: \; \underset{\sim}{y} = \underset{\sim}{x}\cos\theta + (1-\cos\theta)(\underset{\sim}{x}\cdot\underset{\sim}{n})\underset{\sim}{n} + \sin\theta\; \underset{\sim}{n} \wedge \underset{\sim}{x} \;\; \forall\, \underset{\sim}{x} \in E$$

Linearize (*) with respect to θ to arrive at

$$\dot{H}: \; \underset{\sim}{y} = \underset{\sim}{x} + (\underset{\sim}{n}\theta) \wedge \underset{\sim}{x} + O(\theta^2) \;\; \text{as } \theta \to 0 \qquad (2.50)$$

Compare (2.49) and (2.48) with $\underset{\sim}{c} = \underset{\sim}{O}$. Thus, $\underset{\sim}{w} \sim \underset{\sim}{n}\theta$.

$\underset{\sim}{w} = \text{dual}\; \underset{\sim}{\omega}$ ... <u>infinitesimal rotation vector</u> of H

<u>Definition</u>. A homog. deformation H is <u>infinitesimally pure</u> if $\underset{\sim}{c} = \underset{\sim}{0}$ and

$$\underset{\sim}{\omega} = \text{skw}\,\underset{\sim}{\beta} = \underset{\sim}{0} \text{ or } \underset{\sim}{\beta} = \underset{\sim}{\beta}^T \text{ or } \underset{\sim}{\gamma} = \text{sym}\,\underset{\sim}{\beta} = \underset{\sim}{\beta}. \tag{2.51}$$

In this case,

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{\gamma}\underset{\sim}{x} \quad \forall\, \underset{\sim}{x} \in E, \quad \underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T$$

The above def. is motivated by the following theorem.

<u>Thm.</u> 2.11 (Resolution of arbitrary inf. homog. deformats.
Every infinites. homog def. H admits a unique decomposition $H = H'''H''H'$ $\ni$ H', H'', H''' are homog. deformations and H' is inf. pure, H'' inf. rigid def. about O, while H''' a translation. Further, this resolution is commutative and if $\underset{\sim}{\beta}, \underset{\sim}{\beta}', \underset{\sim}{\beta}'', \underset{\sim}{\beta}'''$ and $\underset{\sim}{c}, \underset{\sim}{c}', \underset{\sim}{c}'', \underset{\sim}{c}'''$ are the displt.-gradient tensors and the translation vectors of H, H', H'', H''' respectively, one has

$$\underset{\sim}{\beta}' = \text{sym}\,\underset{\sim}{\beta} = \underset{\sim}{\gamma}, \quad \underset{\sim}{\beta}'' = \text{skw}\,\underset{\sim}{\beta} = \underset{\sim}{\omega}, \quad \underset{\sim}{\beta}''' = \underset{\sim}{0}, \quad \underset{\sim}{c}' = \underset{\sim}{c}'' = \underset{\sim}{0}, \quad \underset{\sim}{c}''' = \underset{\sim}{c}. \tag{2.5}$$

<u>Proof</u>. Immediate from pertinent definitions, (2.46), and Thm. 1.10 (which assures the uniqueness of the above resolution).

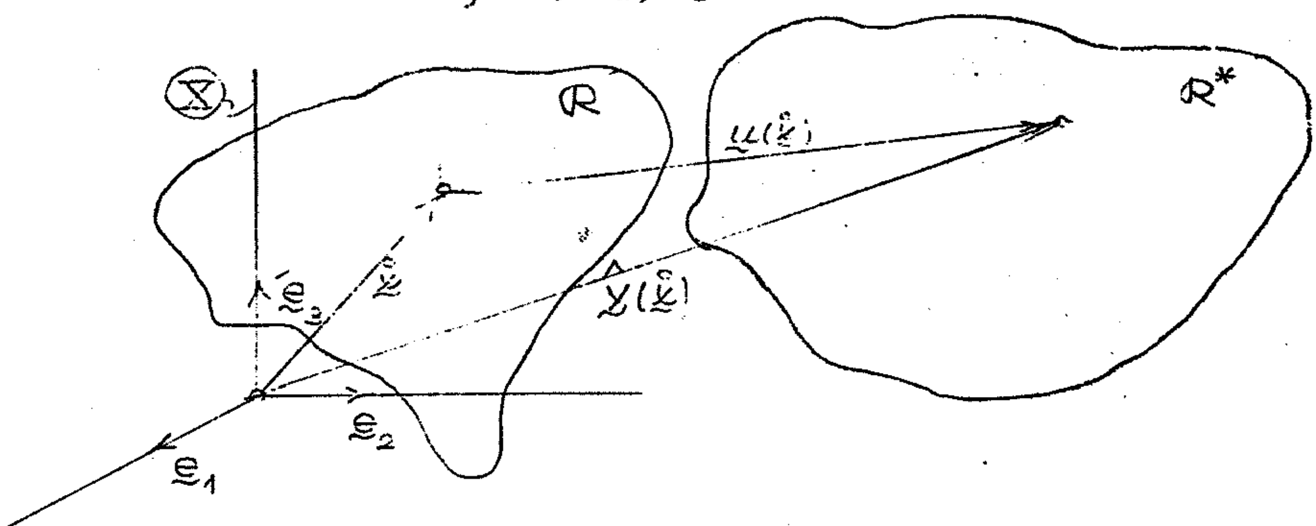
General admissible deformations

Recall definition of an admiss. deform.at. $\hat{\mathbf{y}}: \mathcal{R} \to \mathcal{R}_*$

$$\left.\begin{aligned} \mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \ \forall\ \mathbf{x} \in \mathcal{R},\ \hat{\mathbf{y}} \in C^2(\mathcal{R}) \\ \mathbf{F} = \nabla\hat{\mathbf{y}} = \nabla\mathbf{u} + \mathbf{1},\ J = \det \mathbf{F} > 0 \end{aligned}\right\} \quad (2.53)$$

(Assumptions that $\hat{y}$ is (1,1) not needed at present).

Consider $\mathbb{X} = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathcal{F}$ and choose $\mathring{\mathbf{x}} \in \mathring{\mathcal{R}}$.



Apply Taylor expansion about $\mathring{\mathbf{x}}$ to $\hat{\mathbf{y}}(\mathbf{x})$ and thus obtain

$$y_i = \hat{y}_i(\mathbf{x}) = \hat{y}_i(\mathring{\mathbf{x}}) + \hat{y}_{i,j}(\mathring{\mathbf{x}})(x_j - \mathring{x}_j) + O(|\mathbf{x} - \mathring{\mathbf{x}}|^2) \text{ as } \mathbf{x} \to \mathring{\mathbf{x}}$$

or, in coordinate-free notation, since $\hat{\mathbf{y}}(\mathring{\mathbf{x}}) = \mathring{\mathbf{x}} + \mathbf{u}(\mathring{\mathbf{x}})$,

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathring{\mathbf{x}} + \mathbf{u}(\mathring{\mathbf{x}}) + \nabla\hat{\mathbf{y}}(\mathring{\mathbf{x}})(\mathbf{x} - \mathring{\mathbf{x}}) + O(|\mathbf{x} - \mathring{\mathbf{x}}|^2) \text{ as } \mathbf{x} \to \mathring{\mathbf{x}}$$

Set

$$\mathbf{x}' = \mathbf{x} - \mathring{\mathbf{x}},\quad \mathbf{y}' = \hat{\mathbf{y}}(\mathbf{x}) - \mathring{\mathbf{x}}$$

$$\left.\begin{aligned} &\underline{y}' \doteq \underline{E}(\mathring{\underline{x}})\underline{x}' + \underline{c}(\mathring{\underline{x}}) + O(|\underline{x}'|^2) \quad \text{as } \underline{x}' \to \underline{0} \\ &\underline{E}(\mathring{\underline{x}}) = \nabla\hat{\underline{y}}(\mathring{\underline{x}}),\ \underline{c}(\mathring{\underline{x}}) = \hat{\underline{y}}(\mathring{\underline{x}}) - \mathring{\underline{x}} = \underline{u}(\mathring{\underline{x}})\ ,\ \underline{y}' = \hat{\underline{y}}(\underline{x}) - \mathring{\underline{x}},\ \underline{x}' = \underline{x} - \mathring{\underline{x}} \end{aligned}\right\} (2.5$$

In the above sense <u>every admiss. deformation is locally homog.</u>. Thus the theory of homog. deformations applies locally to arbitrary admiss. deformations.

In view of (2.54) and (2.19), (2.20) we <u>define</u> the right and left deformation-tensor <u>fields</u> of $\hat{\underline{y}}$ through

$$\underline{C} = \underline{E}^T\underline{E} = (\nabla\hat{\underline{y}})^T\nabla\hat{\underline{y}}\ ,\ \underline{G} = \underline{E}\,\underline{E}^T = \nabla\hat{\underline{y}}(\nabla\hat{\underline{y}})^T \text{ on } \mathcal{R} \qquad (2.55)$$

and <u>define</u> Lagrangian strain-tensor <u>field</u> by

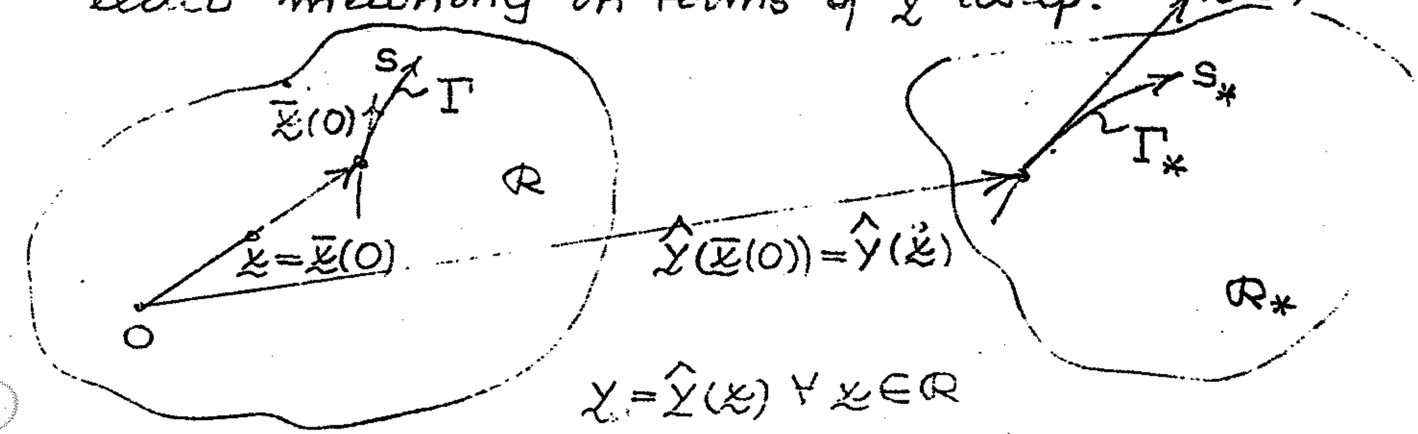
$$\left.\begin{aligned} &\underline{\varphi} = \tfrac{1}{2}(\underline{C} - \underline{1}) = \tfrac{1}{2}(\underline{E}^T\underline{E} - \underline{1}) \text{ or } \underline{\varphi} = \operatorname{sym}\nabla\underline{u} + \tfrac{1}{2}(\nabla\underline{u})^T\nabla\underline{u} \\ &\qquad \text{or } \varphi_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \text{ on } \mathcal{R} \end{aligned}\right\} (2.56$$

Explain geom. significance of φ_{ij} in present context. The last two of (2.56) are one version of the displt.-strain relations in the nonlinear kinematics of continuous media. Explain significance of $J = \det \underline{E}$ as volume-ratio field.

[1] Note that $\underline{E} = \nabla\hat{\underline{y}} = \nabla\{\underline{x} + \underline{u}(\underline{x})\} = \underline{1} + \nabla\underline{u}$

Remark on exact geom. significance of $\boldsymbol{\varphi}$

The kinematic signif. of $\boldsymbol{\varphi}(\mathring{\mathbf{x}})$ in terms of the homog. def. that locally approximates $\hat{\mathbf{y}}$ is clear from earlier analysis [see (2.25),(2.28)]. In addition $\varphi_{ij}(\mathring{\mathbf{x}})$ have exact meaning in terms of $\hat{\mathbf{y}}$ itself.



$$\Gamma: \ \mathbf{x} = \bar{\mathbf{x}}(s) \quad (0 \le s \le \ell), \quad \mathring{\mathbf{x}} = \mathbf{x}(0), \quad \bar{\mathbf{x}} \in C^1([0,\ell])$$

$$\Gamma_* = \hat{\mathbf{y}}(\Gamma): \ \mathbf{y} = \hat{\mathbf{y}}(\bar{\mathbf{x}}(s)) = \bar{\mathbf{y}}(s) \quad (0 \le s \le \ell)$$

$$s_*(s) = \int_0^s \sqrt{\dot{\bar{\mathbf{y}}}(\sigma)\cdot\dot{\bar{\mathbf{y}}}(\sigma)}\, d\sigma,$$ whence by chain-rule

$$\left(\frac{ds_*}{ds}\right)^2 = \dot{\bar{\mathbf{y}}}(s)\cdot\dot{\bar{\mathbf{y}}}(s) = \hat{y}_{k,i}(\bar{\mathbf{x}}(s))\,\dot{\bar{x}}_i(s)\,\hat{y}_{k,j}(\bar{\mathbf{x}}(s))\,\dot{\bar{x}}_j(s) \quad \left(\dot{\bar{x}}_i \equiv \frac{d\bar{x}_i}{ds}\right)$$

$$= \mathbf{E}(\bar{\mathbf{x}}(s))\dot{\bar{\mathbf{x}}}(s)\cdot\mathbf{E}(\bar{\mathbf{x}}(s))\dot{\bar{\mathbf{x}}}(s) = \mathbf{E}^T(\bar{\mathbf{x}}(s))\mathbf{E}(\bar{\mathbf{x}}(s))\dot{\bar{\mathbf{x}}}(s)\cdot\dot{\bar{\mathbf{x}}}(s)$$

But $\boldsymbol{\varphi} = \frac{1}{2}(\mathbf{E}^T\mathbf{E} - \mathbf{1})$ and $\dot{\bar{\mathbf{x}}}(s)\cdot\dot{\bar{\mathbf{x}}}(s) = 1$. Hence,

$$\left(\frac{ds_*}{ds}\right)^2 = 1 + 2\underset{\sim}{\varphi}(\underset{\sim}{x}(s))\dot{\underset{\sim}{x}}(s)\cdot\dot{\underset{\sim}{x}}(s) \quad \text{or}$$

$$\left(\frac{ds_*}{ds}\right)^2 = 1 + 2\varphi_{pq}(\underset{\sim}{x}(s))\dot{x}_p(s)\dot{x}_q(s)$$

Now let $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ and suppose

$$\dot{\underset{\sim}{x}}(0) = \underset{\sim}{e}_i \quad (\text{interpret}).$$

Then,

$$\left.\frac{ds_*}{ds}\right|_{s=0} = \sqrt{1 + 2\varphi_{ii}(\overset{\circ}{\underset{\sim}{x}})} \quad (\text{no sum}),$$

which is local analogue of (2.25). The components $\varphi_{ij}(\overset{\circ}{\underset{\sim}{x}})$ $(i \neq j)$ admit precise local interpretations analogous to (2.28).

In view of (2.39), (2.40) define the following <u>fields</u>:

$$\left.\begin{aligned} \underset{\sim}{\gamma} = \operatorname{sym}\nabla\underset{\sim}{u} \text{ or } \gamma_{ij} = u_{(i,j)}, \ \underset{\sim}{\omega} = \operatorname{skw}\nabla\underset{\sim}{u} \text{ or } \omega_{ij} = u_{[i,j]}, \\ \underset{\sim}{w} = \operatorname{dual}\underset{\sim}{\omega} \text{ or } w_i = \tfrac{1}{2}\varepsilon_{ijk}\omega_{kj}, \\ \vartheta = \operatorname{tr}\underset{\sim}{\gamma} = \gamma_{ii} = I_1(\gamma) \text{ on } \mathcal{R} \end{aligned}\right\} \quad (2.$$

$(2.57) \Rightarrow w_i = \frac{1}{2}\varepsilon_{ijk}u_{[k,j]} = \frac{1}{2}\varepsilon_{ijk}u_{k,j}$ (explain!)

$$\vartheta = u_{(i,i)} = u_{i,i}$$

so that from (1.43),

$$\underset{\sim}{w} = \tfrac{1}{2}\operatorname{curl}\underset{\sim}{u} = \tfrac{1}{2}\nabla\wedge\underset{\sim}{u}, \quad \vartheta = \operatorname{div}\underset{\sim}{u} = \nabla\cdot\underset{\sim}{u} \qquad (2.58)$$

Note that $\underset{\sim}{\gamma}, \underset{\sim}{\omega}, \underset{\sim}{w}, \vartheta$ have no simple geom. significan as yet.

<u>General infinitesimal admiss. deformations</u>

Assumption of infinitesimal deformations: $|\nabla\underset{\sim}{u}| \ll 1$

Then $\hat{\underset{\sim}{\chi}}$ is locally <u>infinitesimally</u> homogeneous (affine)

<u>Remarks</u>. Physical validity of above assumptions ultimately depends on experimental verification; assumption yields useful approximations for wide variety of elas

solids and problems. Important exceptions: highly deformable materials (eg. rubbers), large loads, highly flexible structures (eg. slender beams, thin plates or shells problems of elastic instability, singular problems.

For infinitesimal deformations $\underset{\sim}{\gamma}, \underset{\sim}{\omega}, \underset{\sim}{w}, \vartheta$ assume simple phys. meaning. Explain. Terminology:

$\underset{\sim}{\gamma}$ ... inf. strain tensor field

$\underset{\sim}{\omega}$ ... inf. rotation tensor field

$\underset{\sim}{w}$ ... inf. rotation vector field

ϑ ... inf. dilatation field

Interpret Thm. 2.11 (resolution of arb. inf. H) in the present context. The first of (2.57) are the displt.-strain relations in the kinematics of inf. deformations. Note from (2.56), (2.57) that $\underset{\sim}{L} \sim \underset{\sim}{\gamma} = \text{sym}\, \nabla \underset{\sim}{u}$ upon linearization.

Mention and comment on engineering notation for inf. strain compts.: $(x_1, x_2, x_3) = (x, y, z)$, $\gamma_{11} = \varepsilon_x, \ldots, 2\gamma_{12} = \gamma_{xy},$

Study of the linearized displacement-strain relations

Consider

$$\left.\begin{aligned} &\underset{\sim}{\gamma} = \operatorname{sym}\nabla\underset{\sim}{u} \quad \text{or} \quad \gamma_{ij} = u_{(i,j)} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \\ \text{or} \quad &\gamma_{11} = \frac{\partial u_1}{\partial x_1}\,;\ \ldots,\ \ldots,\ \gamma_{12} = \gamma_{21} = \tfrac{1}{2}(u_{1,2}+u_{2,1}),\ \ldots,\ \ldots \end{aligned}\right\} (2.59)$$

These arose as defining eqs. for the tensor field $\underset{\sim}{\gamma}$ associa with given admiss. $\hat{\underset{\sim}{v}}$. The determination of $\underset{\sim}{\gamma}$ from known $\underset{\sim}{u}$ is a matter of direct computation. For futu purposes it is essential to study (2.59) also as a system of 6 p.d. eqs. for the 3 unknown u_i, assuming γ_{ij} as known. This leads to the following problem:

Given a domain D and a symm. two-tensor field $\underset{\sim}{\gamma}$ on D, find a vector field $\underset{\sim}{u} \ni$ (2.59) hold.

Questions:

(a) Existence issue. Does sol. exist for arbitrary (sufficie smooth) given strain field $\underset{\sim}{\gamma}$? Answer negative (note 6 eqs. in 3 unknowns!)

(b) Uniqueness issue. If sol. does exist, is there more than one sol.? Answer "essentially" positive.

Turn first to the simpler uniqueness issue.

Thm. 2.12. Let D be a domain. Suppose $\underset{\sim}{u}' \in C^2(D)$ and $\underset{\sim}{u}'' \in C^2(D)$ are both solutions of (2.59) for the same $\underset{\sim}{\gamma}$. There $\exists$ a two-tensor $\mathring{\underset{\sim}{\omega}}$ and a vector $\mathring{\underset{\sim}{c}}$ $\ni$

$$\underset{\sim}{u}'(\underset{\sim}{x}) - \underset{\sim}{u}''(\underset{\sim}{x}) = \mathring{\underset{\sim}{\omega}}\underset{\sim}{x} + \mathring{\underset{\sim}{c}} \;\forall\; \underset{\sim}{x} \in D, \; \mathring{\underset{\sim}{\omega}}^T = -\mathring{\underset{\sim}{\omega}}$$

or

$$\underset{\sim}{u}'(\underset{\sim}{x}) - \underset{\sim}{u}''(\underset{\sim}{x}) = \mathring{\underset{\sim}{w}} \wedge \underset{\sim}{x} + \mathring{\underset{\sim}{c}} \;\forall\; \underset{\sim}{x} \in D, \; \mathring{\underset{\sim}{w}} = \text{dual}\, \mathring{\underset{\sim}{\omega}}$$

Thus $\underset{\sim}{u}'$ and $\underset{\sim}{u}''$ differ from each other by the displacement field of an infinitesimally rigid deformation of D

Proof. Define

$$\underset{\sim}{u} = \underset{\sim}{u}' - \underset{\sim}{u}'' \text{ on } D \qquad (1)$$

By hyp.,

$$\text{sym}\,\nabla \underset{\sim}{u} = \underset{\sim}{0} \text{ or } u_{i,j} + u_{j,i} = 0 \text{ on } D \qquad (2)$$

[Note: Thus $\underset{\sim}{\gamma} = \text{sym}\,\nabla\underset{\sim}{u} = \underset{\sim}{0}$ on D. The desired conclusion would be immediate if we knew that $\underset{\sim}{u}$ is the displacement field of a homog. def. of D. This is not obvious.]

By (2) and smoothness hyp.,

$$u_{i,jk} = -u_{j,ik}\ ,\quad u_{i,jk} = u_{i,kj} \qquad (3)$$

(3) $\Rightarrow$

$$\underset{\sim}{u_{i,jk}} = u_{i,kj} = -u_{k,ij} = -u_{k,ji} = u_{j,ki} = u_{j,ik} = -\underset{\sim}{u_{i,jk}}$$

so that

$$u_{i,jk} = 0 \text{ on } D \quad (4)$$

whence

$$u_i(\underset{\sim}{x}) = \mathring{\omega}_{ij} x_j + \mathring{c}_i \ \forall\ \underset{\sim}{x} \in D \ (\mathring{\omega}_{ij}, \mathring{c}_i \ldots \text{consts}) \quad (5)$$

Substitution from (5) into (2) yields $\mathring{\omega}_{ji} = -\mathring{\omega}_{ij}$.
Clearly, $\mathring{\omega}_{ij}$ & $\mathring{c}_i$ are 2-tensor & vector compts (explain).

Remark. The conclusion in Thm. 2.12 remains true if one assumes merely $\underset{\sim}{u}' \in C^1(D)$, $\underset{\sim}{u}'' \in C^1(D)$, as is natural but is harder to reach.

Turn to existence issue raised earlier and first deduce necessary conds. for existence of solution.

Thm. 2.13. Let D be a domain and $\underset{\sim}{\gamma}$ a two-tensor field on D. A necessary condition that $\exists$ a vector field $\underset{\sim}{u} \in C^3(D) \ni \operatorname{sym} \nabla \underset{\sim}{u} = \underset{\sim}{\gamma}$ on D is that $\underset{\sim}{\gamma} \in C^2(D)$, $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T$ on D and that $\underset{\sim}{\gamma}$ on D satisfy the "strain equation of compatibility":

$$\left.\begin{aligned}&\nabla\wedge(\nabla\wedge\underset{\sim}{\gamma})=\underset{\sim}{0} \text{ or } \varepsilon_{ipm}\varepsilon_{jqn}\gamma_{mn,pq}=0 \text{ or}\\ &\frac{\partial^2\gamma_{11}}{\partial x_2^2}+\frac{\partial^2\gamma_{22}}{\partial x_1^2}=2\frac{\partial^2\gamma_{12}}{\partial x_1\partial x_2};\ldots;\frac{\partial^2\gamma_{11}}{\partial x_2\partial x_3}=\frac{\partial}{\partial x_1}\left(-\frac{\partial\gamma_{23}}{\partial x_1}+\frac{\partial\gamma_{31}}{\partial x_2}+\frac{\partial\gamma_{12}}{\partial x_3}\right),\ldots\end{aligned}\right\}(2.6$$

<u>Proof</u>. Suppose $\exists\ \underset{\sim}{u}\in C^3(D)\ni$

$$\frac{1}{2}\{\nabla\underset{\sim}{u}+(\nabla\underset{\sim}{u})^T\}=\underset{\sim}{\gamma} \text{ on } D \quad (1)$$

Then trivially $\underset{\sim}{\gamma}\in C^2(D)$ and $\underset{\sim}{\gamma}=\underset{\sim}{\gamma}^T$. Recall from (1.47),

$$\nabla\wedge(\nabla\underset{\sim}{u})=\underset{\sim}{0}\ ,\ \nabla\wedge(\nabla\underset{\sim}{u})^T=\nabla(\nabla\wedge\underset{\sim}{u}) \quad (2)$$

(1), (2) $\Rightarrow$ $\nabla\wedge\underset{\sim}{\gamma}=\frac{1}{2}\nabla(\nabla\wedge\underset{\sim}{u})$; $\nabla\wedge(\nabla\wedge\underset{\sim}{\gamma})=\underset{\sim}{0}$, from which the indicial version follows on account of (1.4

qed

<u>Remark</u>. Note that the two-tensor $\nabla\wedge(\nabla\wedge\underset{\sim}{\gamma})$ ("in-compatibility <u>tensor</u>") is symmetric[1] (explain). Hen $\nabla\wedge(\nabla\wedge\underset{\sim}{\gamma})=\underset{\sim}{0}$ yields only 6 scalar equations of compatibility.

- <u>Exercise 11</u>. Show that the compatibility equations (2.6 are equivalent to each of the following two sets of equations:

1

$$\varepsilon_{jpm}\varepsilon_{iqn}\gamma_{mn,pq}=\varepsilon_{ipm}\varepsilon_{jqn}\underbrace{\gamma_{nm,qp}}$$

$$\gamma_{ij,kl} + \gamma_{kl,ij} - \gamma_{il,jk} - \gamma_{jk,il} = 0 \qquad (2.61)$$

$$\gamma_{ij,kk} + \gamma_{kk,ij} - \gamma_{ik,jk} - \gamma_{jk,ik} = 0 \qquad (2.62)$$

<u>HINT</u>: To show (2.60) $\Leftrightarrow$ (2.61) and (2.60) $\Leftrightarrow$ (2.62), it suffices to show (2.60) $\Rightarrow$ (2.61) $\Rightarrow$ (2.62) $\Rightarrow$ (2.60). Trivially (2.61) $\Rightarrow$ (2.62) by contraction. To complete proof set, for convenience, $\gamma_{ij,kl} = v_{ijkl}$ and note that $v_{ijkl} = v_{jikl} = v_{ijlk}$. Then deduce from (1.9) the identity

$$\varepsilon_{sik}\,\varepsilon_{spm}\,\varepsilon_{rjl}\,\varepsilon_{rqn}\,v_{pqmn} = v_{ijkl} + v_{klij} - v_{ilkj} - v_{kjil}$$

(Observe that the 81 eqs. (2.61) because of symm. reduce to only 6).

<u>Thm</u> 2.14 (Sufficiency of compat. eqs.). Let D be a <u>simply connected</u> domain and $\underset{\sim}{\gamma} \in C^2(D)$ a two-tensor field $\ni \underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T$ and (2.60) hold on D. Then $\exists$ a vector field $\underset{\sim}{u} \in C^3(D) \ni \underset{\sim}{\gamma} = \operatorname{sym} \nabla \underset{\sim}{u}$ on D.

<u>Proof</u>. By hyp.,

$$\underset{\sim}{\gamma} \in C^2(D), \quad \underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T, \quad \nabla \wedge (\nabla \wedge \underset{\sim}{\gamma}) = \underset{\sim}{0} \text{ on } D. \qquad (1)$$

Let

$$\underset{\sim}{T} = \nabla\wedge\underset{\sim}{\gamma} \quad \text{on } D \qquad (2)$$

(1), (2); (1.46) $\Rightarrow$

$$\underset{\sim}{T}\in C^1(D)\,;\ \nabla\wedge\underset{\sim}{T}=\underset{\sim}{0}\,,\ tr\,\underset{\sim}{T}=\underset{\sim}{0} \ \text{on } D \ (3)$$

(since (1.46) $\Rightarrow T_{ij} = \varepsilon_{ipq}\gamma_{jq,p} \Rightarrow T_{ii} = \varepsilon_{ipq}\gamma_{iq,p}$ & $\gamma_{iq}=\gamma_{qi}$)

(3) & (b) in Thm. 1.17 (p.46) $\exists$ a two-tensor field $\underset{\sim}{W} \ni$

$$\underset{\sim}{W}\in C^2(R)\,,\ \underset{\sim}{W}^T=-\underset{\sim}{W}\,,\ \underset{\sim}{T}=\nabla\wedge\underset{\sim}{W} \ \text{on } D \quad (4)$$

But (2), (4) $\Rightarrow$

$$\nabla\wedge(\underset{\sim}{\gamma}-\underset{\sim}{W})=\underset{\sim}{0} \ \text{on } D. \ (5)$$

(5) & (a) in Thm. 1.17 $\Rightarrow \exists$ a vector field $\underset{\sim}{u}\in C^3(D)\ni$

$$\underset{\sim}{\gamma}-\underset{\sim}{W}=\nabla\underset{\sim}{u} \ \text{on } D. \ (6)$$

From (6), the second of (1) & the second of (4) one has

$$\text{sym}\,(\underset{\sim}{\gamma}-\underset{\sim}{W})=\underset{\sim}{\gamma}=\text{sym}\,\nabla\underset{\sim}{u} \ \text{on } D. /$$

Discussion.

(i) In view of Thms. 2.13, 2.14 the compat. conds. (2.60) are nec. & suff. conds. of integrability for the displ. strain rels. (2.59) — nec. if D is arbitrary, but suff merely if D is simply connected.

(ii) Sketch analogy as to existence & uniqueness of the following two problems:

(a) Given $\underset{\sim}{v}$ on D, find $\phi \ni \nabla\phi = \underset{\sim}{v}$ on D (see Thm.1.16)

(b) Given $\underset{\sim}{\gamma}$ on D, find $\underset{\sim}{u} \ni \operatorname{sym}\nabla\underset{\sim}{u} = \underset{\sim}{\gamma}$ on D (see Thms 2.12, 2.13, 2.14)

(iii) If D is <u>multiply</u> connected and the hyp. of Thm. 2.14 hold <u>otherwise</u>, the conclusion clearly remains valid on any <u>simply connected subdomain</u> of D.

(iv) Suppose D <u>multiply</u> connected, the hyp. of Thm 2.14 hold <u>otherwise</u> and $\exists$ a <u>simply</u> connected $\hat{D} \supset D$ and a two-tensor field $\hat{\underset{\sim}{\gamma}} \ni$

$\hat{\underset{\sim}{\gamma}} \in \mathcal{C}^2(\hat{D})$, $\hat{\underset{\sim}{\gamma}}^T = \hat{\underset{\sim}{\gamma}}$, $\nabla\wedge(\nabla\wedge\hat{\underset{\sim}{\gamma}}) = \underset{\sim}{0}$ on $\hat{D}$, $\hat{\underset{\sim}{\gamma}} = \underset{\sim}{\gamma}$ on D.

Then Thm. 2.15 $\Rightarrow \exists\ \underset{\sim}{u} \in \mathcal{C}^3(D) \ni \operatorname{sym}\nabla\underset{\sim}{u} = \underset{\sim}{\gamma}$ on D.

<u>Example</u>. Consider $\gamma_{ij} = c_{ijk}x_k + d_{ij}\ \forall\ \underset{\sim}{x} \in D$ ($c_{jik} = c_{ijk}$, $d_{ji} = d_{ij}$), D multiply connected.

Clearly, $\underset{\sim}{\gamma} \in \mathcal{C}^2(D)$, $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T$, $\nabla\wedge(\nabla\wedge\underset{\sim}{\gamma}) = \underset{\sim}{0}$ on D.

Take $\hat{D} = E$, $\hat{\gamma}_{ij} = c_{ijk}x_k + d_{ij}\ \forall\ \underset{\sim}{x} \in E$. Note if $c_{ijk} = 0$, one has $\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{\gamma}\underset{\sim}{x}\ \forall\ \underset{\sim}{x} \in D$.

<u>Thm</u> 2.15 (Integral representation for the displacements) Let D be a <u>simply connected</u> domain and let $\underset{\sim}{\gamma}$ satisfy the hypotheses of Thm. 2.14. Let $\underset{\sim}{u} \in C^2(D)$ and sym $\nabla \underset{\sim}{u} = \underset{\sim}{\gamma}$ on D, the existence of such a $\underset{\sim}{u}$ being assured by Thm. 2.14. Then $\underset{\sim}{u}$ admits the representation:

$$u_i(\underset{\sim}{x}) = \int_{\mathring{\underset{\sim}{x}}}^{\underset{\sim}{x}} U_{ij}(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\xi_j + v_i(\underset{\sim}{x}) \ \forall \ \underset{\sim}{x} \in D, \qquad (2.63)$$

$$U_{ij}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \gamma_{ij}(\underset{\sim}{\xi}) + (x_k - \xi_k)\{\gamma_{ij,k}(\underset{\sim}{\xi}) - \gamma_{kj,i}(\underset{\sim}{\xi})\} \ \forall \ (\underset{\sim}{x}, \underset{\sim}{\xi}) \in D \times D \qquad (2.64)$$

where $\mathring{\underset{\sim}{x}}$ is an arbitrarily chosen point in D, $\underset{\sim}{v}$ is the displacement field of an infinitesimally rigid deformation, while the path of integration is <u>any</u> piecewise smooth curve $\Gamma \subset D$ that joins $\mathring{\underset{\sim}{x}}$ and $\underset{\sim}{x}$.

<u>Proof</u>. Suppose

$$u_i \in C^2(D), \quad u_{(i,j)} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \gamma_{ij} \text{ on } D. \ (*)$$

Let $\underset{\sim}{x} \in D$. Choose and fix $\mathring{\underset{\sim}{x}} \in D$. Choose a pw smooth path $\Gamma \subset D$ that joins $\mathring{\underset{\sim}{x}}$ and $\underset{\sim}{x}$.

$\Gamma : \underset{\sim}{\eta} = \underset{\sim}{\xi}(\tau)$ $(\tau_0 \le \tau \le \tau_1)$, $\underset{\sim}{\xi}$ pw. smooth on $[\tau_0, \tau_1]$, $\underset{\sim}{\xi}(\tau_0) = \overset{\circ}{\underset{\sim}{x}}$, $\underset{\sim}{\xi}(\tau_1) = \underset{\sim}{x}$ (1)

Let

$$u_i(\overset{\circ}{\underset{\sim}{x}}) = \overset{\circ}{u}_i \,, \quad u_{[i,j]}(\overset{\circ}{\underset{\sim}{x}}) = \omega_{ij}(\overset{\circ}{\underset{\sim}{x}}) = \overset{\circ}{\omega}_{ij} \qquad (2)$$

(1), (2) and chain-rule $\Rightarrow$

$$u_i(\underset{\sim}{x}) = \overset{\circ}{u}_i + u_i(\underset{\sim}{x}) - u_i(\overset{\circ}{\underset{\sim}{x}}) = \overset{\circ}{u}_i + \int_{\tau_0}^{\tau_1} \frac{d}{d\tau} u_i(\underset{\sim}{\xi}(\tau))\, d\tau$$

$$= \overset{\circ}{u}_i + \int_{\tau_0}^{\tau_1} u_{i,j}(\underset{\sim}{\xi}(\tau))\, \dot{\xi}_j(\tau)\, d\tau$$

Since from (2.57) and (7),

$$u_{i,j} = u_{(i,j)} + u_{[i,j]} = \gamma_{ij} + \omega_{ij} \,, \quad \text{one has}$$

$$u_i(\underset{\sim}{x}) = \overset{\circ}{u}_i + \int_{\tau_0}^{\tau_1} \gamma_{ij}(\underset{\sim}{\xi}(\tau))\, \dot{\xi}_j(\tau)\, d\tau + \int_{\tau_0}^{\tau_1} \omega_{ij}(\underset{\sim}{\xi}(\tau))\, \dot{\xi}_j(\tau)\, d\tau \quad (3)$$

Now seek to eliminate ω_{ij} from (3). By (1), (2) & and integrations by parts,

$$\int_{\tau_0}^{\tau_1} \omega_{ij}(\underset{\sim}{\xi}(\tau))\, \dot{\xi}_j(\tau)\, d\tau = \int_{\tau_0}^{\tau_1} \omega_{ij}(\underset{\sim}{\xi}(\tau)) \frac{d}{d\tau}[\xi_j(\tau) - x_j]\, d\tau$$

$$= \overset{\circ}{\omega}_{ij}(x_j - \overset{\circ}{x}_j) + \int_{\tau_0}^{\tau_1} [x_j - \xi_j(\tau)]\, \omega_{ij,k}(\underset{\sim}{\xi}(\tau))\, \dot{\xi}_k(\tau)\, d\tau \quad (*)$$

Hence, upon making the dummy change $j \leftrightarrow k$ in the last integrand, (3) becomes:

$$u_i(\underset{\sim}{x}) = v_i(\underset{\sim}{x}) + \int_{\tau_0}^{\tau_1} \{ \gamma_{ij}(\underset{\sim}{\xi}(\tau)) + [x_k - \xi_k(\tau)]\, \omega_{ik,j}(\underset{\sim}{\xi}(\tau)) \}\, \dot{\xi}_j(\tau)\, d\tau \quad (4)$$

where

$$v_i(\underset{\sim}{x}) = \mathring{u}_i + \mathring{\omega}_{ij}(x_j - \mathring{x}_j) \quad (\text{inf. rigid!}) \quad (5)$$

In terms of <u>line</u>-integrals, (4) may be written as

$$u_i(\underset{\sim}{x}) = \int_{\mathring{\underset{\sim}{x}}}^{\underset{\sim}{x}} [\gamma_{ij}(\underset{\sim}{\xi}) + (x_k - \xi_k)\, \omega_{ik,j}(\underset{\sim}{\xi})]\, d\xi_j + v_i(\mathring{\underset{\sim}{x}}). \quad (6)$$

Next note that $\underset{\sim}{u} \in C^2(D)$ and (2.57) $\Rightarrow$ the identity

$$\omega_{ik,j} = \tfrac{1}{2}(u_{i,k} - u_{k,i})_{,j} = \tfrac{1}{2}(u_{i,jk} - u_{k,ji}) =$$

$$= \tfrac{1}{2}(u_{i,j} + u_{j,i})_{,k} - \tfrac{1}{2}(u_{k,j} + u_{j,k})_{,i} = \gamma_{ij,k} - \gamma_{kj,i},$$

so that

$$\omega_{ik,j} = \gamma_{ij,k} - \gamma_{kj,i} \quad (2.65)$$

But (2.65), (2.64) and (6) $\Rightarrow$

$$u_i(\underset{\sim}{x}) = \int_{\mathring{\underset{\sim}{x}}}^{\underset{\sim}{x}} U_{ij}(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\xi_j + v_i(\underset{\sim}{x}) \quad \forall\ \underset{\sim}{x} \in D \quad (7)$$

<u>Note</u>: <u>If</u> (2.59) has a solution $\underset{\sim}{u} \in C^2(D)$ for given $\underset{\sim}{\gamma}$, then the sol. must admit the representation (7) regardless of whether or not D is simply connected.

Remarks

(a) The explicit line-integral representation for $\underset{\sim}{u}$ in terms of the known compatible $\underset{\sim}{\gamma}$ is useful in applications, as will emerge later. Discuss effect of altering the choice of $\overset{\circ}{\underset{\sim}{x}}$ in (2.63).

(b) Sketch alternative proof of Thm. 2.14 on the sufficiency of the compat. eqs.: prove path-independence of line integrals in (2.63); define functions u_i through (2.63), (2.64), taking $v_i = 0$ on D; show that u_i so defined satisfy $u_i \in C^3($ $u_{(i,j)} = \gamma_{ij}$ on D.

Example. Let D be an arbitrary domain & suppose

$$\left.\begin{array}{l} \gamma_{ii} = 0 \text{ (no sum)},\ \gamma_{12} = \gamma_{21} = 0 \text{ on } D, \\ \gamma_{23}(\underset{\sim}{x}) = \gamma_{32}(\underset{\sim}{x}) = \dfrac{a x_1}{2},\ \gamma_{31}(\underset{\sim}{x}) = \gamma_{13}(\underset{\sim}{x}) = -\dfrac{a x_2}{2}\ \forall\, \underset{\sim}{x} \in D, \end{array}\right\} \text{(i}$$

where a is a real constant. Find a displacement field $\underset{\sim}{u} \ni \operatorname{sym} \nabla \underset{\sim}{u} = \underset{\sim}{\gamma}$ on D.

Note that since $\underset{\sim}{\gamma}$ is the restriction to D of a C^∞ compatible strain field on E, we know $\exists\ \underset{\sim}{u} \in C^3(D) \ni$

$$\operatorname{sym} \nabla \underset{\sim}{u} = \underset{\sim}{\gamma} \quad \text{or} \quad u_{i,j} + u_{j,i} = 2\gamma_{ij} \text{ on } D. \quad (ii)$$

<u>Solution 1</u> (Direct integration of (ii)).

By (i), (ii) one has

$$u_{1,1} = u_{2,2} = u_{3,3} = 0 \quad (1)$$

$$u_{1,2} + u_{2,1} = 0, \quad u_{2,3} + u_{3,2} = a x_1, \quad u_{3,1} + u_{1,3} = -a x_2 \quad (2)$$

(1) $\Rightarrow$

$$u_1 = u_1(x_2, x_3), \quad u_2 = u_2(x_1, x_3), \quad u_3 = u_3(x_1, x_2) \quad (3)$$

(2),(3) $\Rightarrow$

$$u_{1,2}(x_2, x_3) = -u_{2,1}(x_1, x_3) = f(x_3), \text{ whence}$$

$$u_1 = x_2 f(x_3) + g_1(x_3), \quad u_2 = -x_1 f(x_3) + g_2(x_3) \quad (4)$$

(4) & the last two of (2) $\Rightarrow$

$$-x_1 f'(x_3) + g_2'(x_3) + u_{3,2}(x_1, x_2) = a x_1$$

$$u_{3,1}(x_1, x_2) + x_2 f'(x_3) + g_1'(x_3) = -a x_2$$

$$\left.\begin{aligned} u_3(x_1, x_2) &= a x_1 x_2 + x_1 x_2 f'(x_3) - x_2 g_2'(x_3) + p(x_1) \\ u_3(x_1, x_2) &= -a x_1 x_2 - x_1 x_2 f'(x_3) - x_1 g_1'(x_3) + q(x_2) \end{aligned}\right\} (5)$$

(5) $\Rightarrow$

$$2[a + f'(x_3)] x_1 x_2 + x_1 g_1'(x_3) - x_2 g_2'(x_3) + p(x_1) - q(x_2) = 0$$

Thus,

$$\left.\begin{array}{c} f'(x_3) = -\alpha\,, \quad g_1'(x_3) = \mathring{w}_2\,, \quad g_2'(x_3) = -\mathring{w}_1 \\ p(x_1) + x_1\mathring{w}_2 = q(x_2) - x_2\mathring{w}_1 = \mathring{c}_3\,, \end{array}\right\}(6).$$

$$f(x_3) = -\alpha x_3 - \mathring{w}_3\,, \quad g_1(x_3) = \mathring{w}_2 x_3 + \mathring{c}_1\,, \quad g_2(x_3) = -\mathring{w}_1 x_3 + \mathring{c}_2\,, (7$$

in which $\mathring{w}_i$ and $\mathring{c}_i$ are arbitrary constants. Substitution from (6), (7) into (4), (5) yields

$$\left.\begin{array}{l} u_1(x_2,x_3) = -\alpha x_2 x_3 + \mathring{w}_2 x_3 - \mathring{w}_3 x_2 + \mathring{c}_1 \\ u_2(x_1,x_3) = \alpha x_1 x_3 + \mathring{w}_3 x_1 - \mathring{w}_1 x_3 + \mathring{c}_2 \\ u_3(x_1,x_2) = \mathring{w}_1 x_2 - \mathring{w}_2 x_1 + \mathring{c}_3 \end{array}\right\}(ii)$$

Comment on a.b. additive inf. rigid displt. field
Mention direct verification of (ii) by substitution into (2.5

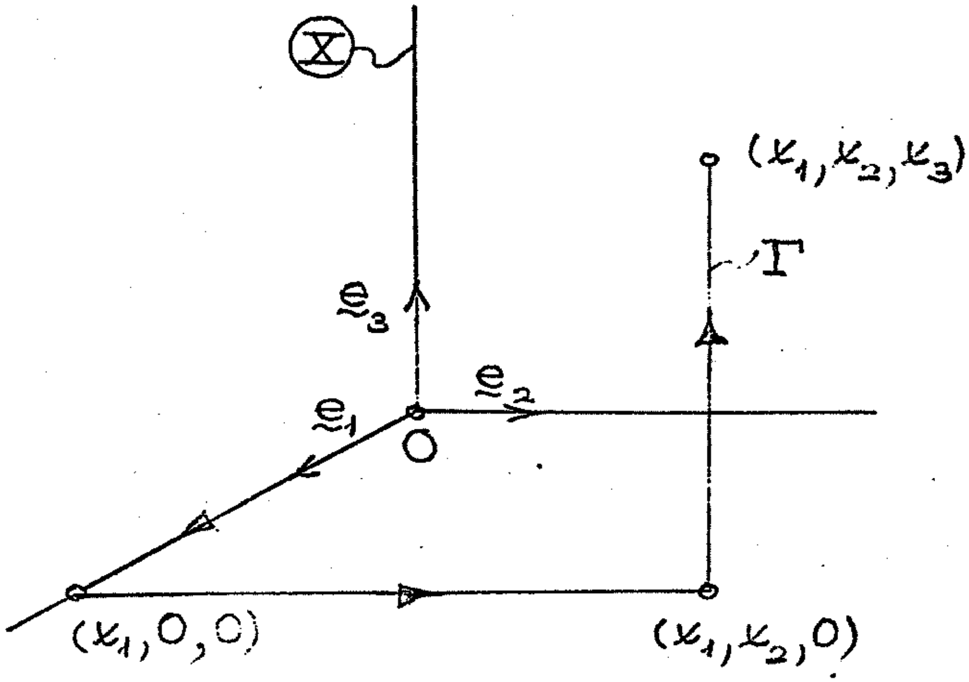
<u>Solution 2</u> (Via line-integrals (2.63), (2.64)).

From (i), (2.64) one has

$$U_{11}(\underset{\sim}{x},\underset{\sim}{\xi}) = 0\,, \quad U_{12}(\underset{\sim}{x},\underset{\sim}{\xi}) = \frac{\alpha}{2}(\xi_3 - x_3)\,, \quad U_{13}(\underset{\sim}{x},\underset{\sim}{\xi}) = \frac{\alpha}{2}(2\xi_2 - 3x$$

etc. (1

Now use (2.63), choosing the path of integration conveniently as follows:



Then (1), (2.63) $\Rightarrow$

$$u_1(\underset{\sim}{x}) = \int_0^{x_2}\left(-\frac{a x_3}{2}\right) d\xi_2 + \int_0^{x_3}\left(-\frac{a x_2}{2}\right) d\xi_3 = -a x_2 x_3$$

Thus and similarly one recovers (ii) with $\mathring{w}_i = \mathring{c}_i = 0$.

Remark on verification of (ii) by substitution into (2.59).

Emphasize once more that every result in the theory of deformations has an immediate counterpart in the theory of motions.