Singular value decomposition of a homogeneous deformation

Synopsis

Homogeneous deformation

- A body deforms from one state to another state.
- The two states are called, respectively, *reference state* and *current state*.

- Consider a set of material particles in the body.
- In the reference state, the set of material particles forms a segment of a straight line, called a *material segment in the reference state*, Y .
- In the current state, the same set of material particles also forms a segment of a straight line, called a *material segment in the current state*, y .
- A homogeneous deformation stretches and rotates a straight segment of material particles, but will keep the segment straight.

- A body has an infinite number of material segments.
- All material segments in the reference state constitute a three-dimensional inner-product space.
- All material segments in the current state constitute another three-dimensional inner-product space.
- The two spaces are distinct vector spaces.

- A *homogeneous deformation* of the body is specified by a linear map, called the *deformation gradient* F .
- F maps the same set of material particles from the segment Y in the reference state to the segment y in the current state:
- $y = FY$.
- The domain of this linear map is the vector space of material segments in the reference state.
- The codomain of this linear map is the vector space of the material segments in the current state.
- In general, F is not a symmetric matrix.

- In the reference state, let $Y = LM$, where L is the length of the material segment, and M is a unit vector.
- In the current state, let $y = lm$, where l is the length of the material segment, and m is a unit vector.

- Define the *stretch* of the material segemen by
- $\lambda = l/L$.
- The deformation gradient F maps the material segment $Y = LM$ in the reference state to the material segment $y = lm$ in the current state:
- $F(LM) = lm$.
- Write the above equation as
- $FM = \lambda m$.
- Given a deformation gradient F, and given a unit vector M in the direction of a line of material particles in the reference state, this equation calculates the stretch and the direction m of the line of material particles in the current state.

- When F maps a material segment Y in the reference state to a material segment y in the current state, $y = FY$, the inner product of the material segment in the current state is $y^T y = Y^T CY$, where
- $C = F^T F$.
- C is a symmetric, positive-definite matrix.
- The expression $Y^T CY$ is called a *quadratic form* in the reference state.
- The inner product $y^T y$ is the square of the length of the material segment in the current state, $y^T y = l^2$.

Exercise. A body undergoes a homogeneous deformation described by the deformation gradient

$$\begin{bmatrix} 4 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

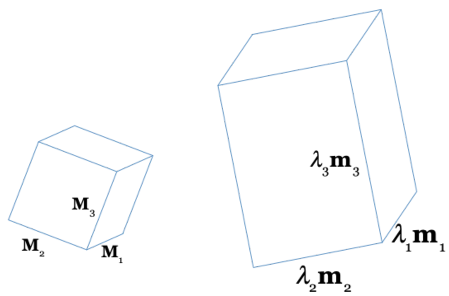
Consider a set of material particles that forms a segment of a straight line. In the reference state, the material particles at the two ends of the segment are at two points $(1,2,3)$ and $(2,3,6)$. Calculate the stretch of the material segment.

Exercise. In the above exercise, another material segment in the reference state is defined by two endpoints $(1,2,2)$ and $(1,3,6)$. Calculate the angle between the two material segments in the current state.

Singular value decomposition

- A body undergoes a homogeneous deformation F from a reference state to a current state.
- A unit cube in the reference state deforms to a parallelepiped in the current state.
- The shape of the parallelepiped in the current state depends on the orientation of the unit cube in the reference state.

- *Theorem of singular value decomposition*. For any homogeneous deformation F, there exists a unit cube in the reference state that deforms to a rectangular block in the current state.



Write singular value decomposition as a sum of outer products

- We have just stated the theorem by a geometric picture.
- Transcribe this geometric picture into an algebraic equation:
- $F = \lambda_1 m_1 M_1^T + \lambda_2 m_2 M_2^T + \lambda_3 m_3 M_3^T$.
- M_1, M_2, M_3 are the unit vectors that define the edges of the unit cube in the reference state, and are called *singular material segments in the reference state*.
- m_1, m_2, m_3 are the unit vectors that define the directions of edges of the rectangular box in the current state, and are called *singular material segments in the current state*.
- λ_1, λ_2, λ_3 are the lengths of the edges of the rectangular box in the current state, and are called *singular stretches*.
- $m_1 M_1^T$ is called an *outer product* of the two vectors, and is a three-by-three matrix.
- The above equation expresses the deformation gradient F as a sum of outer products.

Exercise. A body dilates by a stretch λ in all directions. Write the deformation gradient and its singular value decomposition.

Exercise. A sheet of an incompressible material is stretched by a ratio λ in all directions in the plane of the sheet. Write the deformation gradient and its singular value decomposition.

Exercise. A body undergoes a rigid-body rotation by an angle $\pi/6$ around an axis. Write the deformation gradient and its singular value decomposition.

Project a material segment onto singular material segments

- A set of material particles forms a material segment Y in the reference state, and forms a material segment y in the current state.

- M_1, M_2, M_3 are an orthonormal basis for the vector space of material segments in the reference state. Write

- $Y = Y_1 M_1 + Y_2 M_2 + Y_3 M_3$.
- Y_1, Y_2, Y_3 are the components of the vector Y relative to the basis M_1, M_2, M_3.

- m_1, m_2, m_3 are an orthonormal basis for the vector space of material segments in the reference state. Write
- $y = y_1 m_1 + y_2 m_2 + y_3 m_3$.
- y_1, y_2, y_3 are the components of the vector y relative to the basis m_1, m_2, m_3.

- Confirm that
- $y = FY = \lambda_1 Y_1 m_1 + \lambda_2 Y_2 m_2 + \lambda_3 Y_3 m_3$.
- Thus,
- $y_1 = \lambda_1 Y_1$,
- $y_2 = \lambda_2 Y_2$,
- $y_3 = \lambda_3 Y_3$.

- $Y_1 M_1$ is a set of material particles, which forms a material segment in the reference state.
- $y_1 m_1$ is the same set of material particles, which forms a material segment in the current state.

Exercise. Give a geometric interpretation of the following two sums of outer products:
$m_1 M_1^{\mathsf{T}} + m_2 M_2^{\mathsf{T}} + m_3 M_3^{\mathsf{T}}$,
$M_1 m_1^{\mathsf{T}} + M_2 m_2^{\mathsf{T}} + M_3 m_3^{\mathsf{T}}$.

Determine singular value decomposition in the following steps:

1. Specify a homogeneous deformation by a deformation gradient F.
2. Calculate $C = F^{\mathsf{T}} F$.
3. Find three orthonormal eigenvectors of C: M_1, M_2, M_3.
4. Use $FM_1 = \lambda_1 m_1$ to calculate λ_1 and m_1. Similarly calculate λ_2 and m_2, as well as λ_3 and m_3.

Exercise. A body undergoes a homogeneous deformation specified by the deformation gradient

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

What does this homogeneous deformation look like? Determine its singular value decomposition. Interpret the results in geometric terms.

Exercise. Given a homogeneous deformation F, is the set of singular vectors in the reference state unique?

Write singular value decomposition as a product of matrices

- We have written the singular decomposition as a sum:
- $F = \lambda_1 m_1 M_1^T + \lambda_2 m_2 M_2^T + \lambda_3 m_3 M_3^T$.

- We next write the singular decomposition as a product of matrices.
- Introduce three matrices:
- $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$,
- $G = [M_1, M_2, M_3]$,
- $g = [m_1, m_2, m_3]$.

- Λ is a diagonal matrix.
- G is an orthogonal matrix, $G^T G = GG^T = I$.
- g is an orthogonal matrix, $g^T g = gg^T = I$.

- Confirm that the singular value decomposition can be written as
- $F = g\Lambda G^T$.

Exercise. Confirm the following equations.

- $F^T = G\Lambda g^T$
- $F^T = \lambda_1 M_1 m_1^T + \lambda_2 M_2 m_2^T + \lambda_3 M_3 m_3^T$
- $F^T F = G\Lambda^2 G^T$
- $F^T F = \lambda_1^2 M_1 M_1^T + \lambda_2^2 M_2 M_2^T + \lambda_3^2 M_3 M_3^T$
- $FF^T = g\Lambda^2 g^T$
- $FF^T = \lambda_1^2 m_1 m_1^T + \lambda_2^2 m_2 m_2^T + \lambda_3^2 m_3 m_3^T$
- $F^{-1} = G\Lambda^{-1} g^T$.

Polar decomposition 1

- Write the singular value decomposition $F = g\Lambda G^T$ as
- $F = gG^T G\Lambda G^T$.
- Denote $R = gG^T$.
- Confirm that R is an orthogonal matrix, $R^T R = RR^T = I$.
- Confirm that $R = m_1 M_1^T + m_2 M_2^T + m_3 M_3^T$.
- Denote $U = G\Lambda G^T$.
- Confirm that U is a symmetric matrix, $U = U^T$.
- Confirm that $U = \lambda_1 M_1 M_1^T + \lambda_2 M_2 M_2^T + \lambda_3 M_3 M_3^T$.
- With these definitions, we write the singular value decomposition as
- $F = RU$.
- This equation is called the polar decomposition of the deformation gradient.

- This polar decomposition has the following geometric interpretation.
- The principal vectors M_1, M_2, M_3 define a unit cube in the reference state.
- U maps this unit cube in the reference state to a rectangular box in an intermediate state.
- The edges of the rectangular box in the intermediate state are $\lambda_1 M_1$, $\lambda_2 M_2$, $\lambda_3 M_3$.
- The rectangular box in the intermediate state has the same orientation as the unit cube in the reference state.
- U is called the stretch tensor.

- R maps the rectangular box in the intermediate state to a rectangular box, of edges in the current state.
- The edges of the rectangular box in the current state are $\lambda_1 m_1$, $\lambda_2 m_2$, $\lambda_3 m_3$.
- The rectangular box in the current state and the rectangular box in the intermediate state have the same shape but different orientations.
- R is called the rotation tensor.

Polar decomposition 2
- Write the singular value decomposition $F = g\Lambda G^{\mathsf{T}}$ as
- $F = g\Lambda g^{\mathsf{T}} g G^{\mathsf{T}}$.
- Denote $R = gG^{\mathsf{T}}$.
- Denote $V = g\Lambda g^{\mathsf{T}}$.
- Confirm that $V = \lambda_1 m_1 m_1^{\mathsf{T}} + \lambda_2 m_2 m_2^{\mathsf{T}} + \lambda_3 m_3 m_3^{\mathsf{T}}$.
- With these definitions, we write the singular value decomposition as
- $F = VR$.
- This equation is another polar decomposition of the deformation gradient.

Exercise. Give a geometric interpretation of this polar decomposition.

Outer product

We next review facts of linear algebra, and derive singular value decomposition.

Outer product of two vectors in the same inner-product space
- Consider an inner-product space.
- Let a and b be two vectors in the three-dimensional inner-product space.
- ab^{T} is called the *outer product* of the two vectors.
- The outer product is a there-by-three matrix.

- This outer product is a linear map that maps one vector in the inner-product space to another vector in the same inner-product space.

- Let Y be a vector in the inner-product space.
- What is $ab^{\mathsf{T}}Y$?
- $b^{\mathsf{T}}Y$ is the inner product of the two vectors, and is a scalar, which is the projection of the vector Y onto the vector b.
- $ab^{\mathsf{T}}Y$ is a vector in the same direction as the vector a, but with a different length.

Exercise. Give a geometric interpretation of aa^{T}.

Orthonormal basis for an inner-product space

- Let M_1, M_2, M_3 be an *orthonormal basis* for a three-dimensional inner-product space.
- Thus,
- $M_1^{\mathsf{T}}M_2 = M_2^{\mathsf{T}}M_3 = M_2^{\mathsf{T}}M_3 = 0$.
- $M_1^{\mathsf{T}}M_1 = M_2^{\mathsf{T}}M_2 = M_3^{\mathsf{T}}M_3 = 1$.

- Here is a fact of linear algebra:
- The sum of outer products,
- $M_1M_1^{\mathsf{T}} + M_2M_2^{\mathsf{T}} + M_3M_3^{\mathsf{T}}$,
- is an identity matrix.

- *Proof*
- Let Y be a vector in the inner-product space.
- Write
- $Y = Y_1M_1 + Y_2M_2 + Y_3M_3$.
- Y_1, Y_2, Y_3 are the components of Y onto the basis M_1, M_2, M_3.
- Confirm that
- $(M_1M_1^{\mathsf{T}} + M_2M_2^{\mathsf{T}} + M_3M_3^{\mathsf{T}})Y = Y$.
- That is, the sum of the outer products is an identity matrix.
- Write the identity matrix as
- $M_1M_1^{\mathsf{T}} + M_2M_2^{\mathsf{T}} + M_3M_3^{\mathsf{T}} = I$.

Outer product of vectors in two inner-product spaces

- Let a be a vector in one inner-product space, and B be a vector in another inner-product space.
- In general, the two inner-product spaces are distinct, and may even have different dimensions.
- aB^{T} is the outer product of the two vectors in two inner-product spaces.

- Let Y be a vector in the second inner-product space.
- $B^{\mathsf{T}}Y$ is an inner product, which is a scalar.
- $aB^{\mathsf{T}}Y$ is a vector in the first inner-product space, and is in the same direction as the vector a.

Eigenvalues of a symmetric matrix

Orthonormal eigenvectors

- Recall a fact of the linear algebra of symmetric matrix.
- A symmetric matrix C on a three-dimensional inner-product space has three orthonormal eigenvectors.

- A matrix C is symmetric if $C^{\mathrm{T}} = C$.

- Let α be an eigenvalue of C, and M be the corresponding eigenvector, so that
- $CM = \alpha M$.

- For a symmetric matrix C, all eigenvalues are real numbers.
- For simplicity, assume that C has three distinct eigenvalues, $\alpha_1, \alpha_2, \alpha_3$.
- Denote the corresponding eigenvectors by M_1, M_2, M_3.
- Thus,
- $CM_1 = \alpha_1 M_1$,
- $CM_2 = \alpha_2 M_2$,
- $CM_3 = \alpha_3 M_3$.

- For a symmetric matrix C, the three eigenvectors are orthogonal to one another:
- $M_1^{\mathrm{T}} M_2 = M_2^{\mathrm{T}} M_3 = M_2^{\mathrm{T}} M_3 = 0$.

- Make each eigenvector a unit vector:
- $M_1^{\mathrm{T}} M_1 = M_2^{\mathrm{T}} M_2 = M_3^{\mathrm{T}} M_3 = 1$.

- Thus, the three eigenvectors of a symmetric matrix C are *orthonormal*.

Spectral decomposition of a symmetric matrix

- Let C be a symmetric matrix on a three-dimensional inner-product space.
- Let M_1, M_2, M_3 be three orthonormal eigenvectors of C, and $\alpha_1, \alpha_2, \alpha_3$ be the corresponding eigenvalues.
- Recall that
- $CM_1 = \alpha_1 M_1$,
- $CM_2 = \alpha_2 M_2$,
- $CM_3 = \alpha_3 M_3$.

- Form outer products and then sum:
- $CM_1 M_1^{\mathrm{T}} + CM_2 M_2^{\mathrm{T}} + CM_2 M_2^{\mathrm{T}} = \alpha_1 M_1 M_1^{\mathrm{T}} + \alpha_2 M_2 M_2^{\mathrm{T}} + \alpha_3 M_3 M_3^{\mathrm{T}}$.
- On the left side of the above equation, apply the identity

- $M_1 M_1^\mathsf{T} + M_2 M_2^\mathsf{T} + M_3 M_3^\mathsf{T} = I$,
- and we obtain that
- $C = \alpha_1 M_1 M_1^\mathsf{T} + \alpha_2 M_2 M_2^\mathsf{T} + \alpha_3 M_3 M_3^\mathsf{T}$.
- This equation is called the *spectral decomposition* of the symmetric matrix C.

Geometric interpretation of spectral decomposition

- Spectral decomposition has the following geometric interpretation.
- Let Y be a vector in the inner-product space.
- Project the vector Y onto the three orthonormal vectors:
- $Y = Y_1 M_1 + Y_2 M_2 + Y_3 M_3$.
- Here Y_1, Y_2, Y_3 are the projections of Y onto M_1, M_2, M_3.
- Confirm that
- $CY = \alpha_1 Y_1 M_1 + \alpha_2 Y_2 M_2 + \alpha_3 Y_3 M_3$.
- Thus, C maps a vector Y to a new vector CY, whose projection onto the eigenvectors are scaled by the eigenvalues.

Deformation gradient

A linear map from reference state to current state

- A body deforms from one state to another state.
- The two states are called, respectively, reference state and current state.

- A *homogeneous deformation* is specified by a linear map, called the deformation gradient F.
- Consider a set of material particles in the body.
- When the body is in the reference state, the set of material particles form a material segment Y, which is a vector.
- When the body is in the current state, the set of material particles form a material segment y, which is another vector.
- F maps Y to y:
- $y = FY$.

Homogeneous deformation stretches and rotates a material segment

- Write $Y = LM$, where L is the length of the vector Y, and M is the unit vector in the direction of the vector Y.
- Write $y = lm$, where l is the length of the vector y, and m is the unit vector in the direction of the vector y.

- $\lambda = l/L$ defines the *stretch* of the material segment.
- The angle θ between the two vectors Y and y defines the *rotation* of the material segment, $\cos\theta = m^\mathsf{T} M$.

- The equation $y = FY$ becomes
- $\lambda m = FM$.
- Given a deformation gradient, F, and the direction of a material segment in the reference state, M, the above equation calculates the stretch, λ, and the direction of the material segment in the current state, m.

Singular material segments and singular stretches

- Let $C = F^{\mathsf{T}} F$.
- Confirm that C is a symmetric matrix:
- $C^{\mathsf{T}} = C$.

- Let α be an eigenvalue of C, and M be the corresponding eigenvector.
- The eigenvector corresponds to a material segment.
- M is the unit vector in the direction of the material segment in the reference state,
- What does the eigenvalue α mean?

- Recall that $\lambda m = FM$.
- M is the unit vector in the direction of a material segment in the reference state.
- m is the unit vector in the direction of the material segment in the current state.
- λ is the stretch of the material segment.

- The square of the length of the material segment in the current state is given by the inner product:
- $(\lambda m)^{\mathsf{T}} (\lambda m) = (FM)^{\mathsf{T}} (FM)$.
- Recall the definitions $C = F^{\mathsf{T}} F$ and $CM = \alpha M$.
- The above equation simplifies to
- $\lambda^2 = \alpha$.
- Thus, every eigenvalue α of C is nonnegative, and equals the corresponding stretch squared.

- Let α_1 and α_2 be two eigenvalues of C, and M_1 and M_2 be the corresponding orthonormal eigenvectors.
- The two eigenvectors correspond to two material segments.
- Let $\lambda_1 = \sqrt{\alpha}_1$, $\lambda_2 = \sqrt{\alpha}_2$.
- Recall that
- $FM_1 = \lambda_1 m_1$,
- $FM_2 = \lambda_2 m_2$.
- Here m_1 and m_2 are the corresponding orthonormal vectors in the current state.

- In the reference state, the two material segments are orthogonal:
- $M_1^{\mathsf{T}} M_2 = 0$.
- Taking inner product:
- $(\lambda_1 m_1)^{\mathsf{T}} (\lambda_2 m_2) = (F_1 M_1)^{\mathsf{T}} (F_2 M_2)$.
- Because the right side vanishes, the left side also vanishes.
- Thus, the two material segments are also orthogonal in the current state:
- $m_1^{\mathsf{T}} m_2 = 0$.

- Let M_1, M_2, M_3 be three orthonormal eigenvectors of C, and $\alpha_1, \alpha_2, \alpha_3$ be the corresponding eigenvalues.
- The three eigenvectors correspond to three material segments.
- Let $\lambda_1 = \sqrt{\alpha}_1, \lambda_2 = \sqrt{\alpha}_2, \lambda_3 = \sqrt{\alpha}_3$.
- Recall that
- $FM_1 = \lambda_1 m_1$,
- $FM_2 = \lambda_2 m_2$,
- $FM_3 = \lambda_3 m_3$.
- Here m_1, m_2, m_3 are the corresponding orthonormal vectors in the current state.

- M_1, M_2, M_3 are called the singular vectors (material segments) in the reference state.
- m_1, m_2, m_3 are called the singular vectors (material segments) in the current state.
- $\lambda_1, \lambda_2, \lambda_3$ are called the singular values (stretches).

Exercise. Prove the following fact. For any given deformation gradient, of all material segments, the singular material segments have the largest and smallest stretches.

Write singular decomposition as a sum of outer products

- Form outer products and then sum:
- $FM_1 M_1^{\mathsf{T}} + FM_2 M_2^{\mathsf{T}} + FM_3 M_3^{\mathsf{T}} = \lambda_1 m_1 M_1^{\mathsf{T}} + \lambda_2 m_2 M_2^{\mathsf{T}} + \lambda_3 m_3 M_3^{\mathsf{T}}$.
- On the left side of the above equation, apply the identity
- $M_1 M_1^{\mathsf{T}} + M_2 M_2^{\mathsf{T}} + M_3 M_3^{\mathsf{T}} = I$,
- and we obtain that
- $F = \lambda_1 m_1 M_1^{\mathsf{T}} + \lambda_2 m_2 M_2^{\mathsf{T}} + \lambda_3 m_3 M_3^{\mathsf{T}}$.
- This equation is called the singular value decomposition of the deformation gradient F.