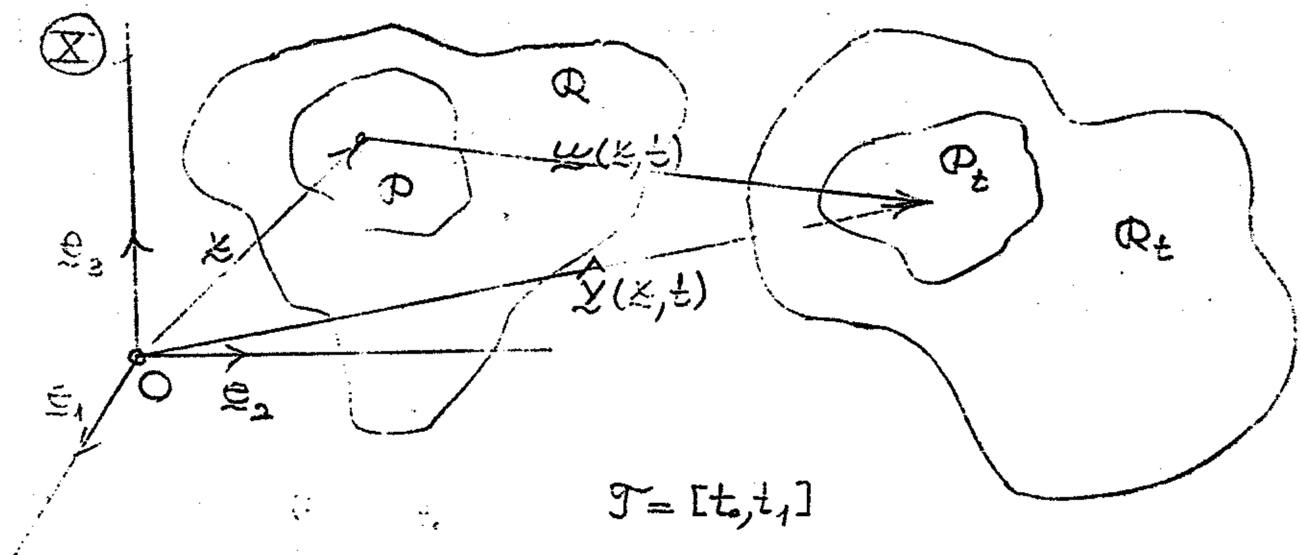
3. Kinetics of continuous media. Analysis of stress

Objective. Exposition of basic principles of Newtonian kinetics of deformable continua, conservation of mass, momentum principles, notion of stress, equations of motion. Emphasize lack of constitutive assumptions.

Recall definition of admiss. motion $\hat{y}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le$



$\mathcal{T} = [t_0, t_1]$

$$y = \hat{y}(x,t) = x + u(x,t) \ \forall \ (x,t) \in \mathcal{R}\times\mathcal{T}, \ \hat{y}(\mathcal{R},t) = \mathcal{R}_t \ \forall t \in \mathcal{T}$$

$$\hat{y} \in C^2(\mathcal{R}\times\mathcal{T}), \ \hat{y}(\cdot,t) \text{ is one-to-one on } \mathcal{R} \ \forall t \in \mathcal{T}$$

$$J = \det \nabla\hat{y} = \det(1 + \nabla\hat{u}) > 0 \text{ on } \mathcal{R}\times\mathcal{T}$$

(3.1)

$$\hat{y}^{-1}(\cdot,t) = \hat{x}(\cdot,t) \text{ on } \mathcal{R}_t \ \forall t \in \mathcal{T} \qquad (3.2)$$

Recall velocity and acceleration fields:

$$\left.\begin{aligned}&\underset{\sim}{v}(\underset{\sim}{x},t)=\frac{\partial}{\partial t}\hat{\underset{\sim}{\chi}}(\underset{\sim}{x},t)=\dot{\underset{\sim}{u}}(\underset{\sim}{x},t),\ \underset{\sim}{a}(\underset{\sim}{x},t)=\ddot{\underset{\sim}{u}}(\underset{\sim}{x},t)\ \forall\ (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T}\\&\overline{v}(\underset{\sim}{y},t)=\underset{\sim}{q}(\underset{\sim}{y},t)=\underset{\sim}{v}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t),\ \overline{\underset{\sim}{a}}(\underset{\sim}{y},t)=\underset{\sim}{a}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t)\\&\qquad\qquad\forall\ \underset{\sim}{y}\in\mathcal{R}_t\ (t_0\le t\le t_1)\end{aligned}\right\}(3.$$

Mass density, conservation of mass (eq. of continuity)

We assume $\exists$ a scalar-valued function $\rho(\cdot,t)$ defined on $\mathcal{R}_t$ $(t_0\le t\le t_1)$ {mass density of body at time t} with the properties:

(i) $\rho(\cdot,t)\in\mathcal{C}(\mathcal{R}_t)$, $\rho(\underset{\sim}{y},t)>0\ \forall\ \underset{\sim}{y}\in\mathcal{R}_t\ (t_0\le t\le t_1)$; (3.4)

(ii) if $\mathcal{P}\subset\mathcal{R}$ is a compact region & $\mathcal{P}_t=\hat{\underset{\sim}{\chi}}(\mathcal{P},t)$ $\forall\ t\in\mathcal{T}$, then

$$m(\mathcal{P}_t)=\int_{\mathcal{P}_t}\rho(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}}\quad\forall\ t\in\mathcal{T}\qquad(3.5)$$

is the mass of the material occupying $\mathcal{P}_t$ at the time t.

$$\dim\rho=[ML^{-3}]$$

Postulate of mass conservation

$$m(\mathcal{P}_t) = m(\mathcal{P}_{t_0}) \quad (t_0 \le t \le t_1) \quad \forall \text{ compact } \mathcal{P} \subset \mathcal{R} \qquad (3.6)$$

Thus $m(\mathcal{P}_t)$ is time-independent.

Claim. Define

$$\varphi(\underset{\sim}{x},t) = \rho(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t)\, J(\underset{\sim}{x},t) \quad \forall\ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}. \qquad (a)$$

Then,

$$\varphi(\underset{\sim}{x},t) = \varphi(\underset{\sim}{x},t_0) \equiv \rho_0(\underset{\sim}{x})\ \forall\ (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T}\ \text{(time-indept.)}, \qquad (b)$$

$$\therefore m(\mathcal{P}_t) = \int_{\mathcal{P}} \rho_0(\underset{\sim}{x})\, dV_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1), \qquad (3.7)$$

and

$$\rho(\underset{\sim}{y},t)\, J(\underset{\sim}{x},t) = \rho_0(\underset{\sim}{x}) \quad \forall\ (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T},\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) \qquad (3.8)$$

Proof: Change the variable of integ. in (3.5) and use (a) to infer

$$m(\mathcal{P}_t) = \int_{\mathcal{P}} \varphi(\underset{\sim}{x},t)\, dV_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1) \qquad (1)$$

Now (1), (3.6) $\Rightarrow$,

$$m(\mathcal{P}_t) - m(\mathcal{P}_{t_0}) = \int_{\mathcal{P}} \{\varphi(\underset{\sim}{x},t) - \varphi(\underset{\sim}{x},t_0)\}\, dV_{\underset{\sim}{x}} = 0 \quad (t_0 \le t \le t_1)$$

$$\underline{\underline{\forall}} \text{ compact } \mathcal{P}\subset\mathcal{R} \qquad (2)$$

But, by (a), $\varphi(\cdot,t) \in \mathcal{C}(R)\ \forall t \in \mathcal{T}$, whence (b) follows. Also, (1), (b) $\Rightarrow$ (3.7) and (a), (b) $\Rightarrow$ (3.8). /

One calls ρ_0 the "nominal (reference) mass density"* and (3.8) "the equation of continuity".

Note that conversely (3.8), (3.5) $\Rightarrow$ (3.6).

Remarks

(i) Incompressible material. Here $J = 1$ on $R \times \mathcal{T}$ and thus

$$\rho(\underset{\sim}{y}, t) = \rho_0(\underset{\sim}{x})\ \forall\ (\underset{\sim}{x}, t) \in R \times \mathcal{T},\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}, t)$$

(ii) Infinitesimal deformations. In this instance $J = \det(\underset{\sim}{1} + \nabla \underset{\sim}{u}) \sim 1$ and thus

$$\rho(\underset{\sim}{y}, t) \sim \rho_0(\underset{\sim}{x})\ \forall\ (\underset{\sim}{x}, t) \in R \times \mathcal{T},\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x}, t) \quad (3.9)$$

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* Note that $\rho_0 = \rho(\cdot, t_0)$ (initial mass density) only if the initial config. is taken as the reference configuration.

~~Exercise 13. Show that for an admissible motion $\hat{\underset{\sim}{y}}(\cdot,t): R \to R$~~

~~$\dot{J}(\underset{\sim}{x},t) = J(\underset{\sim}{x},t)\,\nabla\cdot\underset{\sim}{q}(\underset{\sim}{y},t)\ \forall (\underset{\sim}{x},t)\in R\times T,\ \underset{\sim}{y}=\hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$, (*)~~

~~where $\underset{\sim}{q}$ is the Eulerian velocity field. Use (*) to show that the cont eq. (3.8) is equivalent to~~

~~$\dot{\rho}+\nabla\cdot(\rho\underset{\sim}{q})=0$ or $\dot{\rho}+(\rho q_i)_{,i}=0$ on R_t $(t_0\le t\le t_1)$ (**)~~

~~Note that (**) reduces to $\nabla\cdot\underset{\sim}{q}=0$ if the medium is incompressible. (Clarify notation used in (**))~~

<u>Body forces and contact forces, surface tractions, stre</u>

Let $\hat{\underset{\sim}{y}}$, P, P_t be as before. We admit two kinds of forces acting on the material occupying P_t $(t_0\le t\le t_1)$:

(a) <u>Body forces</u> (extrinsic or assigned forces). These are field forces due to agencies outside R_t (e.g. gravi magnetic field forces) acting on the material poin (particles) in R_t.

(b) <u>Contact forces</u>. These arise from contact of the material in P_t with surrounding material or, in case $P=R$, from contact with other bodies.

Remark on exclusion of "mutual forces"

Body-force density

We assume $\exists$ a vector-valued function $\underset{\sim}{b}(\cdot,t)$ (body-force density per unit current volume) defined on $\mathcal{R}_t$ $(t_0 \le t \le t_1)$ with the properties:

(i) $\underset{\sim}{b}(\cdot,t) \in \mathcal{C}(\mathcal{R}_t)$ $(t_0 \le t \le t_1)$

(ii) the resultant force and the resultant moment about O due to the body forces acting on the material occupying $\mathcal{P}_t$ at time t are given by

$$\int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}}\,, \quad \int_{\mathcal{P}_t} \underset{\sim}{y} \wedge \underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} \quad (t_0 \le t \le t_1)$$

Note that $\dim \underset{\sim}{b} = [FL^{-3}]$.

Contact forces, surface tractions

From here on require that $\mathcal{P} \subset \mathcal{R}$ be the closure of a bounded regular subregion of $\mathcal{R}$ and set $\mathcal{P}_t = \hat{\underset{\sim}{y}}(\mathcal{P},t)\ \forall\, t \in \mathcal{T}$, ~~$\mathcal{P}_{t_0} = \mathcal{P}$~~. Describe regularity of $\partial\mathcal{P}_t$.

We assume $\exists$ a vector-valued function $\underset{\sim}{t}(\cdot,\cdot,t)$ (surface-force density per unit current area, actual surface traction, Cauchy traction) defined on $\mathcal{R}_t \times \mathcal{U}$, $(t_0 \le t \le t_1)$, where $\mathcal{U}$ is the set of all unit vectors, which has the properties:

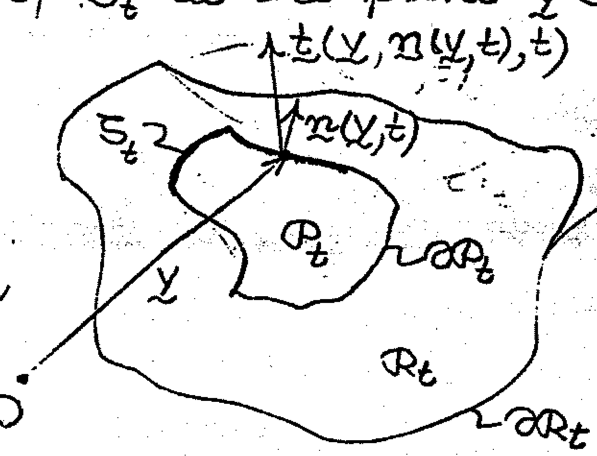
(i) $\underset{\sim}{t}(\cdot,\cdot,t) \in \mathcal{C}(\mathcal{R}_t \times \mathcal{U})\ \forall\, t \in \mathcal{T}$

(ii) if $\mathcal{S} \subset \partial\mathcal{P}$ is a regular subsurface of $\partial\mathcal{P}$ & $\mathcal{S}_t = \hat{\underset{\sim}{y}}(\mathcal{S},t)\ \forall\, t \in \mathcal{T}$, then the resultant force and the resultant moment about O due to the contact forces acting over $\mathcal{S}_t$ on the material occupying $\mathcal{P}_t$ at time t are given by

$$\int_{\mathcal{S}_t} \underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}(\underset{\sim}{y},t),t)\, dA_{\underset{\sim}{y}}\,, \quad \int_{\mathcal{S}_t} \underset{\sim}{y} \wedge \underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}(\underset{\sim}{y},t),t)\, dA_{\underset{\sim}{y}} \quad (t_0 \le t \le t_1)$$

provided $\underset{\sim}{n}(\underset{\sim}{y},t)$ is the unit outer normal vector of $\mathcal{S}_t$ at the point $\underset{\sim}{y} \in \mathcal{S}_t$.

Schematic two-dim. diagram



Note that $\underset{\sim}{n}(\underset{\sim}{y},t)$ is not defined at corners or edges of $\mathcal{S}_t$.

$\dim \underset{\sim}{t} = [FL^{-2}]$

Discussion

(a) $\underset{\sim}{t}(\cdot,\cdot,t)$ is not just a vector field. Explain.

(b) Elaborate on implications of above assumptions: tractio[n] at $\underset{\sim}{y} \in \mathcal{R}_t$ is the same on all oriented surface element[s] that have a common normal vector.

<u>Simplified notation</u>. For fixed, given $\mathcal{P}$ with image $\mathcal{P}_t$ $(t_0 \le t \le t_1)$, we shall write

$$\underset{\sim}{t}(\underset{\sim}{y}; \underset{\sim}{n}(\underset{\sim}{y},t),t) \equiv \underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t) \; \forall \; \underset{\sim}{y} \in \partial\mathcal{P}_t \quad (t_0 \le t \le t_1)$$

<u>The principles of linear & angular momentum (postulate[s])</u>

Let a $\hat{\underset{\sim}{y}}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ be an admiss. motion rel. to Newtonian E. Then, for every $\mathcal{P}_t$ $(t_0 \le t \le t_1)$ as specified earlier,

(*)
$$\int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y},t)\, dV_{\underset{\sim}{y}} + \int_{\partial\mathcal{P}_t} \underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\, dA_{\underset{\sim}{y}} = \frac{d}{dt}\int_{\mathcal{P}_t} \rho(\underset{\sim}{y},t)\, \overline{\underset{\sim}{v}}(\underset{\sim}{y},t)\, dV_{\underset{\sim}{y}} \quad (t_0 \le t \le t_1) \tag{3.10}$$

(**)
$$\int_{\mathcal{P}_t} \underset{\sim}{y} \wedge \underset{\sim}{b}(\underset{\sim}{y},t)\, dV_{\underset{\sim}{y}} + \int_{\partial\mathcal{P}_t} \underset{\sim}{y} \wedge \underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\, dA_{\underset{\sim}{y}} = \frac{d}{dt}\int_{\mathcal{P}_t} \underset{\sim}{y} \wedge \rho(\underset{\sim}{y},t)\, \overline{\underset{\sim}{v}}(\underset{\sim}{y},t)\, dV_{\underset{\sim}{y}} \quad (t_0 \le t \le t_1) \tag{3.11}$$

Here again $\overline{\underset{\sim}{v}}(\underset{\sim}{y},t) = \underset{\sim}{v}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t)$. Explain terms in (3.10), (3.11)

<u>Remarks</u>. The above two independent axioms constitute Newton's laws of motion for continuous media. Note:

the mechanics of discrete particle systems, the principle of angular momentum is deducible from the principle of linear momentum if one assumes that the mutual forces between the constituent particles occur in balanced pairs.

The existence of the time derivatives involved in (3.10), (3.11) is assured by our previous regularity hyp., as will become clear soon. In order to cast (3.10), (3.11) into a more convenient form, we require the following auxiliary theorem.

<u>Thm.</u> 3.1. Let $\hat{\underset{\sim}{y}}(\cdot,t):\mathcal{R}\to\mathcal{R}_t$ $(t_0\le t\le t_1)$ be an admiss. motion of a body. Let $\mathcal{P}_t$ $(t_0\le t\le t_1)$ and $\rho(\cdot,t)$ be as before. Suppose $\underset{\sim}{g}\in C^1(\mathcal{R}\times\mathcal{J})$ is vector-valued & define

$$\bar{\underset{\sim}{g}}(y,t)=\underset{\sim}{g}(\hat{\underset{\sim}{x}}(y,t),t)\ \forall\ \underset{\sim}{y}\in\mathcal{R}_t\quad(t_0\le t\le t_1).$$

Then,

$$\frac{d}{dt}\int_{\mathcal{P}_t}\rho(y,t)\bar{\underset{\sim}{g}}(y,t)\,dV_y=\int_{\mathcal{P}_t}\rho(y,t)\dot{\underset{\sim}{g}}(\underset{\sim}{x},t)\,dV_y\ ,\ \underset{\sim}{x}=\hat{\underset{\sim}{x}}(y,t)\quad(3.12)$$

$$(t_1\le t\le t_2).$$

Proof

$$\bar{\underset{\sim}{g}}(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t) = \underset{\sim}{g}(\underset{\sim}{x},t) \;\forall\; (x,t) \in \mathcal{R}\times\mathcal{T}$$

From this identity, mass conservation, & Calculus one has

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho(\underset{\sim}{y},t)\,\bar{\underset{\sim}{g}}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} = \frac{d}{dt}\int_{\mathcal{P}} \underbrace{\rho(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t))\,J(\underset{\sim}{x},t)}_{\rho_0(\underset{\sim}{x})}\,\underset{\sim}{g}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

$$= \frac{d}{dt}\int_{\mathcal{P}} \rho_0(\underset{\sim}{x})\,\underset{\sim}{g}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}} = \int_{\mathcal{P}} \rho_0(\underset{\sim}{x})\,\dot{\underset{\sim}{g}}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}} = \int_{\mathcal{P}} \rho(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t))\,J(\underset{\sim}{x},t)\,\dot{\underset{\sim}{g}}(\underset{\sim}{x},t)\,dV$$

$$= \int_{\mathcal{P}_t} \rho(\underset{\sim}{y},t)\,\dot{\underset{\sim}{g}}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t)\,dV_{\underset{\sim}{y}}$$

Remark on obvious generalization of above theorem to tensors of arbitrary order.

Now apply Thm. 3.1 to (3.10), (3.11), taking

$\underset{\sim}{g}(\underset{\sim}{x},t) = \underset{\sim}{y}(\underset{\sim}{x},t)$ in (3.10), $\underset{\sim}{g}(\underset{\sim}{x},t) = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)\wedge\underset{\sim}{v}(\underset{\sim}{x},t)$ in (3.11)

so that

$\dot{\underset{\sim}{g}}(\underset{\sim}{x},t) = \underset{\sim}{a}(\underset{\sim}{x},t)$ in (3.10), $\dot{\underset{\sim}{g}}(\underset{\sim}{x},t) = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)\wedge\underset{\sim}{a}(\underset{\sim}{x},t)$ in (3.1

$\overline{\underset{\sim}{g}}(\underset{\sim}{y},t) = \overline{\underset{\sim}{v}}(\underset{\sim}{y},t)$ in (3.10), $\overline{\underset{\sim}{g}}(\underset{\sim}{y},t) = \underset{\sim}{y}\wedge\overline{\underset{\sim}{v}}(\underset{\sim}{y},t)$ in (3.11)

Thus (3.10), (3.11) are equivalent to

(§) $$\int_{\mathcal{P}_t}\underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} + \int_{\partial\mathcal{P}_t}\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\,dA_{\underset{\sim}{y}} = \int_{\mathcal{P}_t}\rho(\underset{\sim}{y},t)\,\overline{\underset{\sim}{a}}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} \quad \forall\, t\in\mathcal{T} \quad (3.$$

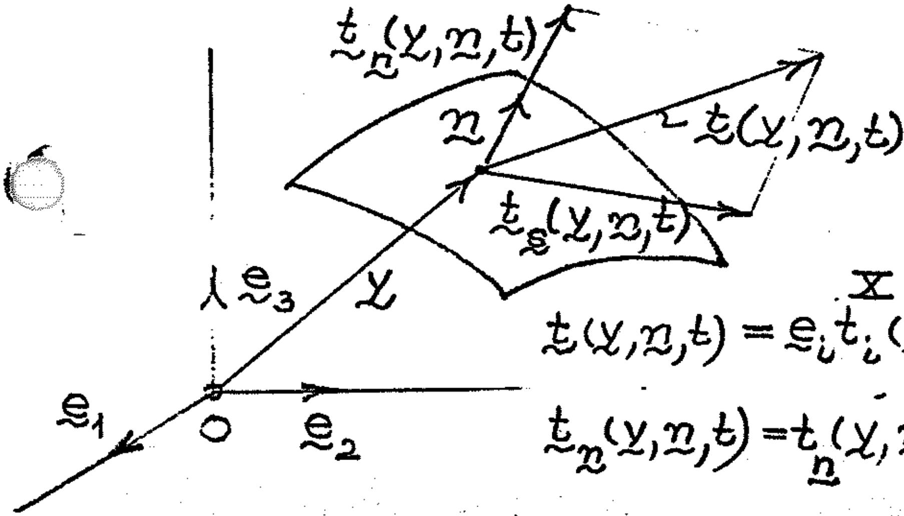
(§) $$\int_{\mathcal{P}_t}\underset{\sim}{y}\wedge\underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} + \int_{\partial\mathcal{P}_t}\underset{\sim}{y}\wedge\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\,dA_{\underset{\sim}{y}} = \int_{\mathcal{P}_t}\underset{\sim}{y}\wedge\rho(\underset{\sim}{y},t)\,\overline{\underset{\sim}{a}}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} \quad \forall\, t\in\mathcal{T} \quad (3.$$

where $\overline{\underset{\sim}{a}}(\underset{\sim}{y},t) = \underset{\sim}{a}(\hat{\underset{\sim}{x}}(\underset{\sim}{y},t),t)$ is the Eulerian accel. field.

<u>The notion of stress; study of the instantaneous local state of st</u>

Let $\hat{\underset{\sim}{y}}(\cdot,t): \mathcal{R}\to\mathcal{R}_t$ $(t_0\le t\le t_1)$ be an admiss. motion. Fix $t\in\mathcal{T}$ and consider a point $\underset{\sim}{y}\in\mathcal{R}_t$. We wish to explore the dependence of $\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)$ upon the unit-vector $\underset{\sim}{n}$ for fixed $t\in\mathcal{T}$ and fixed $\underset{\sim}{y}\in\mathcal{R}_t$.

Choose $\mathfrak{X} = \{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}\in\mathcal{F}$;



$$\left.\begin{array}{l} \mathbf{t}(\mathbf{y},\mathbf{n},t) = \mathbf{e}_i \overset{X}{t_i}(\mathbf{y},\mathbf{n},t) = \mathbf{t}_{\mathbf{n}}(\mathbf{y},\mathbf{n},t) + \mathbf{t}_{\mathbf{s}}(\mathbf{y},\mathbf{n},t) \\ \mathbf{t}_{\mathbf{n}}(\mathbf{y},\mathbf{n},t) = t_{\mathbf{n}}(\mathbf{y},\mathbf{n},t)\,\mathbf{n}\ ,\ \mathbf{t}_{\mathbf{s}}(\mathbf{y},\mathbf{n},t)\cdot\mathbf{n} = 0 \end{array}\right\} (3$$

$$t_{\mathbf{n}}(\mathbf{y},\mathbf{n},t) = \mathbf{t}(\mathbf{y},\mathbf{n},t)\cdot\mathbf{n} = \mathbf{t}_{\mathbf{n}}(\mathbf{y},\mathbf{n},t)\cdot\mathbf{n} \qquad (3.16)$$

Terminology:

$\mathbf{t}(\mathbf{y},\mathbf{n},t)$ ... traction or stress vector at $\mathbf{y}$ on a surface element with oriented unit normal vector $\mathbf{n}$, at time t.

$\overset{X}{t_i}(\mathbf{y},\mathbf{n},t)$ ... scalar components in X of $\mathbf{t}(\mathbf{y},\mathbf{n},t)$

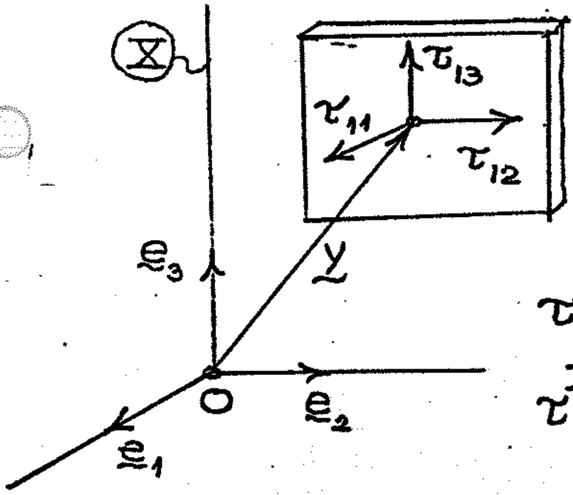
$\mathbf{t}_{\mathbf{n}}(\mathbf{y},\mathbf{n},t)$, $\mathbf{t}_{\mathbf{s}}(\mathbf{y},\mathbf{n},t)$ ... vectorial normal and shear components of $\mathbf{t}(\mathbf{y},\mathbf{n},t)$

$t_{\mathbf{n}}(\mathbf{y},\mathbf{n},t)$ ... scalar normal stress; tensile if $t_{\mathbf{n}} > 0$, compressive if $t_{\mathbf{n}} < 0$.

Define $\overset{X}{\boldsymbol{\tau}_i}(\mathbf{y},t)$ and $\overset{X}{\tau_{ij}}(\mathbf{y},t)$ as follows:

$$\left.\begin{array}{l} (1)\ \overset{X}{\boldsymbol{\tau}_i}(\mathbf{y},t) = \mathbf{t}(\mathbf{y},\mathbf{e}_i,t)\ ;\ (2)\ \overset{X}{\tau_{ij}}(\mathbf{y},t) = \boldsymbol{\tau}_i(\mathbf{y},t)\cdot\mathbf{e}_j \ \text{ or} \\ \qquad \overset{X}{\boldsymbol{\tau}_i}(\mathbf{y},t) = \overset{X}{\tau_{ij}}(\mathbf{y},t)\,\mathbf{e}_j \end{array}\right\} (3.17)$$

$\overset{X}{\tau_{ij}}(\mathbf{y},t)$ ... cartesian compts. of stress at $\mathbf{y}$ and time t.



$\tau^{X}_{ij}(y,t)$ ... cartes. compts. of actual (Cauchy) stress in X at y & t

$\tau^{X}_{ii}(y,t)$ (no sum) ... normal stress com

$\tau^{X}_{ij}(y,t)$ $(i \neq j)$... shear stress component

From (3.16), (3.17) one has

$$\left.\begin{aligned} \tau^{X}_{ii}(y,t) = \underset{\sim}{t}(y,e_i,t)\cdot e_i = t_n(y,e_i,t) \quad \text{(no sum)} \\ e_2\tau^{X}_{12}(y,t) + e_3\tau^{X}_{13}(y,t) = \underset{\sim}{t}_s(y,e_1,t) \end{aligned}\right\} \quad (3.18)$$

Thus $\forall X \in \mathcal{F}$, $\forall y \in \mathcal{R}_t$, $\forall t \in \mathcal{T}$ one has 9 cartesian components of stress $[\tau^{X}_{ij}(y,t)]$. Raise questions associated with instantaneous local state of stress.*

Thm. 3.2. Let $\hat{y}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ be an admissible motion and suppose $\underset{\sim}{t}(y,n,t)$ $\forall$ $(y,n) \in \mathcal{R}_t \times \mathcal{U}$ $(t_0 \le t \le t_1)$ is the associated surface-traction field. Then,

(a) $\underset{\sim}{t}(y,-n,t) = -\underset{\sim}{t}(y,n,t)$ $\forall$ $(y,n) \in \mathcal{R}_t \times \mathcal{U}$. $(t_0 \le t \le t_1)$

(b) $\exists$ a two-tensor field $\tau(\cdot,t)$ defined on $\mathcal{R}_t$ $\forall t \in \mathcal{T}$.

$$\underset{\sim}{t}(y,n,t) = \tau^T(y,t)\,n \quad \forall \ (y,n) \in \mathcal{R}_t \times \mathcal{U} \ (t_0 \le t \le t_1).$$

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* We do not know as yet that $\tau^{X}_{ij}(y,t)$ are components

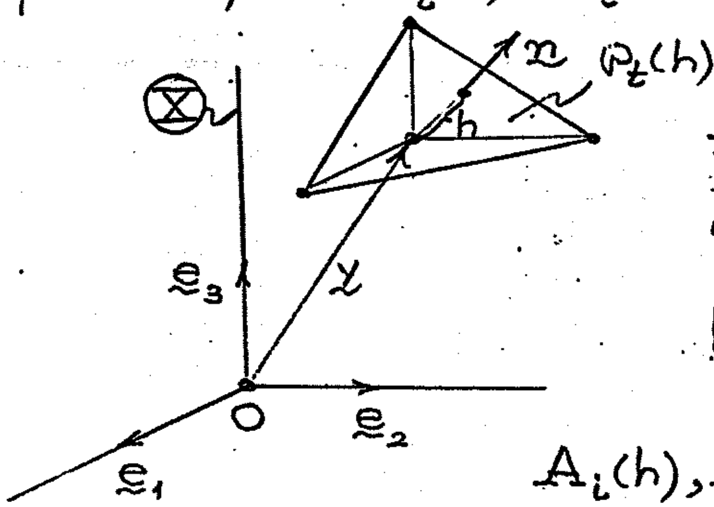
Moreover, $\overset{X}{\tau_{ij}}(\underset{\sim}{y},t)=\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{e}_i;t)\cdot\underset{\sim}{e}_j\ \forall\, X\in\mathcal{F}$, so that $\overset{X}{\tau_{ij}}$ are the cartesian components of stress defined before

Proof. By our previous assumptions we have

$\rho(\cdot,t)\in\mathcal{C}(\mathcal{R}_t),\ \underset{\sim}{b}(\cdot,t)\in\mathcal{C}(\mathcal{R}_t),\ \underset{\sim}{t}(\cdot,\cdot,t)\in\mathcal{C}(\mathcal{R}_t\times\mathcal{U}),\ \overline{\underset{\sim}{a}}(\cdot,t)\in\mathcal{C}(\mathcal{R}_t$ $\forall t\in\mathcal{J}$ (1

Choose $X=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}\in\mathcal{F}$ and fix $t\in\mathcal{J}$.

Assume first $\underset{\sim}{y}\in\mathring{\mathcal{R}}_t$ and $\overset{X}{n_i}>0$ (explain). Apply (3.13) (Principle of linear momentum) to the following famil of tetrahydra $\mathcal{P}_t(h)\subset\mathring{\mathcal{R}}_t$ $(0<h<h_0)$:



$\Lambda_i(h),\Lambda(h)$ ... faces with outer unit normal $-\underset{\sim}{e}_i,\underset{\sim}{n}$, respectivel

h ... altitude of $\Lambda(h)$

$A_i(h),A(h)$ ... areas of $\Lambda_i(h),\Lambda(h)$

$V(h)$ ... volume of $\mathcal{P}_t(h)$.

Thus,

$$A_i(h)=A(h)\overset{X}{n_i}\ ,\quad V(h)=\tfrac{1}{3}\,h\,A(h) \tag{2}$$

Next, $\forall\ \underset{\sim}{\eta}\in\mathcal{R}_t$ write

$$\rho(\underset{\sim}{\eta})=\rho(\underset{\sim}{\eta},t),\ \underset{\sim}{b}(\underset{\sim}{\eta})=\underset{\sim}{b}(\underset{\sim}{\eta},t),\ \underset{\sim}{t}(\underset{\sim}{\eta},\underset{\sim}{n})=\underset{\sim}{t}(\underset{\sim}{\eta},\underset{\sim}{n},t),\ \overline{\underset{\sim}{a}}(\underset{\sim}{\eta})=\overline{\underset{\sim}{a}}(\underset{\sim}{\eta},t \tag{3}$$

Then, from (3.13),

$$\int_{\mathcal{P}_t(h)} \underset{\sim}{b}(\underset{\sim}{\eta})\, dV_{\eta} + \int_{\Lambda(h)} \underset{\sim}{t}(\underset{\sim}{\eta}, \underset{\sim}{n})\, dA_{\eta} + \sum_{j=1}^{3} \int_{\Lambda_j(h)} \underset{\sim}{t}(\underset{\sim}{\eta}, -\underset{\sim}{e}_j)\, dA_{\eta} = \int_{\mathcal{P}_t(h)} \rho(\underset{\sim}{\eta})\, \overline{\underset{\sim}{a}}(\underset{\sim}{\eta})\, dV_{\eta}.$$

Hence, by (1), (2) and the mean-value theorem,

$$[\underset{\sim}{b}(\underset{\sim}{y}) + \underset{\sim}{\alpha}(h)] \frac{hA(h)}{3} + [\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}) + \underset{\sim}{\beta}(h)] A(h) + \sum_{j=1}^{3} [\underset{\sim}{t}(\underset{\sim}{y}, -\underset{\sim}{e}_j) + \underset{\sim}{\beta}_j(h)] A(h) n_j^{X}$$

$$= [\rho(\underset{\sim}{y}) \overline{\underset{\sim}{a}}(\underset{\sim}{y}) + \underset{\sim}{\mu}(h)] \frac{hA(h)}{3} \quad (0<h<h_0), \quad (4)$$

where, by continuity properties (1),

$$\underset{\sim}{\alpha}(h) = o(1), \ \underset{\sim}{\beta}(h) = o(1), \ \underset{\sim}{\beta}_j(h) = o(1), \ \underset{\sim}{\mu}(h) = o(1) \text{ as } h \to 0 \quad (5)$$

Divide (4) by A(h) and let $h \to 0$, using (5). Thus,

$$\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}) = -\sum_{j=1}^{3} \underset{\sim}{t}(\underset{\sim}{y}, -\underset{\sim}{e}_j)\, n_j^{X} \quad \forall\ \underset{\sim}{n} \in \mathcal{U} \ni n_i^{X} > 0 \quad (6)$$

Proceed to the limit in (6) as $\underset{\sim}{n} \to \underset{\sim}{e}_1$, i.e. as $n_j^{X} \to \delta_{j1}$. Thus and from the continuity of $\underset{\sim}{t}(\underset{\sim}{y}, \cdot)$ on $\mathcal{U}$,

$\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{e}_1) = -\underset{\sim}{t}(\underset{\sim}{y}, -\underset{\sim}{e}_1)$, which implies (a) since X is an arbitrarily chosen frame. In view of (a) and (3.17) one may write (6) as

$$\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}) = \sum_{j=1}^{3} \underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{e}_j)\, n_j^{X} = \underset{\sim}{\tau}_j^{X}(\underset{\sim}{y})\, n_j^{X} \quad \forall\ \underset{\sim}{y} \in \mathring{R}_t \quad (7)$$
$$\forall\ \underset{\sim}{n} \in \mathcal{U} \ni n_i^{X} \geq 0$$

So far (7) is known to hold only $\forall\, \underset{\sim}{n} \in \mathcal{U} \ni \overset{X}{n_i} \geq 0$ & $\forall\, \underset{\sim}{y} \in \mathring{R}$. To see that (7) holds $\underline{\underline{\forall}}\, \underset{\sim}{n} \in \mathcal{U}$ consider analogous families of tetrahydra in remaining octants; To see that (7) holds $\forall\, \underset{\sim}{y} \in \underline{\underline{R_t}}$ use continuity of $\underset{\sim}{t}(\cdot, \underset{\sim}{n})$ on R_t.

Dot-multiply (7) by $\underset{\sim}{e}_i$ & use (3.17) to get

$$\overset{X}{t_i}(\underset{\sim}{y}, \underset{\sim}{n}) = \overset{X}{\tau_{ji}}(\underset{\sim}{y})\, \overset{X}{n_j} \quad \forall\, \underset{\sim}{y} \in R_t\,, \forall\, \underset{\sim}{n} \in \mathcal{U}\,, \forall\, X \in \mathcal{F}$$

(b) now follows from Thm. 1.4 (Quotient rule)/

Discussion. Elaborate on content of above thm. The instantaneous local state of stress at $\underset{\sim}{y} \in R_t$ is completely determined by $\overset{X}{\tau_{ij}}(\underset{\sim}{y}, t)$ or equivalently by $\underset{\sim}{\tau}_j(\underset{\sim}{y}, t) \equiv \underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{e}_j, t)$ for a single $X \in \mathcal{F}$. Also, $\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}, t)$ depends linearly upon $\underset{\sim}{n}$; indeed, $\underset{\sim}{t}(\underset{\sim}{y}, \cdot, t)$ is the restriction to $\mathcal{U}$ of a linear vector function, whose generating tensor is $\underset{\sim}{\tau}^T(\underset{\sim}{y}, t)$. One calls $\underset{\sim}{\tau}(\cdot, t)$ the actual (Cauchy) stress tensor field.

Our next aim is to use Thm. 3.2 to convert the momentum principles into p.d.e. form.

Thm. 3.3. Let $\hat{y}(\cdot,t): \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ be an admissible motion. Suppose $\rho(\cdot,t) \in \mathcal{C}(\mathcal{R}_t)$, $\underline{b}(\cdot,t) \in \mathcal{C}(\mathcal{R}_t)$ and let $\underline{\tau}(\cdot,t) \in \mathcal{C}^1(\mathcal{R}_t)$ $(t_0 \le t \le t_1)$ be the associated mass density, body-force density, and Cauchy stress-tensor field. Then a necessary condition that the momentum equats. (3.13), (3.14) hold $\forall\, \mathcal{P}_t = \hat{y}(\mathcal{P},t)$ $(t_0 \le t \le t_1) \ni \mathcal{P}$ is the closure of a bounded regular subregion of $\mathcal{R}$, is that

$$\left.\begin{aligned} &\nabla_{y}\cdot \underline{\tau}(\underline{y},t) + \underline{b}(\underline{y},t) = \rho(\underline{y},t)\,\underline{\bar{a}}(\underline{y},t)\,,\quad \underline{\tau}(\underline{y},t) = \underline{\tau}^T(\underline{y},t)\quad \forall\, \underline{y} \in \mathcal{R}_t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t_0 \le t \le t_1) \\ &\text{or}\qquad \tau_{ij,j} + b_i = \rho\,\bar{a}_i\,,\quad \tau_{ij} = \tau_{ji} \\ &\text{or}\qquad \frac{\partial \tau_{11}}{\partial y_1} + \frac{\partial \tau_{12}}{\partial y_2} + \frac{\partial \tau_{13}}{\partial y_3} = \rho\,\bar{a}_1, \ldots, \ldots, \tau_{12} = \tau_{21}, \ldots, \ldots \end{aligned}\right\} (3.$$

Conversely, (3.19) and (b) in Thm. 3.2 $\Rightarrow$ (3.13), (3.14).

Proof

Re necessity. (3.13) and (b) in Thm. 3.2 $\Rightarrow$

$$\int_{\mathcal{P}_t} b_i\, dV + \int_{\partial\mathcal{P}_t} \tau_{ji}\, n_j\, dA = \int_{\mathcal{P}_t} \rho\, \bar{a}_i\, dV$$

Apply Thm. 1.15 (divergence thm) to the surface integral[1]:

1 One can show that this application of Thm. 1.15 is legitimate under our hyp. concerning $\mathcal{P}$.

$$\int_{\mathcal{P}_t} (\tau_{ji,j} + b_i - \rho \bar{a}_i)\, dV = 0,$$

so that since $\mathcal{P}_t$ is arbitrary and the integrand in $\mathcal{C}(\mathcal{P}_t)$,

$$\tau_{ji,j} + b_i = \rho \dot{\bar{a}}_i \quad \text{on } \mathcal{R}_t \quad (t_0 \le t \le t_1) \qquad (1)$$

Next, (3.14), (1.12), and (b) in Thm. 3.2 $\Rightarrow$

$$\int_{\mathcal{P}_t} \varepsilon_{ijk}\, y_j b_k\, dV + \int_{\partial\mathcal{P}_t} \underbrace{\varepsilon_{ijk}\, y_j\, \tau_{pk}}_{W_{ip}} n_p\, dA = \int_{\mathcal{P}_t} \rho\, \varepsilon_{ijk}\, y_j \bar{a}_k\, dV$$

Hence from Thm. 1.15,

$$\int_{\mathcal{P}_t} \varepsilon_{ijk} [y_j b_k + (y_j \tau_{pk})_{,p} - \rho y_j \bar{a}_k]\, dV = 0 \quad \text{or}$$

$$\int_{\mathcal{P}_t} \varepsilon_{ijk} [y_j \underbrace{(b_k + \tau_{pk,p} - \rho \bar{a}_k)}_{\text{zero by (1)}} + \delta_{jp} \tau_{pk}]\, dV = 0 \quad \text{or}$$

$$\int_{\mathcal{P}_t} \varepsilon_{ijk} \tau_{jk}\, dV = 0, \quad \text{whence}$$

$$\varepsilon_{ijk} \tau_{jk} = 0 \quad \text{or} \quad \tau_{ji} = \tau_{ij} \quad \text{on } \mathcal{R}_t \quad (t_0 \le t \le t_1) \qquad (2)$$

(1), (2) $\Rightarrow$ (3.19).

* [illegible] τ [illegible] $\in \mathcal{C}^1(\mathcal{R}_t)$ [illegible]

Re converse. Reverse above argument to deduce (3.13), (3.14) from (3.19). qed.

Discussion. Eqs. (3.19) are the equations of motion for a cont. medium. It is clear from proof of Thm. 3.3 that (3.13) (princ. of linear momentum), Thm. 3.2, and symm of $\underset{\sim}{T}$ $\Rightarrow$ (3.14) (princ. of angular momentum). Explain.

Remark on heuristic derivation of eqs. of motion.

From $\underset{\sim}{T}^T = \underset{\sim}{T}$ and (b) in Thm. 3.2 one now has

$$\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}, t) = \underset{\sim}{T}(\underset{\sim}{y}, t)\,\underset{\sim}{n} \quad \forall\, (\underset{\sim}{y}, \underset{\sim}{n}) \in \mathcal{R}_t \times \mathcal{U} \quad (t_0 \le t \le t_1) \qquad (3.20)$$

(Cauchy traction-stress relation).

Thus $\underset{\sim}{t}(\underset{\sim}{y}, \cdot, t)$ is the restriction to $\mathcal{U}$ of the linear vector function generated by the symm. two-tensor $\underset{\sim}{T}(\underset{\sim}{y}, t)$. Mention traction boundary conds.

Further study of the instantaneous local state of stress.

Fix $t \in \mathcal{J}$, fix $\underset{\sim}{y} \in \mathcal{R}_t$ and write

$$\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}, t) = \underset{\sim}{t}(\underset{\sim}{n}), \quad \underset{\sim}{T}(\underset{\sim}{y}, t) = \underset{\sim}{T}$$

whence (3.19), (3.20) give

$$\left.\begin{array}{ll} & \underset{\sim}{t}(\underset{\sim}{n}) = \underset{\sim}{\tau}\underset{\sim}{n} \ \forall \ \underset{\sim}{n} \in \mathcal{U}, \ \underset{\sim}{\tau}^T = \underset{\sim}{\tau} \\ \text{or} & t_i(\underset{\sim}{n}) = \tau_{ij} n_j \ \forall \ \underset{\sim}{n} \in \mathcal{U}, \ \tau_{ji} = \tau_{ij} \end{array}\right\} \quad (3.21)$$

(3.21), (3.23) $\Rightarrow \underset{\sim}{n}' \cdot (\underset{\sim}{\tau}\underset{\sim}{n}) = (\underset{\sim}{\tau}^T \underset{\sim}{n}') \cdot \underset{\sim}{n} = (\underset{\sim}{\tau}\underset{\sim}{n}') \cdot \underset{\sim}{n}$ or

$$\underset{\sim}{t}(\underset{\sim}{n}) \cdot \underset{\sim}{n}' = \underset{\sim}{t}(\underset{\sim}{n}') \cdot \underset{\sim}{n} \ \forall \ (\underset{\sim}{n}, \underset{\sim}{n}') \in \mathcal{U} \times \mathcal{U} \qquad (3.22)$$

Interpret the reciprocal relation (3.22). In particular,

(3.22), (3.17) $\Rightarrow \underset{\sim}{t}(\underset{\sim}{e}_i) \cdot \underset{\sim}{e}_j = \underset{\sim}{t}(\underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i$ or $\tau_{ij} = \tau_{ji}$.

Recall that λ is a p. value and $\underset{\sim}{n}$ an associated p. direction of the tensor $\underset{\sim}{\tau}$ if

$$\underset{\sim}{t}(\underset{\sim}{n}) = \underset{\sim}{\tau}\underset{\sim}{n} = \lambda \underset{\sim}{n} \Leftrightarrow \underset{\sim}{t}_{\underset{\sim}{s}}(\underset{\sim}{n}) = \underset{\sim}{0}, \ t_{\underset{\sim}{n}}(\underset{\sim}{n}) = \lambda \qquad (3.23)$$

Use (3.23) to explain physical signif. of "principal stresses" and "principal planes". Since $\underset{\sim}{\tau}$ is symm., $\exists \ \mathbb{X} \in \mathcal{F} \ni$

$$[\overset{\mathbb{X}}{\tau_{ij}}] = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}, \quad \overset{\mathbb{X}}{t_i} = \tau_i \overset{\mathbb{X}}{n_i} \ (\text{no sum}) \ \forall \ \underset{\sim}{n} \in \mathcal{U}, \ (3.2$$

where $\tau_i = \lambda_i(\underset{\sim}{\tau})$. Review determination τ_i & p. frame $\mathbb{X}$

Motivate and cite without proof the following theorems.

Thm. Let $\underset{\sim}{\tau}$ be a symm. two-tensor. A nec. & suff. cond. that $\exists\ X^* = \{O; \underset{\sim}{e}^*_1, \underset{\sim}{e}^*_2, \underset{\sim}{e}^*_3\} \in \mathcal{F} \ni \tau^{X^*}_{ii} = 0$ (no sum) is that $\operatorname{tr} \underset{\sim}{\tau} = \tau_{ii} = 0$.

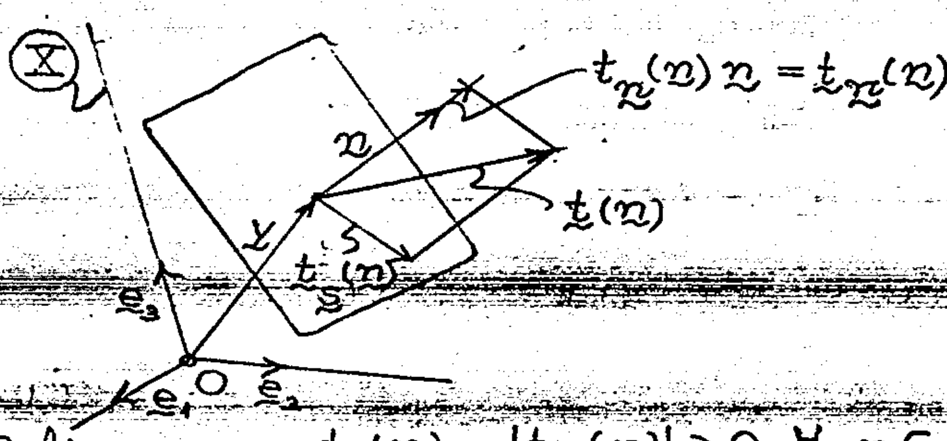
Remark. Necessity obvious, sufficiency not. If $\operatorname{tr}\underset{\sim}{\tau} = 0$, one calls $\underset{\sim}{\tau}$ a deviatoric tensor (refer to Thm. 1.12).

Mohr's circles[1]

Objective: A geometric model for the instantaneous local state of stress that leads to additional results.

Recall from (3.15), if $\underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}, t) = \underset{\sim}{t}(\underset{\sim}{n})$, $\underset{\sim}{\tau}(\underset{\sim}{y}, t) = \underset{\sim}{\tau}$ etc.

$$\underset{\sim}{t}(\underset{\sim}{n}) = t_{\underset{\sim}{n}}(\underset{\sim}{n})\underset{\sim}{n} + \underset{\sim}{t}_{\underset{\sim}{s}}(\underset{\sim}{n}), \quad \underset{\sim}{t}_{\underset{\sim}{s}}(\underset{\sim}{n}) \cdot \underset{\sim}{n} = 0, \quad t_{\underset{\sim}{n}}(\underset{\sim}{n}) = \underset{\sim}{t}(\underset{\sim}{n}) \cdot \underset{\sim}{n} \qquad (3.25)$$



Define $$t_{\underset{\sim}{s}}(\underset{\sim}{n}) = |\underset{\sim}{t}_{\underset{\sim}{s}}(\underset{\sim}{n})| \geq 0 \ \forall\ \underset{\sim}{n} \in \mathcal{U} \qquad (3.26)$$

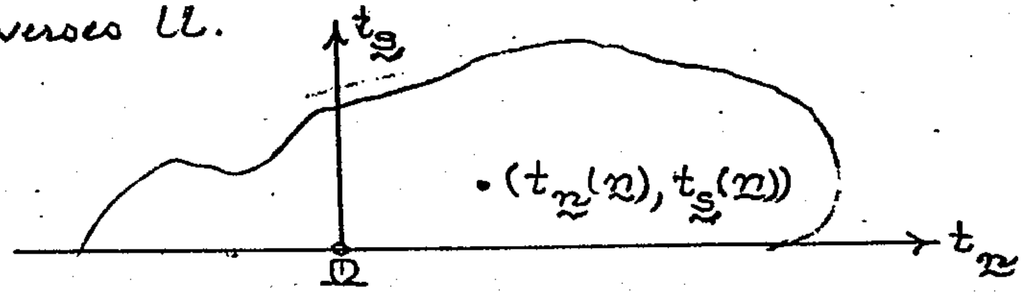
---

[1] Otto Mohr, 1868.

Let $\overset{+}{\Pi}$ be the "stress half-plane" defined by

$$\overset{+}{\Pi} = \{(t_{\mathbf{n}}, t_{\mathbf{s}}) \mid -\infty < t_{\mathbf{n}} < \infty,\ 0 \le t_{\mathbf{s}} < \infty\}$$

Given $\boldsymbol{\tau}$, we seek the locus Λ of all points in $\overset{+}{\Pi}$ with the rectangular cartesian coordinates $(t_{\mathbf{n}}(\mathbf{n}), t_{\mathbf{s}}(\mathbf{n}))$ as $\mathbf{n}$ traverses $\mathcal{U}$.



$$\Lambda(\boldsymbol{\tau}) = \{(t_{\mathbf{n}}, t_{\mathbf{s}}) \mid (\boldsymbol{\tau}\mathbf{n})^2 = t_{\mathbf{n}}^2 + t_{\mathbf{s}}^2,\ (\boldsymbol{\tau}\mathbf{n})\cdot\mathbf{n} = t_{\mathbf{n}} \text{ for some } \mathbf{n} \in \mathcal{U}$$

Clearly, $\Lambda(\boldsymbol{\tau}) \subset \overset{+}{\Pi}$.

For convenience assume $\mathfrak{X} = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathcal{F}$ is principal for $\boldsymbol{\tau}$ $\therefore$ $\boldsymbol{\tau}\mathbf{e}_i = \tau_i \mathbf{e}_i$ (no sum). Write $\overset{\mathfrak{X}}{n_i} \equiv n_i$. By (3.24

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau}\mathbf{n} = \mathbf{e}_1\tau_1 n_1 + \mathbf{e}_2\tau_2 n_2 + \mathbf{e}_3\tau_3 n_3$$

and thus (3.25), (3.26) give

$$\left.\begin{aligned} \mathbf{t}^2(\mathbf{n}) = t_{\mathbf{n}}^2(\mathbf{n}) + t_{\mathbf{s}}^2(\mathbf{n}) &= \tau_1^2 n_1^2 + \tau_2^2 n_2^2 + \tau_3^2 n_3^2 \\ \mathbf{t}(\mathbf{n})\cdot\mathbf{n} = t_{\mathbf{n}}(\mathbf{n}) &= \tau_1 n_1^2 + \tau_2 n_2^2 + \tau_3 n_3^2 \\ \mathbf{n}^2 = 1 &= n_1^2 + n_2^2 + n_3^2 \end{aligned}\right\} \quad (3.27)$$

Assume without loss of generality

$$\tau_3 \le \tau_2 \le \tau_1 \qquad (3.27)$$

The first two of (3.27) enable one to compute $t_{\underset{\sim}{n}}(\underset{\sim}{n})$, $t_{\underset{\sim}{s}}(\underset{\sim}{n})$ $\forall\ \underset{\sim}{n} \in \mathcal{U}$. Conversely, $t_{\underset{\sim}{n}}(\underset{\sim}{n})$ and $t_{\underset{\sim}{s}}(\underset{\sim}{n})$ actually occur on an oriented plane with the unit normal vector $\underset{\sim}{n}$ if and only if the system (3.27) has a sol. for n_1^2, n_2^2, n_3^2 with $n_i^2 \geq 0$.

Case 1: $\tau_3 < \tau_2 < \tau_1$; Case 2: Say, $\tau_3 < \tau_2 = \tau_1$; Case 3: $\tau_1 = \tau_2 =$

Re Case 1. (non-degenerate)

Here (3.27) have a non-zero determinant and their solution is easily found to be

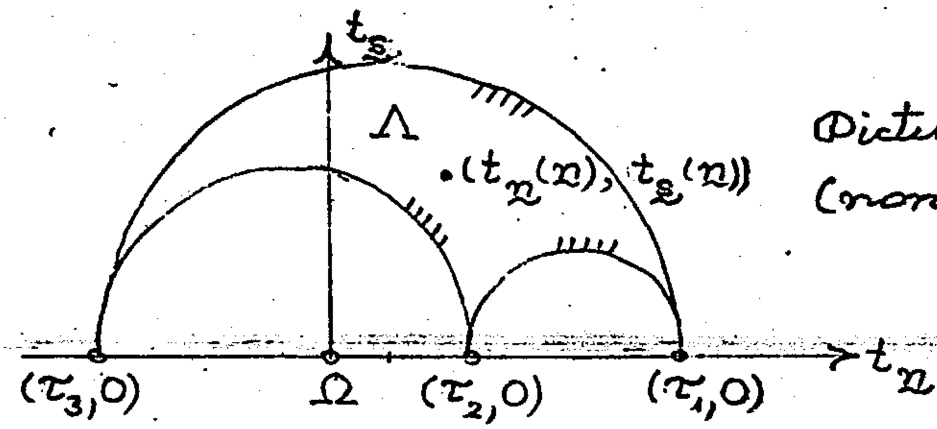
$$\left.\begin{aligned} n_1^2 &= \frac{t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_2)(t_{\underset{\sim}{n}} - \tau_3)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)} \\ n_2^2 &= \frac{t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_3)(t_{\underset{\sim}{n}} - \tau_1)}{(\tau_2 - \tau_3)(\tau_2 - \tau_1)} \\ n_3^2 &= \frac{t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_1)(t_{\underset{\sim}{n}} - \tau_2)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)} \end{aligned}\right\} \quad (3.28)$$

Thus Λ at present is characterized by

$$t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_2)(t_{\underset{\sim}{n}} - \tau_3) \geq 0, \ t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_3)(t_{\underset{\sim}{n}} - \tau_1) \leq 0, \ t_{\underset{\sim}{s}}^2 + (t_{\underset{\sim}{n}} - \tau_1)(t_{\underset{\sim}{n}} - \tau_2)$$

Therefore,

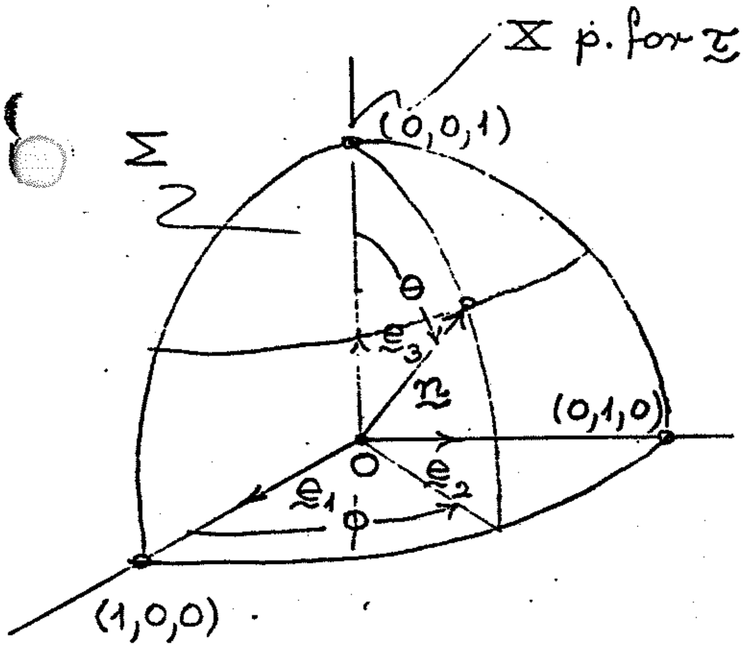
$$\left.\begin{aligned}
&\Lambda(\tau) \doteq \overset{*}{\Lambda}(\tau_1,\tau_2,\tau_3) = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \cap \overset{+}{\Pi} \\
&\Lambda_1:\ [t_n - \tfrac{1}{2}(\tau_2+\tau_3)]^2 + t_s^2 \geq \tfrac{1}{4}(\tau_2-\tau_3)^2 \\
&\Lambda_2:\ [t_n - \tfrac{1}{2}(\tau_3+\tau_1)]^2 + t_s^2 \leq \tfrac{1}{4}(\tau_3-\tau_1)^2 \\
&\Lambda_3:\ [t_n - \tfrac{1}{2}(\tau_1+\tau_2)]^2 + t_s^2 \geq \tfrac{1}{4}(\tau_1-\tau_2)^2
\end{aligned}\right\} \quad (3.29)$$

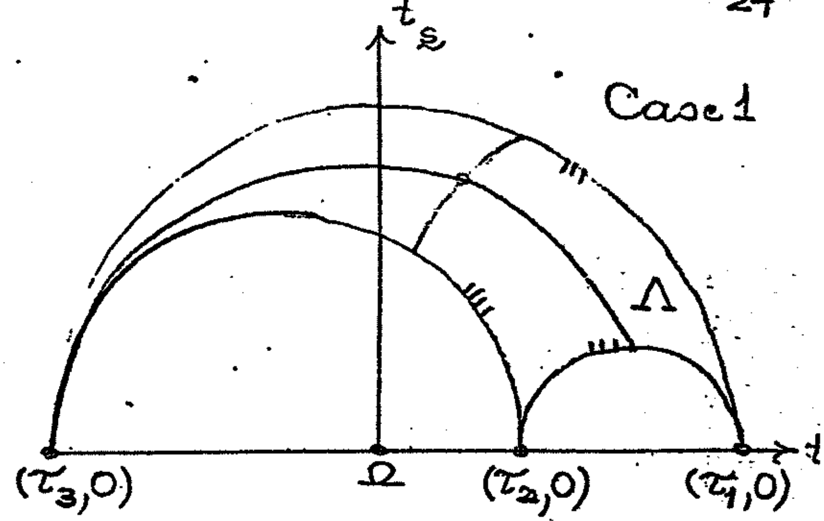


Picture for Case
(non-degenerate

One confirms easily that (3.29) continue to hold in Case 2 (see Exercise 15); (3.29) hold trivially in Case 3.

Discussion. ~~The first two of~~ (3.27) may be regarded as a mapping of the unit direction sphere Σ (whose poin have the pos. vector n) onto the region Λ. This mappg. is not (1,1) since the right-hand members in (3.27) are invariant under reflections about the planes $n_i = 0$ In Case 1 the inverse mapping, here given by (3.28), becomes single-valued if n is confined to the first octant ($n_i \geq 0$) of Σ.





<u>Exercise</u> 15. With reference to the Mohr mapping show that (3.29) continue to hold in Case 2. Show further that in Case 1 the images of the circles of longitude and latitude on Σ are circular arcs in Λ. HINT: Use spherical coor[dinates] on Σ.

<u>Theorem</u> 3.4 (Extremum properties of principal stresses). Let $\underset{\sim}{\tau}$ be the stress tensor generating an instantaneous local state of stress and assume the p. stresses ordered $\ni$ $\tau_3 \le \tau_2 \le \tau_1$. Then,

$$\tau_1 = \max_{\underset{\sim}{n} \in \mathcal{U}} t_{\underset{\sim}{n}}(\underset{\sim}{n}) = \max_{X \in \mathcal{F}} \tau_{ii}^{X}\,, \quad \tau_3 = \min_{\underset{\sim}{n} \in \mathcal{U}} t_{\underset{\sim}{n}}(\underset{\sim}{n}) = \min_{X \in \mathcal{F}} \tau_{ii}^{X} \qquad (3.3$$

(no sums)

Re remark at top of p.25

Let $\underset{\sim}{V}$ be a symm. two-tensor and $V_i = \lambda_i(\underset{\sim}{V})$ the p. values of V. Assume $V_3 \leq V_2 \leq V_1$. Define

$$w(\underset{\sim}{n}) = \underset{\sim}{n} \cdot \underset{\sim}{V}\underset{\sim}{n} \quad \forall\, \underset{\sim}{n} \in \mathcal{U}$$

and note that $w(\underset{\sim}{n}) = t_{\underset{\sim}{n}}(\underset{\sim}{n})$ if $\underset{\sim}{V} = \underset{\sim}{T}$. Thus,

$$V_1 = \max_{\underset{\sim}{n}\in\mathcal{U}} w(\underset{\sim}{n}) \;,\; V_3 = \min_{\underset{\sim}{n}\in\mathcal{U}} w(\underset{\sim}{n})$$

Now consider $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$ and note that

$$w(\underset{\sim}{e}_i) = \underset{\sim}{e}_i \cdot \underset{\sim}{V}\underset{\sim}{e}_i = \delta_{ik} V^{X}_{kj}\, \delta_{ij} = V^{X}_{ii} \quad \text{(no sum over } i\text{)}$$

Hence,

$$V_1 = \max_{X\in\mathcal{F}} V_{ii} \;,\; V_3 = \min_{X\in\mathcal{F}} V_{ii} \quad \text{(no sum)},$$

as asserted in Theorem 1.8.

<u>Proof.</u> Immediate from (3.29).

<u>Remark.</u> The above proof of Thm. 3.4, stripped of its particular physical context, supplies a proof of Thm. 1.8 concerning the extremum properties of the p. values of an arb. symm. 2-tensor. The question as to the max. $t_{\underset{\sim}{s}}(\mathfrak{n})$ and as to the planes on which it occurs is answered by the following theorem:

<u>Theorem</u> 3.5 (Maximum shear stress). Let $\mathfrak{X} = \{0; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be p. for an instantaneous local stress tensor $\underset{\sim}{\tau}$ and let $\underset{\sim}{e}_i$ be the p. direction vectors corresponding to the p. stresses τ_i with $\tau_3 \leq \tau_2 \leq \tau_1$. Then,

$$\max_{\mathfrak{n} \in \mathcal{U}} |\underset{\sim}{t}_{\underset{\sim}{s}}(\mathfrak{n})| = \frac{\tau_1 - \tau_3}{2} \qquad (3.31)$$

and this maximum occurs for $n_i \overset{\mathfrak{X}}{\equiv} n_i$ given by

$$n_1 = \pm \frac{1}{\sqrt{2}}, \quad n_2 = 0, \quad n_3 = \pm \frac{1}{\sqrt{2}} \qquad (3.32)^1$$

Further, if $\mathfrak{n}$ satisfies (3.32), one has

$$t_{\mathfrak{n}}(\mathfrak{n}) = \frac{\tau_1 + \tau_3}{2} \qquad (3.33)$$

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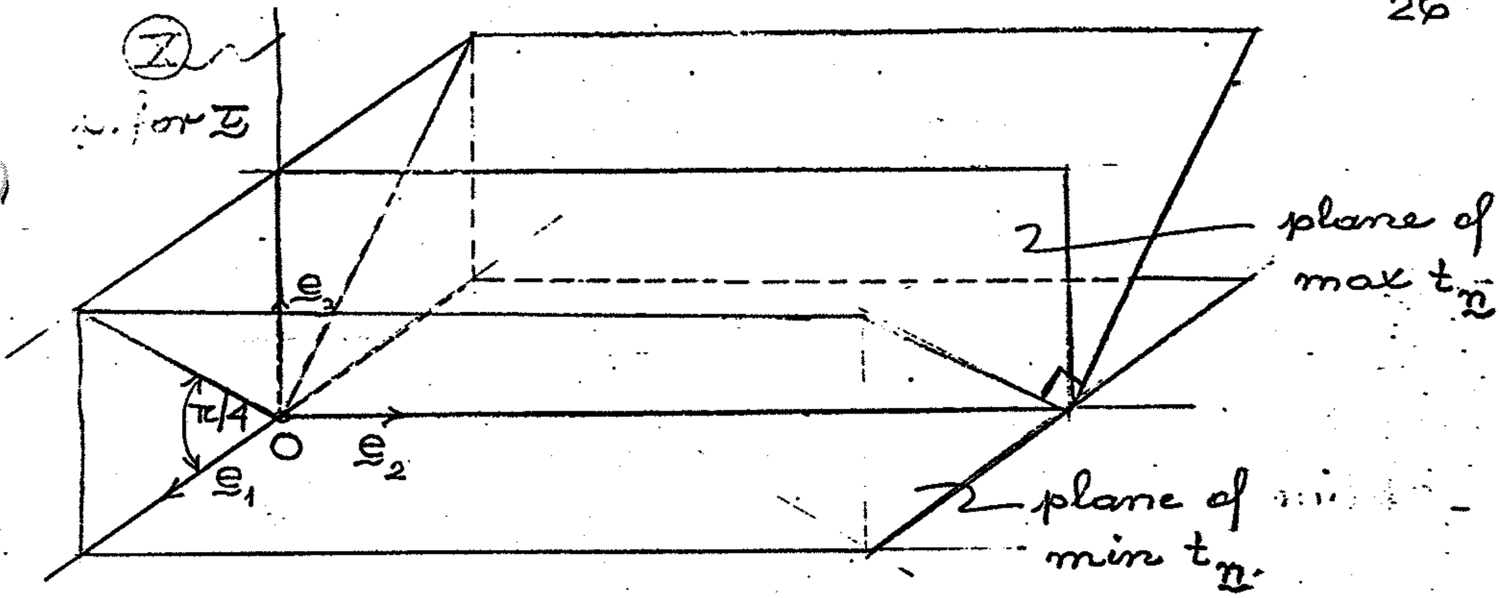
1 All sign combinations (there are four)



red ... p. planes , green ... planes of $\max|\underline{t}_{\underline{s}}|$

"The planes of max. shear bisect the right angle between the planes of max. & min. normal stress."

<u>Proof</u>. (3.31), (3.33) are immediate from (3.29) (see Mohr's diagram). To confirm (3.32) for Case 1, substitute $t_{\underline{n}} = \frac{1}{2}(\tau_1+\tau_3)$, $t_{\underline{s}} = \frac{1}{2}(\tau_1-\tau_3)$ into (3.28). Proceed analogously in Case 2. ~~(3.32)~~ (3.32) holds trivially in Case 3.

<u>Claim</u> . Let $X \in F$ be p. for an instantaneous local stress tensor $\underline{\tau}$. Then for $n_i \equiv \overset{X}{n_i}$ given by

$$n_1 = \pm\frac{1}{\sqrt{3}}\ ,\ n_2 = \pm\frac{1}{\sqrt{3}}\ ,\ n_3 = \pm\frac{1}{\sqrt{3}} \qquad (3.34)^1$$

one has

$$\left.\begin{aligned} t_{\underset{\sim}{n}}(\underset{\sim}{n}) &= \frac{1}{3}(\tau_1+\tau_2+\tau_3) = \frac{1}{3}\operatorname{tr}\underset{\sim}{\tau} = \frac{1}{3}I_1(\underset{\sim}{\tau}) \\ t_{\underset{\sim}{s}}(\underset{\sim}{n}) &\stackrel{\prime}{=} \tau_0 = \frac{1}{3}\sqrt{(\tau_1-\tau_2)^2+(\tau_2-\tau_3)^2+(\tau_3-\tau_1)^2} = \frac{\sqrt{2}}{3}\sqrt{I_1^2(\underset{\sim}{\tau})-3I_2(\underset{\sim}{\tau})} \end{aligned}\right\} (3.35)$$

<u>Remarks</u>. $t_{\underset{\sim}{n}}(\underset{\sim}{n}), t_{\underset{\sim}{s}}(\underset{\sim}{n})$ of (3.35) are called the "octahydral normal and shear stress" associated with $\underset{\sim}{\tau}$. Explain why. Allude to energetic interpretations of $t_{\underset{\sim}{s}}(\underset{\sim}{n}) \equiv \tau_0$ within context of linear elasticity theory (see later). Mention role of τ_0 in deformation theory of plasticity (Hencky-Mises flow condition).

<u>Re proof of above claim</u>

(3.35) follow from (3.34), (3.27) together with

$$I_1 = \tau_1+\tau_2+\tau_3\ ,\ I_2 = \tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1 .$$

---

[1] All sign combinations (there are eight)

Nominal body-force density, traction, and stress field

Recall following continuum-mechanical ingredien

Admissible motions:

$$\hat{\underset{\sim}{y}}(\cdot,t) : \mathcal{R} \to \mathcal{R}_t \quad (t_0 \le t \le t_1) \;;\; \mathcal{T} = [t_0, t_1]$$

$\hat{\underset{\sim}{y}} \in C^2(\mathcal{R}\times\mathcal{T})$, $\hat{\underset{\sim}{y}}(\cdot,t)$ is (1,1) on $\mathcal{R}$ $\forall t \in \mathcal{T}$, $J = \det \nabla \hat{\underset{\sim}{y}} > 0$ on $\mathcal{R}$

Conservation of mass:

$$\rho(\underset{\sim}{y},t)\, J(\underset{\sim}{x},t) = \rho_0(\underset{\sim}{x}) \quad \forall\, (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}, \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t),$$

Recall meaning of ρ_0 and $\rho(\cdot,t)$.

Body-force density: $\underset{\sim}{b}(\underset{\sim}{y},t)\ \forall\, \underset{\sim}{y} \in \mathcal{R}_t \quad (t_0 \le t \le t_1)$

Traction field: $\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\ \forall\, (\underset{\sim}{y},\underset{\sim}{n}) \in \mathcal{R}_t \times \mathcal{U} \quad (t_0 \le t \le t_1)$

Principles of linear & angular momentum:

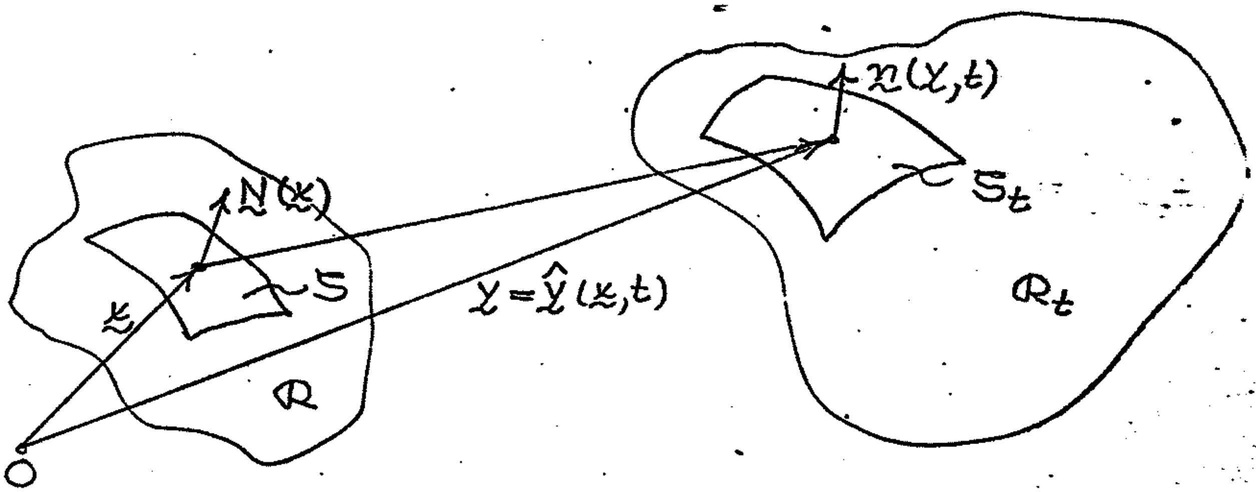
These principles were found to yield

$$\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t) = \underset{\sim}{\tau}(\underset{\sim}{y},t)\,\underset{\sim}{n} \quad \forall\, (\underset{\sim}{y},\underset{\sim}{n}) \in \mathcal{R}_t \times \mathcal{U} \quad (t_0 \le t \le t_1)$$

$$\nabla_{\underset{\sim}{y}} \cdot \underset{\sim}{\tau}(\underset{\sim}{y},t) + \underset{\sim}{b}(\underset{\sim}{y},t) = \rho(\underset{\sim}{y},t)\,\overline{\underset{\sim}{a}}(\underset{\sim}{y},t), \quad \underset{\sim}{\tau}^T(\underset{\sim}{y},t) = \underset{\sim}{\tau}(\underset{\sim}{y},t)$$

$$\forall\, \underset{\sim}{y} \in \mathcal{R}_t \quad (t_0 \le t \le t_1)$$

From now on refer to $\underset{\sim}{b}$, $\underset{\sim}{t}$, and $\underset{\sim}{\tau}$ as the <u>actual</u> body-force density, traction, and stress fields.



Let $S \subset R$ be an oriented regular surface element with the unit normal vector $\underset{\sim}{N}(\underset{\sim}{x})\ \forall\, \underset{\sim}{x} \in S$. Let $S_t = \hat{\underset{\sim}{y}}(S,t)$ $(t_0 \le t \le t_1)$ and call $\underset{\sim}{n}(\underset{\sim}{y},t)$ the unit normal vector at $\underset{\sim}{y} \in R_t$ $(t_0 \le t \le t_1)$. We seek a relation between $\underset{\sim}{N}(\underset{\sim}{x})$ and $\underset{\sim}{n}(\underset{\sim}{y},t)$ if $\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$.

Assume S admits the following parametrization.

$S: \underset{\sim}{x} = \bar{\underset{\sim}{x}}(\xi,\eta)\ \forall\,(\xi,\eta) \in \Sigma,\ \bar{\underset{\sim}{x}} \in C^1(\Sigma),\ \bar{\underset{\sim}{x}}_\xi \wedge \bar{\underset{\sim}{x}}_\eta \neq \underset{\sim}{0}$ on Σ, (1)

Here Σ is a closed region in the (ξ,η)-plane and $\bar{\underset{\sim}{x}}_\xi \equiv \partial\bar{\underset{\sim}{x}}/\partial\xi$, $\bar{\underset{\sim}{x}}_\eta \equiv \partial\bar{\underset{\sim}{x}}/\partial\eta$ are evidently tangent vectors of S. The last of (1) assures that for fixed $(\xi,\eta) \in \Sigma$, i.e. for fixed $\underset{\sim}{x} \in S$, these two tangent vectors are not parallel to each other.

The unit normal vector of S is given by

$$\underset{\sim}{N}(\underset{\sim}{x}) = \frac{\bar{\underset{\sim}{x}}_\xi \wedge \bar{\underset{\sim}{x}}_\eta}{|\bar{\underset{\sim}{x}}_\xi \wedge \bar{\underset{\sim}{x}}_\eta|}\ \ \forall\,(\xi,\eta) \in \Sigma,\ \underset{\sim}{x} = \bar{\underset{\sim}{x}}(\xi,\eta), \text{etc.} \quad (2)$$

From (1),

$$S_t = \hat{\underset{\sim}{y}}(S,t):\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\overline{\underset{\sim}{x}}(\xi,\eta),t) = \overline{\underset{\sim}{y}}(\xi,\eta,t)\ \forall\ (\xi,\eta)\in\Sigma\ (t_0\le t\le t_1) \quad (3)$$

Clearly, $\overline{\underset{\sim}{y}}(\cdot,\cdot,t)\in\mathcal{C}^1(\Sigma)$. Also, (3) & chain rule $\Rightarrow$

$$\frac{\partial\overline{y}_i}{\partial\xi} = \frac{\partial\hat{y}_i}{\partial x_j}\frac{\partial\overline{x}_j}{\partial\xi} = F_{ij}\frac{\partial\overline{x}_j}{\partial\xi}.$$ Thus,

$$\left.\begin{aligned}&\overline{\underset{\sim}{y}}_\xi = \underset{\sim}{E}(\underset{\sim}{x},t)\overline{\underset{\sim}{x}}_\xi,\ \overline{\underset{\sim}{y}}_\eta = \underset{\sim}{E}(\underset{\sim}{x},t)\overline{\underset{\sim}{x}}_\eta\ \forall\ (\xi,\eta)\in\Sigma\ (t_0\le t\le t_1)\\ &\underset{\sim}{x} = \overline{\underset{\sim}{x}}(\xi,\eta),\ \overline{\underset{\sim}{x}}_\xi = \overline{\underset{\sim}{x}}_\xi(\xi,\eta),\ \overline{\underset{\sim}{x}}_\eta = \overline{\underset{\sim}{x}}_\eta(\xi,\eta),\ \underset{\sim}{E} = \nabla\hat{\underset{\sim}{y}}\end{aligned}\right\}\ (4)$$

Evidently, $\overline{\underset{\sim}{y}}_\xi, \overline{\underset{\sim}{y}}_\eta$ are tangent-vectors of S_t. Now note the identity: (see Fr[illegible]) :

$$\underset{\sim}{a}\wedge\underset{\sim}{b} = \frac{1}{J}\underset{\sim}{E}^T\{(\underset{\sim}{E}\underset{\sim}{a})\wedge(\underset{\sim}{E}\underset{\sim}{b})\},\quad J = \det\underset{\sim}{E}, \quad (5)$$

which holds for any pair of vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and for every nonsingular two-tensor $\underset{\sim}{E}$. See proof on back of p

Apply (5) with $\underset{\sim}{a} = \overline{\underset{\sim}{x}}_\xi(\xi,\eta)$, $\underset{\sim}{b} = \overline{\underset{\sim}{x}}_\eta(\xi,\eta)$, $\underset{\sim}{E} = \underset{\sim}{E}(\overline{\underset{\sim}{x}}(\xi,\eta),$ and use (4) to obtain:

$$\left.\begin{aligned}&\overline{\underset{\sim}{x}}_\xi\wedge\overline{\underset{\sim}{x}}_\eta = \frac{1}{J(\underset{\sim}{x},t)}\underset{\sim}{E}^T(\underset{\sim}{x},t)\,\overline{\underset{\sim}{y}}_\xi\wedge\overline{\underset{\sim}{y}}_\eta\ \forall\ (\xi,\eta)\in\Sigma\ (t_0\le t\le t_1),\\ &\underset{\sim}{x} = \overline{\underset{\sim}{x}}(\xi,\eta),\ \overline{\underset{\sim}{x}}_\xi = \overline{\underset{\sim}{x}}_\xi(\xi,\eta),\ \text{etc.}\end{aligned}\right\}\ (6)$$

(6), (1) $\Rightarrow$ $\overline{\underset{\sim}{y}}_\xi\wedge\overline{\underset{\sim}{y}}_\eta \neq 0$ on Σ since $\underset{\sim}{E}$ is nonsingular.

The unit normal vector of S_t is thus given by

$$\underset{\sim}{n}(\underset{\sim}{y},t) = \frac{\bar{\underset{\sim}{y}}_\xi \wedge \bar{\underset{\sim}{y}}_\eta}{|\bar{\underset{\sim}{y}}_\xi \wedge \bar{\underset{\sim}{y}}_\eta|} \quad \forall\, (\xi,\eta)\in\Sigma\,,\ \underset{\sim}{y} = \bar{\underset{\sim}{y}}(\xi,\eta,t);\ \text{etc.} \qquad (7)$$

(6), (7), (2) $\Longrightarrow$

$$\boxed{\begin{gathered} \underset{\sim}{N}(\underset{\sim}{x})\,|\bar{\underset{\sim}{x}}_\xi \wedge \bar{\underset{\sim}{x}}_\eta| = \frac{1}{J(\underset{\sim}{x},t)}\,\underset{\sim}{E}^T(\underset{\sim}{x},t)\,\underset{\sim}{n}(\underset{\sim}{y},t)\,|\bar{\underset{\sim}{y}}_\xi \wedge \bar{\underset{\sim}{y}}_\eta| \quad \forall\,(\xi,\eta)\in\Sigma \\ (t_0 \le t \le t_1) \\ \underset{\sim}{x} = \bar{\underset{\sim}{x}}(\xi,\eta)\,,\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\bar{\underset{\sim}{x}},t) = \bar{\underset{\sim}{y}}(\xi,\eta,t)\,,\ \text{etc.} \end{gathered}} \qquad (i)$$

~~We now digress and linearize (i) for *infinitesimal deformations*. Since $\underset{\sim}{E} = \underset{\sim}{1} + \nabla\underset{\sim}{u}$ on $R\times T$ and $|\nabla\underset{\sim}{u}| \ll 1$, one has with the aid of (4):~~

$$\cancel{\bar{\underset{\sim}{y}}_\xi = (\underset{\sim}{1}+\nabla\underset{\sim}{u})\,\bar{\underset{\sim}{x}}_\xi \approx \bar{\underset{\sim}{x}}_\xi\,,\ \bar{\underset{\sim}{y}}_\eta \approx \bar{\underset{\sim}{x}}_\eta\,,\ J = \det(\underset{\sim}{1}+\nabla\underset{\sim}{u}) \approx 1}$$

~~so that (i) gives~~

$$\boxed{\cancel{\underset{\sim}{N}(\underset{\sim}{x}) \approx \underset{\sim}{n}(\underset{\sim}{y},t) \ \forall\, \underset{\sim}{x}\in S \ (t_0\le t\le t_1),\ \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)}}$$

~~Return to the finite theory and recall from Calculus the following result concerning surface integrals.~~

Postpone! see p. 37

<u>Lemma</u>. Let $\mathcal{S}: \underset{\sim}{x} = \overline{\underset{\sim}{x}}(\xi,\eta)\ \forall\ (\xi,\eta) \in \Sigma$, $\overline{\underset{\sim}{x}} \in C^1(\Sigma)$, and $\overline{\underset{\sim}{x}}_\xi \wedge \overline{\underset{\sim}{x}}_\eta \neq 0$ on Σ, where Σ is a closed region in the (ξ,η)-plane. Suppose $\underset{\sim}{\varphi} \in C(\mathcal{S})$ is vector-valued. Then,

$$\int_{\mathcal{S}} \underset{\sim}{\varphi}(\underset{\sim}{x})\, dA_{\underset{\sim}{x}} = \iint_{\Sigma} \underset{\sim}{\varphi}(\overline{\underset{\sim}{x}}(\xi,\eta))\, |\overline{\underset{\sim}{x}}_\xi(\xi,\eta) \wedge \overline{\underset{\sim}{x}}_\eta(\xi,\eta)|\, d\xi\, d\eta .$$

See, for example, Courant, Volume II, ch. V. The above is true for scalar-valued φ as well.

Let $\underset{\sim}{T}(\cdot,t) \in C^1(R_t)$ be the actual stress field associated with the admiss. motion $\hat{\underset{\sim}{y}}(\cdot,t): R \to R_t$ $(t_0 \le t \le t_1)$. From the above lemma and (i),

$$\int_{\mathcal{S}_t} \underset{\sim}{T}(\underset{\sim}{y},t)\, \underset{\sim}{n}(\underset{\sim}{y},t)\, dA_{\underset{\sim}{y}} = \iint_{\Sigma} \underset{\sim}{T}(\overline{\underset{\sim}{y}},t)\, \underset{\sim}{n}(\overline{\underset{\sim}{y}},t)\, |\overline{\underset{\sim}{y}}_\xi \wedge \overline{\underset{\sim}{y}}_\eta|\, d\xi\, d\eta$$

$$= \iint_{\Sigma} \underset{\sim}{T}(\overline{\underset{\sim}{y}},t)\, J(\overline{\underset{\sim}{x}},t)\, \underset{\sim}{F}^{-T}(\overline{\underset{\sim}{x}},t)\, \underset{\sim}{N}(\overline{\underset{\sim}{x}})\, |\overline{\underset{\sim}{x}}_\xi \wedge \overline{\underset{\sim}{x}}_\eta|\, d\xi\, d\eta =$$

$$= \int_{\mathcal{S}} \underset{\sim}{T}(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t)\, J(\underset{\sim}{x},t)\, \underset{\sim}{F}^{-T}(\underset{\sim}{x},t)\, \underset{\sim}{N}(\underset{\sim}{x})\, dA_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1).$$

Thus,

$$\int_{S_t} \underset{\sim}{\tau}(y,t)\,\underset{\sim}{n}(y,t)\,dA_y = \int_{S} \underbrace{J(x,t)\,\underset{\sim}{\tau}(\hat{y}(x,t),t)\,\underset{\sim}{F}^{-T}(x,t)}_{\underset{\sim}{\sigma}(x,t)}\,\underset{\sim}{N}(x)\,dA_x \;\forall t \in \mathcal{T}$$

Define

$$\left.\begin{aligned} \underset{\sim}{\sigma}(x,t) &\equiv J(x,t)\,\underset{\sim}{\tau}(y,t)\,\underset{\sim}{F}^{-T}(x,t) \;\forall\,(x,t)\in\mathcal{R}\times\mathcal{T},\; y=\hat{y}(x,t) \\ \sigma_{ij}(x,t) &\equiv J(x,t)\,\tau_{ik}(y,t)\,F^{-1}_{jk}(x,t) \;\forall\,(x,t)\in\mathcal{R}\times\mathcal{T},\; y=\hat{y}(x,t) \end{aligned}\right\} (ii$$

Then,

$$\left.\begin{aligned} \underset{\sim}{\tau}(y,t) &= \frac{1}{J(x,t)}\,\underset{\sim}{\sigma}(x,t)\,\underset{\sim}{F}^{T}(x,t) \\ \int_{S_t}\underset{\sim}{\tau}(y,t)\,\underset{\sim}{n}(y,t)\,dA_y &= \int_{S}\underset{\sim}{\sigma}(x,t)\,\underset{\sim}{N}(x)\,dA_x \quad (t_0\le t\le t_1) \end{aligned}\right\} (iii)$$

Note $\underset{\sim}{\sigma}(\cdot,t)\in C^1(\mathcal{R})\;\forall t\in\mathcal{T}$, $\underset{\sim}{\sigma}$ is 2-tensor-valued. Also, $\dim \underset{\sim}{\sigma} = \dim \underset{\sim}{\tau} = [FL^{-2}]$. One calls $\underset{\sim}{\sigma}(\cdot,t)$ the <u>nominal</u> (Piola) <u>stress-tensor field</u> on $\mathcal{R}$. Note that $\underset{\sim}{\sigma}(x,t)$ is in general not symmetric, but that the symmetry of $\underset{\sim}{\tau}(y,t)$ gives

$$\underset{\sim}{\sigma}\,\underset{\sim}{F}^{T} = \underset{\sim}{F}\,\underset{\sim}{\sigma}^{T} \text{ on } \mathcal{R}\times\mathcal{T} \quad (iv)$$

Define the <u>nominal traction field</u> $\underset{\sim}{s}(\cdot,\cdot,t)$ on $\mathcal{R}\times\mathcal{U}\;\forall t\in\mathcal{T}$ by means of

$$\underset{\sim}{s}(x,\underset{\sim}{N},t) = \underset{\sim}{\sigma}(x,t)\,\underset{\sim}{N} \;\forall\,(x,\underset{\sim}{N})\in\mathcal{R}\times\mathcal{U} \quad (t_0\le t\le t_1) \quad (v)$$

(iii) now becomes

$$\int_{S_t} \underset{\sim}{t}\, dA_y = \int_{S} \underset{\sim}{s}\, dA_x \quad (t_0 \le t \le t_1),\ \underset{\sim}{t} = \underset{\sim}{t}(\underset{\sim}{y}, \underset{\sim}{n}, t),\ \underset{\sim}{s} = \underset{\sim}{s}(\underset{\sim}{x}, \underset{\sim}{N}, t) \quad (vi)$$

where $\underset{\sim}{n} = \underset{\sim}{n}(\underset{\sim}{y}, t)$ & $\underset{\sim}{N} = \underset{\sim}{N}(\underset{\sim}{x})$ are the respective unit normal vectors of S_t and of S.

(vi) and the physical significance of the actual traction vector allow us to intepret the nominal traction vector. Explain.

Return to (v) and note that if $\mathfrak{X} = \{0; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \in \mathcal{F}$, one has

$$s_i(\underset{\sim}{x}, \underset{\sim}{e}_j, t) = \sigma_{ik}(\underset{\sim}{x}, t)\, \delta_{kj} = \sigma_{ij}(\underset{\sim}{x}, t) \quad \forall (\underset{\sim}{x}, t) \in R \times \mathcal{T} \quad (vii)$$

Interpret σ_{ij} on the basis of (vii). In contrast,

$$t_i(\underset{\sim}{y}, \underset{\sim}{e}_j, t) = \tau_{ji}(\underset{\sim}{y}, t) = \tau_{ij}(\underset{\sim}{y}, t) = t_j(\underset{\sim}{y}, \underset{\sim}{e}_i, t) \quad \forall\, \underset{\sim}{y} \in R_t \quad (t_0 \le t \le t_1) \quad (viii)$$

Now let $P \subset R$ be an arbitrary bounded regular region. & $P_t = \hat{\underset{\sim}{y}}(P, t)$ $(t_0 \le t \le t)$. Illustrate with a picture.

From the principle of linear momentum we deduced

$$\int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} + \int_{\partial\mathcal{P}_t} \underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\,dA_{\underset{\sim}{y}} = \int_{\mathcal{P}_t} \rho(\underset{\sim}{y},t)\,\underset{\sim}{\bar{a}}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} \quad (t_0 \le t \le t_1) \quad (\S)$$

$$\int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} = \int_{\mathcal{P}} \underbrace{\underset{\sim}{b}(\underset{\sim}{\hat{y}}(\underset{\sim}{x},t),t)\,J(\underset{\sim}{x},t)}_{\underset{\sim}{f}(\underset{\sim}{x},t)}\,dV_{\underset{\sim}{x}}$$

Define

$$\underset{\sim}{f}(\underset{\sim}{x},t) = \underset{\sim}{b}(\underset{\sim}{y},t)J(\underset{\sim}{x},t) \;\; \forall\, (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T},\;\; \underset{\sim}{y} = \underset{\sim}{\hat{y}}(\underset{\sim}{x},t) \quad (ix)$$

$V_{\underset{\sim}{x}}$

Then,

$$\int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} = \int_{\mathcal{P}} \underset{\sim}{f}(\underset{\sim}{x},t)\,dA_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1) \quad (x)$$

One calls $\underset{\sim}{f}(\cdot,t)$ the <u>nominal body-force density</u> field at time t. Explain why. Clearly, $\underset{\sim}{f}(\cdot,t) \in \mathcal{C}(\mathcal{R})\;\forall\, t \in \mathcal{T}$ and $\dim \underset{\sim}{f} = [FL^{-3}]$.

Next, from (vi), (v), and the divergence theorem,

$$\int_{\partial\mathcal{P}_t} \underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t)\,dA_{\underset{\sim}{y}} = \int_{\partial\mathcal{P}} \underset{\sim}{s}(\underset{\sim}{x},\underset{\sim}{N},t)\,dA_{\underset{\sim}{x}} = \int_{\partial\mathcal{P}} \underset{\sim}{\sigma}(\underset{\sim}{x},t)\,\underset{\sim}{N}(\underset{\sim}{x})\,dA_{\underset{\sim}{x}}$$

$$= \int_{\mathcal{P}} \nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{\sigma}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1) \quad (\alpha)$$

Finally, from mass-conservation

$$\int_{\Phi_t} \rho(\underset{\sim}{y},t)\,\bar{\underset{\sim}{a}}(\underset{\sim}{y},t)\,dV_{\underset{\sim}{y}} = \int_{\Phi} \underbrace{\rho(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t)\,J(\underset{\sim}{x},t)}_{\rho_0(\underset{\sim}{x})}\,\underset{\sim}{a}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

$$= \int_{\Phi} \rho_0(\underset{\sim}{x})\,\underset{\sim}{a}(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}} \quad (t_0 \le t \le t_1) \qquad \text{(b)}$$

(§), (*), (a), (b) $\Rightarrow$

$$\int_{\Phi} [\underset{\underset{\sim}{x}}{\nabla}\cdot\underset{\sim}{\sigma}'(\underset{\sim}{x},t) + \underset{\sim}{f}(\underset{\sim}{x},t) - \rho_0(\underset{\sim}{x})\,\underset{\sim}{a}(\underset{\sim}{x},t)]\,dV_{\underset{\sim}{x}} = 0 \quad (t_0 \le t \le t_1),$$

which must hold for <u>every</u> bound regular $\Phi \subset \mathcal{R}$. Since the above integrand is continuous on $\mathcal{R}$ at every $t \in \mathcal{T}$, we arrive at

$$\boxed{\underset{\underset{\sim}{x}}{\nabla}\cdot\underset{\sim}{\sigma}(\underset{\sim}{x},t) + \underset{\sim}{f}(\underset{\sim}{x},t) = \rho_0(\underset{\sim}{x})\,\underset{\sim}{a}(\underset{\sim}{x},t) \quad \forall\ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}} \qquad (xi)$$

(xi) is the equation of motion in terms of <u>nominal</u> stress & body-force density. Compare with the eq. of motion in terms of <u>actual</u> stress & body-force density. Re-emphasize $\underset{\sim}{\tau}^T = \underset{\sim}{\tau}$, but $\underset{\sim}{\sigma}^T \neq \underset{\sim}{\sigma}$.

Linearization for infinitesimal deformations

Assume $|\nabla \underset{\sim}{u}| \ll 1$

Thus,

$$\underset{\sim}{F} = \underset{\sim}{1} + \nabla \underset{\sim}{u} \sim \underset{\sim}{1}, \quad J = \det \underset{\sim}{F} \sim 1 + \nabla \cdot \underset{\sim}{u} \quad \text{on } \mathcal{R}\times\mathcal{T} \quad (*)$$

Recall the equation of mass conservation:

$$J(\underset{\sim}{x},t)\,\rho(\underset{\sim}{y},t) = \rho_0(\underset{\sim}{x}) \quad \forall\, (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}, \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$$

So, now

$$\rho(\underset{\sim}{y},t) \sim \rho_0(\underset{\sim}{x}) \quad \forall\, (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}, \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$$

Next, $(*)$, $(4) \Rightarrow \overline{\underset{\sim}{y}}_\xi \sim \overline{\underset{\sim}{x}}_\xi$, $\overline{\underset{\sim}{y}}_\eta \sim \overline{\underset{\sim}{x}}_\eta$. Hence from $(*)$, (i) follow

$$\underset{\sim}{n}(\underset{\sim}{y},t) \sim \underset{\sim}{N}(\underset{\sim}{x}) \quad \forall\, \underset{\sim}{x} \in \mathcal{S} \quad (t_0 \le t \le t_1), \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$$

Further, from (ii), $(*) \Rightarrow$

$$\boxed{\underset{\sim}{T}(\underset{\sim}{y},t) \sim \underset{\sim}{S}(\underset{\sim}{x},t) \quad \forall\, (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T}, \quad \underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)}$$

Thus the distinction between the actual and the nominal stress tensor disappears for infinitesimal deformations! In particular, now $\underset{\sim}{S} \sim \underset{\sim}{S}^T$ on $\mathcal{R}\times\mathcal{T}$, as is also clear from (IV), $(*)$. Next, from above,

$$\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t) = \underset{\sim}{T}(\underset{\sim}{y},t)\,\underset{\sim}{n}(\underset{\sim}{y},t) \sim \underset{\sim}{S}(\underset{\sim}{x},t)\,\underset{\sim}{N}(\underset{\sim}{x}) = \underset{\sim}{s}(\underset{\sim}{x},\underset{\sim}{N},t) \quad \text{so that}$$

$$\underset{\sim}{t}(\underset{\sim}{y},\underset{\sim}{n},t) \sim \underset{\sim}{s}(\underset{\sim}{x},N,t) \;\forall\, (\underset{\sim}{x},t)\in R\times T,\; \underset{\sim}{y}=\hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$$

$$\underset{\sim}{n}=\underset{\sim}{n}(\underset{\sim}{y},t) \;\forall\, \underset{\sim}{y}\in S_t \;(t_0\le t\le t_1),\; \underset{\sim}{N}=\underset{\sim}{N}(\underset{\sim}{x}) \;\forall\, \underset{\sim}{x}\in S,\; S_t=\hat{\underset{\sim}{y}}(S,t)$$

(ix), (*) $\Rightarrow$

$$\underset{\sim}{b}(\underset{\sim}{y},t)\sim \underset{\sim}{f}(\underset{\sim}{x},t) \;\forall\, (\underset{\sim}{x},t)\in R\times T,\; \underset{\sim}{y}=\hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$$

No distinction between actual and nominal body force density.

From here on all our considerations will be confined to <u>infinitesimal deformations</u>. To simplify notation we henceforth write ρ, $\underset{\sim}{n}$ instead of ρ_0, $\underset{\sim}{N}$ and adopt following terminology:

$\rho(\underset{\sim}{x}) \;\forall\, \underset{\sim}{x}\in R$ ... mass density

$\underset{\sim}{f}(\underset{\sim}{x},t) \;\forall\, (\underset{\sim}{x},t)\in R\times T$ ... body force density

$\underset{\sim}{s}(\underset{\sim}{x},\underset{\sim}{n},t) \;\forall\, (\underset{\sim}{x},\underset{\sim}{n})\in R\times U \;(t_0\le t\le t_1)$ ... traction vector

$\underset{\sim}{\sigma}(\underset{\sim}{x},t) \;\forall\, (\underset{\sim}{x},t)\in R\times T$ ... stress tensor

The <u>equation of motion</u> now reads:

$$\nabla\cdot\underset{\sim}{\sigma}(\underset{\sim}{x},t)+\underset{\sim}{f}(\underset{\sim}{x},t)=\rho(\underset{\sim}{x})\underset{\sim}{a}(\underset{\sim}{x},t),\; \underset{\sim}{\sigma}(\underset{\sim}{x},t)=\underset{\sim}{\sigma}^T(\underset{\sim}{x},t) \;\forall\, (\underset{\sim}{x},t)\in R\times T$$

The <u>traction-stress relation</u> is:

$$\underset{\sim}{s}(\underset{\sim}{x},\underset{\sim}{n},t)=\underset{\sim}{\sigma}(\underset{\sim}{x},t)\underset{\sim}{n} \;\forall\, (\underset{\sim}{x},\underset{\sim}{n},t)\in R\times U\times T$$