Bending of plates

1. Introduction

A plate is a two-dimensional structural element, i.e., one of the dimensions (the plate thickness h) is small compared to the in-plane dimensions a and b. The load on the plate is applied perpendicular to the center plane of the plate. In plate theory, one generally distinguishes the following cases:

1. Thick plates with a three-dimensional stress state. These can only be described by the full set of differential equations we derived in Chapter 1. As a rule of thumb, plates with $b/h < 5$ and $a > b$ fall in this category.

2. Thin plates with small deflections. In this case, the membrane stresses generated by the deflection are small compared to the bending stresses and this simplifies the analysis considerably. As a rule of thumb, plates with $b/h > 5$ and $w < h/5$ fall in this category. These are the plates we will study here.

3. Thin plates with large deflections. In this case, the membrane stresses generated by the deflection are significant compared to the bending stresses and the plate behaves nonlinearly. As a rule of thumb, plates with $b/h > 5$ and $w > h/5$ fall in this category. We will not address plates with large deflections here.

2. Elastic theory of thin isotropic plates with small deflections

1. Basic assumptions

We make the following assumptions in our analysis.

1. The plate material is linear elastic and follows Hooke's law
2. The plate material is homogeneous and isotropic. Its elastic deformation is characterized by Young's modulus E and Poisson's ratio ν.
3. The thickness of the plate is small compared to its lateral dimensions. The normal stress in the transverse direction can be neglected compared to the normal stresses in the plane of the plate.

4. *Points that lie on a line perpendicular to the center plane of the plate remain on a straight line perpendicular to the center plane after deformation.*
5. The deflection *w* of the plate is small compared to the plate thickness. The curvature of the plate after deformation can then be approximated by the second derivative of the deflection *w*.
6. The center plane of the plate is stress free, i.e., we can neglect the membrane stresses in the plate.
7. Loads are applied in a direction perpendicular to the center plane of the plate.

These assumptions are known as the Love-Kirchhoff hypotheses and they allow us to reduce the elasticity equations to one differential equation describing the plate-bending problem

2. Stresses and strains

Assumption 4 makes it possible to derive a simple relationship between the deflection $w(x,y,0)$ of the center plane of the plate and the displacements $u(x,y,z)$ and $v(x,y,z)$. Indeed we find that

$$u = -z\frac{\partial w}{\partial x},$$

$$v = -z\frac{\partial w}{\partial y}.$$

From the definitions of the strain components we find then that

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2},$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2},$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}.$$

In other words, the strain components vary linearly through the plate thickness and are zero on the center plane. From Hooke's law

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right),$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right),$$

$$\gamma_{xy} = \frac{2\left(1+\nu\right)}{E}\sigma_{xy},$$

we get the stress components

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right),$$

$$\sigma_{xy} = \frac{E}{2\left(1+\nu\right)}\varepsilon_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}.$$

Like the strain components, the stress components vary linearly through the plate thickness. From these equations, it is obvious that if we know the deflection of the plate, we can calculate most relevant stress and strain components. Note that we haven't mentioned the vertical shear stresses yet. These are important as they will ensure vertical equilibrium of the plate, but we cannot yet calculate them at this point.

3. Resultant forces and moments

Our analysis can be further simplified if we replace the stresses with the corresponding resultant forces. In particular, we define the following quantities:

$$m_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z dz \qquad m_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z dz,$$

$$m_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \qquad m_{yx} = \int_{-h/2}^{h/2} \sigma_{yx} z dz,$$

$$q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz \qquad q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz.$$

It's clear from the definitions that m_{xx} and m_{yy} are bending moments per unit length, parallel to the x and y-axes respectively; m_{xy} and m_{yx} are twisting moments per unit length; q_x and q_y are transverse forces per unit length. We have taken these quantities to be positive if they cause a positive stress at a point with positive z-coordinate.

Since we have expressions for the σ_{xx}, σ_{yy}, and σ_{xy} in terms of the deflection of the plate, we can write

$$m_{xx} = -\frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\int_{-h/2}^{h/2} z^2 dz = -\frac{EI}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

where I is the moment of inertia per unit length of the plate. It is usually more convenient to use the bending stiffness of the plate

$$k = \frac{EI}{1-\nu^2} = \frac{Eh^3}{12\left(1-\nu^2\right)}.$$

With this notation we get

$$m_{xx} = -K\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

$$m_{yy} = -K\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right),$$

$$m_{xy} = m_{yx} = -\left(1-\nu\right)K\frac{\partial^2 w}{\partial x \partial y}.$$

We cannot calculate the transverse forces, because, as I pointed out earlier, we do not have expressions for the vertical shear stresses in terms of the plate deflection. We can say something about these transverse forces, though, if we consider the rotational equilibrium of an infinitesimal plate segment h dx dy. Consider the condition for rotational equilibrium around the y-axis:

$$\frac{\partial m_{xx}}{\partial x} dxdy + \frac{\partial m_{xy}}{\partial y} dxdy - q_x dydx + \frac{1}{2} pdxdydx = 0,$$

or

$$q_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y}.$$

From the condition for rotational equilibrium around the *x*-axis, we find similarly:

$$q_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x}.$$

If we now substitute the expressions for the bending and twisting moments, we find

$$\begin{aligned} q_x &= -K\left[\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2} - \nu\frac{\partial^3 w}{\partial x \partial y^2}\right] \\ &= -K\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] = -K\frac{\partial \Delta w}{\partial x} \end{aligned}.$$

on the condition that the plate bending stiffness is constant. Similarly

$$q_y = -K\frac{\partial \Delta w}{\partial y}.$$

4. The governing differential equation

Thus far, we have not yet required vertical equilibrium. Again considering an infinitesimal plate segment *h* d*x* d*y*, we find

$$q_x dy - \left(q_x + \frac{\partial q_x}{\partial x}dx\right)dy + q_y dx - \left(q_y + \frac{\partial q_y}{\partial y}dy\right)dx - pdxdy = 0,$$

which leads to

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p,$$

after simplification. One final step remains; we need to fill out the expressions for the transverse forces in terms of the plate deflection. This operation easily leads to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w = \frac{p}{K}.$$

Solving this equation gives us the deflection of the plate; once we know the deflection of the plate we can calculate the bending and twisting moments and the stress distributions. In order to solve this equation, however, we need to know the boundary conditions. In the next section we focus on deriving the proper boundary conditions.

4. The boundary conditions

Consider a point on the plate boundary given by $x = a$. At this point we can calculate the three quantities m_{xx}, m_{xy}, and q_x. These quantities must be in equilibrium with the externally applied bending moment, twisting moment and transverse force. But now we run into a problem: the differential equation is a fourth order equation in two variables, i.e., we have 8 integration constants, or two integration constants for each edge of the plate. In other words, we can satisfy only two conditions. The reason for this discrepancy is that we neglected the deformation induced by the vertical shear stresses when we derived our theory. Kirchhoff came up with a solution of the discrepancy through the introduction of Ersatz-transverse forces. These Ersatz forces are statically equivalent to twisting moment applied to the edge of the plate. Let's derive an expression for these Ersatz forces:

Consider three points spaced a distance dy apart on the plate edge given by $x = a$. At these three points we have twisting moments

$$\left(m_{xy} - \frac{\partial m_{xy}}{\partial y}dy\right)dy,\ \ m_{xy}dy,\ \ \left(m_{xy} + \frac{\partial m_{xy}}{\partial y}dy\right)dy\ ,$$

respectively. At each of these three points, we can replace these distributed twisting moments by a couple of forces with lever arm dy

$$m_{xy} - \frac{\partial m_{xy}}{\partial y}dy,\ \ m_{xy},\ \ m_{xy} + \frac{\partial m_{xy}}{\partial y}dy\ .$$

We see that the forces m_{xy} cancel each other – only the force

$$\frac{\partial m_{xy}}{\partial y}dy$$

remains. If we now distribute this force uniformly along the edge of the plate, we have obtained a distributed transverse force caused by the twisting moments. We have replaced the distributed twisting moment by a distributed transverse force – hence the name Ersatz force. The expressions for the transverse forces including the Ersatz forces as a function of the plate deflection are then:

$$\bar{q}_x = q_x + \frac{\partial m_{xy}}{\partial y} = -K\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] - (1-\nu)K\frac{\partial^3 w}{\partial x \partial y^2}$$
$$= -K\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial y^2}\right]$$

and

$$\bar{q}_y = q_y + \frac{\partial m_{xy}}{\partial x} = -K\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2} + (2-\nu)\frac{\partial^2 w}{\partial x^2}\right].$$

If external transverse forces and twisting moments are applied to the edge of the plate, they also need to be reduced an equivalent distributed transverse force taking into account the Ersatz forces generated by the twisting moments.

We are now in a position to write down the boundary conditions for the most common plate edge conditions.

1. The edge x= a is free and not supported

$$m_{xx} = 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0,$$

$$\bar{q}_x = 0 \quad \text{or} \quad \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial y^2}\right] = 0.$$

2. The edge x= a is free and simply supported

$$w = 0,$$

$$m_{xx} = 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0.$$

3. The edge x= a is clamped

$$w = 0,$$

$$\frac{\partial w}{\partial x} = 0 .$$

Along the edge, we also have

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x \partial y} = 0 ,$$

so that there is no twisting moment along the edge, i.e., the transverse forces are equal to the reaction forces (no Ersatz force!).

4. The corner of a plate

Consider a rectangular plate. At the corner of the plate we have that $\sigma_{xy} = \sigma_{yx}$, or $m_{xy} = m_{yx}$. If we convert the twisting moment to Ersatz forces, there remains a concentrated force in the corner:

$$R = m_{xy} + m_{yx} = 2m_{xy} .$$

So, in cases such as the simply supported plate where we have a twisting moment along the edge, there is a concentrated force at the corners of the plate. This force will tend to turn up the corners of the plate, unless the support can pull the corners down

5. Solution of the plate equation

The general solution of the plate equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = \Delta\Delta w = \frac{p}{K} ,$$

consists of a particular solution w_p and a general solution w_h of the homogeneous equation $\Delta\Delta w = 0$. This homogeneous equation is identical to the biharmonic equation we have encountered in plane-stress and plane-strain problems, so any solutions we have encountered in that context also applies here. We now solve the problem of a simply supported rectangular plate subjected to a uniform pressure distribution.

The general solution has the form

$$w = w_p + w_h .$$

A particular solution is easily found in this case:

$$w_p = \frac{p}{24K}\left(x^4 - 2ax^3 + a^3 x\right).$$

This function clearly satisfies the boundary conditions at the edges $x = 0$ and $x = a$:

$$w = 0,$$

$$\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x^2} = 0.$$

We now seek the homogeneous solution in the following form

$$w_h = \sum_{n=1,3,5}^{\infty} Y_n(y)\sin\left(\frac{n\pi x}{a}\right).$$

This function also satisfies the boundary conditions at the edges $x = 0$ and $x = a$. We now need to determine the functions Y_n such that w_h satisfies the biharmonic equation and the boundary conditions on the two other edges of the plate. If we substitute the expression for w_h into the biharmonic equation, we find that

$$Y_n^{(IV)} - \frac{2n^2\pi^2}{a^2}Y_n^{(II)} + \frac{n^4\pi^4}{a^4}Y_n = 0.$$

The general solution of this equation is

$$Y_n(y) = \frac{pa^4}{K}\left(A_n\cosh\frac{n\pi y}{a} + B_n\frac{n\pi y}{a}\sinh\frac{n\pi y}{a} + C_n\sinh\frac{n\pi y}{a} + D_n\frac{n\pi y}{a}\cosh\frac{n\pi y}{a}\right).$$

If we assume the plate is symmetric around the x-axis, it follows immediately that C_n and D_n are zero. The solution is then

$$w = \frac{p}{24K}\left(x^4 - 2ax^3 + a^3 x\right) + \frac{pa^4}{K}\sum_{n=1,3,5}^{\infty}\left(A_n\cosh\frac{n\pi y}{a} + B_n\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a},$$

where we still need to determine two integration constants. The easiest way to do this, is to develop the develop the particular solution in a Fourier series and absorb it into the Fourier series for the homogeneous solution. One can show that

$$w_p = \frac{p}{24K}\left(x^4 - 2ax^3 + a^3 x\right) = \frac{4pa^4}{\pi^5 K}\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a},$$

so that the expression for the deflection of the plate becomes

$$w = \frac{pa^4}{K} \sum_{n=1,3,5}^{\infty} \left(\frac{4}{\pi^5 n^5} + A_n \cosh \frac{n\pi y}{a} + B_n \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a}.$$

We now need to satisfy the boundary conditions at the edges $y = \pm b / 2$:

$$w = 0,$$

$$\frac{\partial^2 w}{\partial y^2} = 0.$$

These conditions lead to

$$\frac{4}{\pi^5 n^5} + A_n \cosh \alpha_n + \alpha_n B_n \sinh \alpha_n = 0,$$
$$\left(A_n + 2B_n \right) \cosh \alpha_n + \alpha_n B_n \sinh \alpha_n = 0,$$

where

$$\alpha_n = \frac{n\pi b}{2a}.$$

Solving these two equations eventually gives us

$$A_n = -\frac{2\left(\alpha_n \tanh \alpha_n + 2 \right)}{\pi^5 n^5 \cosh \alpha_n},$$
$$B_n = \frac{2}{\pi^5 n^5 \cosh \alpha_n}.$$

The deflection of the plate is then given by

$$w = \frac{4pa^4}{\pi^5 K} \sum_{n=1,3,5}^{\infty} \frac{1}{n^5} \left(1 - \frac{\alpha_n \tanh \alpha_n + 2}{2\cosh \alpha_n} \cosh \frac{n\pi y}{a} + \frac{n\pi y}{2a \cosh \alpha_n} \sinh \frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a}.$$

All other quantities of interest such as maximum deflection, stresses, reaction forces, etc., can be calculated from this formula using the expressions derived earlier. Can you derive an expression for the concentrated reaction forces in the corners of the plate and show how these reaction forces vary with plate aspect ratio.

6. Circular plates

6.1 Differential equation in polar coordinates

If the plate is axisymmetric, it makes sense to introduce polar coordinates. We have seen before that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2},$$

so that the plate equation in polar coordinates is given by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right)w = \frac{p}{K}.$$

The displacements u and v are as before:

$$u = -z\frac{\partial w}{\partial r} \quad \text{and} \quad v = -z\frac{1}{r}\frac{\partial w}{\partial \vartheta}.$$

If we substitute these expressions into the definitions of the strain components in polar coordinates, apply Hooke's law and integrate, we find the following expressions for the resultant bending and twisting moments:

$$m_{rr} = \int_{-h/2}^{h/2} \sigma_{rr} z dz = -K\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right]$$

$$m_{\vartheta\vartheta} = \int_{-h/2}^{h/2} \sigma_{\vartheta\vartheta} z dz = -K\left[\nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$m_{r\vartheta} = \int_{-h/2}^{h/2} \sigma_{r\vartheta} z dz = -(1-\nu)K\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \vartheta}\right)$$

The transverse forces q_{rr} and $q_{\theta\theta}$ can be derived from the expressions for q_{xx} and q_{yy}, if we replace x and y with the axis r and the axis t perpendicular to r, with $dt = r\,d\theta$:

$$q_{rr} = -K\frac{\partial \Delta w}{\partial r}; \quad q_{\vartheta\vartheta} = -K\frac{1}{r}\frac{\partial \Delta w}{\partial \vartheta}.$$

6.2 The axisymmetric case

If both the load on a circular plate and its support are axisymmetric, the deflection of the plate is independent of θ. This allows us to simplify the general formulas considerably:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)w=\frac{p}{K}.$$

The solution of the homogeneous part of this equation was derived in the chapter on plane elasticity problems. The general solution of the axisymmetric plate equation is then

$$w=w_p+c_1+c_2r^2+c_3r^2\ln r+c_4\ln r,$$

where w_p is a particular solution of the heterogeneous equation. If the applied pressure p is constant, w_p is readily determined. For instance:

Let $\Delta w=M$, then the plate equation becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)M=\frac{p}{K},$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}M=\frac{p}{K}.$$

Integrating this expression, we find that

$$M=\frac{pr^2}{4K}=\Delta w.$$

Integrating the right hand side of this equation in a similar fashion, we finally get

$$w_p=\frac{pr^4}{64K}.$$

The full solution for a uniformly loaded circular plate is given by

$$w=\frac{pr^4}{64K}+c_1+c_2r^2+c_3r^2\ln r+c_4\ln r,$$

where the integration constants are determined from the boundary conditions.

6.3 Solution for a simply supported circular plate with a uniform load

As a simple application of the axisymmetric case, let's consider a simply supported circular plate of radius a. The boundary conditions are:

$$w = 0 \qquad \text{for } r = a$$

$$m_{rr} = 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial r^2} + \nu \frac{1}{r}\frac{\partial w}{\partial r} = 0 \qquad \text{for } r = a$$

At the center of the plate, the deflection and the curvature of the plate must remain finite. This implies that the logarithmic terms in the general solution must be zero, i.e., $c_3 = c_4 = 0$. Substituting the expression for w into the boundary conditions gives

$$\frac{pa^4}{64K} + c_1 + c_2 a^2 = 0,$$
$$\frac{3pa^2}{16K} + 2c_2 + \nu\left(\frac{pa^2}{16K} + 2c_2\right) = 0.$$

Solving these equations finally yields:

$$c_1 = \frac{5+\nu}{1+\nu}\frac{pa^4}{64K},$$
$$c_2 = -\frac{3+\nu}{1+\nu}\frac{pa^2}{32K} = 0.$$

The deflection of a uniformly loaded circular plate that is simply supported is then

$$w = \frac{pa^4}{64K}\left(\left(\frac{r}{a}\right)^4 - 2\frac{3+\nu}{1+\nu}\left(\frac{r}{a}\right)^2 + \frac{5+\nu}{1+\nu}\right)$$

The bending moments can be readily determined from this expression and the equations given on page 11. The maximum bending moment occurs at the center of the plate and takes the value

$$\left(m_{rr}\right)_{\max} = \left(3+\nu\right)\frac{pa^2}{16}.$$

Note that for circular plates with a hole in the center, there is no reason why the logarithmic terms in the general expression of the deflection should be zero. The additional integration constants in this case are determined from the boundary condition on the edge of the hole.