8. Stress functions and displacement potentials

Introduction. Here we aim at general (or complete) solutions of (a) the stress eqs. of equilibrium, (b) the displacement equations of equil. or motion in the linear theory of homog. & isotropic elastic solids. Topic (a) belongs to general equil. theory of cont. media and the results discussed under (a) are not confined in their applicability to infinitesimal deformations or elastic solids.

General solutions of the stress equations of equilibrium

Recall from Thm. 3.3 that the exact eqs. of motion in the equil. case reduce to

$$\nabla\cdot\underset{\sim}{\sigma}+\underset{\sim}{f}=\underset{\sim}{0},\ \underset{\sim}{\sigma}=\underset{\sim}{\sigma}^T \text{ or } \sigma_{ij,j}+f_i=0,\ \sigma_{ij}=\sigma_{ji}, \tag{8.1}$$

if $\underset{\sim}{\sigma}$ is the stress-tensor field regarded as a function of the Eulerian coordinates on the equil. configuration of the body. Also, according to (3.43), eqs. (8.1) are the equil eqs. in the presence of infinitesimal deformation if $\underset{\sim}{\sigma}$ is interpreted as the stress field referred to Lagrangian coordinates on the reference configuration.

We assume first that the body forces are absent and later show that this entails no essential loss of generality. Thus consider

$$\nabla \cdot \underset{\sim}{\sigma} = \underset{\sim}{0}\,,\quad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \quad \text{or} \quad \sigma_{ij,j} = 0,\ \sigma_{ij} = \sigma_{ji}. \qquad (8.2$$

<u>Thm</u>. 8.1 (Beltrami's solution). Let R be an open region and $\underset{\sim}{\phi} \in \mathcal{C}^3(R)$ a two-tensor field with $\underset{\sim}{\phi} = \underset{\sim}{\phi}^T$ on R. Let

$$\underset{\sim}{\sigma} = \nabla \wedge \nabla \wedge \underset{\sim}{\phi} \quad \text{or} \quad \sigma_{ij} = \varepsilon_{ipm}\varepsilon_{jqn}\phi_{mn,pq} \ \text{on } R \qquad (8.$$

Then $\underset{\sim}{\sigma} \in \mathcal{C}^1(R)$ and $\underset{\sim}{\sigma}$ satisfies (8.2) on R.

<u>Proof.</u> Recall from (1.46) that if $\underset{\sim}{W} \in \mathcal{C}^1(R)$ is a two-tensor field,

$$\underset{\sim}{U} = \nabla \wedge \underset{\sim}{W} \Leftrightarrow U_{ij} = \varepsilon_{ipq} W_{jq,p}\,, \qquad (1).$$

which confirms the equivalence asserted in (8.3). Clearly, $\underset{\sim}{\sigma} \in \mathcal{C}^1(R)$.

Recall from (1.47) (Exercise 7) that if $\underset{\sim}{W} \in C^2(R)$,

$$\left.\begin{aligned} &\nabla\cdot\nabla\wedge\underset{\sim}{W} = \nabla\wedge\nabla\cdot\underset{\sim}{W}^T,\ \nabla\cdot(\nabla\wedge\underset{\sim}{W})^T = \underset{\sim}{0}, \\ &(\nabla\wedge\nabla\wedge\underset{\sim}{W})^T = \nabla\wedge\nabla\wedge\underset{\sim}{W}^T \end{aligned}\right\} \quad (2)$$

(8.3) & first two of (2) $\Rightarrow$

$$\nabla\cdot\underset{\sim}{\sigma} = \nabla\cdot(\nabla\wedge\nabla\wedge\underset{\sim}{\Phi}) = \nabla\wedge\nabla\cdot(\nabla\wedge\underset{\sim}{\Phi})^T = \underset{\sim}{0} \text{ on } R$$

Also, from (8.3), the last of (1), & symmetry of $\underset{\sim}{\Phi}$,

$$\underset{\sim}{\sigma}^T = (\nabla\wedge\nabla\wedge\underset{\sim}{\Phi})^T = \nabla\wedge\nabla\wedge\underset{\sim}{\Phi}^T = \nabla\wedge\nabla\wedge\underset{\sim}{\Phi} = \underset{\sim}{\sigma}. \text{ qed}$$

$\underset{\sim}{\Phi}$ .. Beltrami's stress function (stress potential).

<u>Remarks.</u>

(1) Thm. 8.1 enables one to generate an infinity of solns. to (8.2). Useful in constructing statically admissible stress fields.

(2) The analogue of Thm. 8.1 in vector analysis is: $\underset{\sim}{w} \in C^2(R)$, $\underset{\sim}{v} = \nabla\wedge\underset{\sim}{w}$ on $R \Rightarrow \nabla\cdot\underset{\sim}{v} = \underset{\sim}{0}$ on R.

Raise two questions:

(a) Completeness of Beltrami's solution. (Existence issue, vital for applications)

(b) Uniqueness of generating Φ (non-uniqueness would suggest possibility of simplification of solution, less important issue.)

Before dealing with above questions note following special cases of Beltrami's solution.

(i) <u>Airy's solution</u>.

$$[\Phi] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \phi \end{bmatrix} \Leftrightarrow \Phi_{mn} = \delta_{3m}\delta_{3n}\phi$$

(8.3) reduces to:

$$\sigma_{11} = \phi_{,22}\ ,\ \sigma_{22} = \phi_{,11}\ ,\ \sigma_{12} = -\phi_{,12}\ ,\ \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$

(ii) <u>Maxwell's solution</u>

$$[\Phi] = \begin{bmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{bmatrix}$$

(8.3) reduces to:

$$\sigma_{11} = \chi_{2,33} + \chi_{3,22}\ ,\ \sigma_{12} = -\chi_{3,12}\ ,\ \text{etc.}$$

(iii) <u>Morera's solution</u>

$$[\Phi] = \begin{bmatrix} 0 & \Psi_3 & \Psi_2 \\ \Psi_3 & 0 & \Psi_1 \\ \Psi_2 & \Psi_1 & 0 \end{bmatrix}$$

(8.3) reduces to:

$$\sigma_{11} = -2\Psi_{1,23}\,,\quad \sigma_{12} = (\Psi_{1,1}+\Psi_{2,2}-\Psi_{3,3})_{,3}\,,\ \text{etc.}$$

<u>Thm.</u> 8.2 (Stokes). Let R be an open region and $\underset{\sim}{v} \in \mathcal{C}^1($ a vector field on R. Then, given any closed regular surface $S \subset R$,

$$\int_S (\nabla \wedge \underset{\sim}{v}) \cdot \underset{\sim}{n}\, dA = 0 \quad \text{or} \quad \int_S \varepsilon_{ijk} v_{k,j} n_i \, dA = 0 \qquad (*)$$

For a proof see, e.g., Courant II, Chapter 5.

Note that (*) follows from divergence thm. if $\underset{\sim}{v} \in \mathcal{C}^2(R$ and S contains only pts. of R in its interior. Explain

<u>Def.</u> Let R be an open region.

(a) $\underset{\sim}{\sigma}$ is an <u>equilibrated stress field</u> on R if $\underset{\sim}{\sigma} \in \mathcal{C}^1(R)$ and (8.2) hold.

(b) $\underset{\sim}{\sigma}$ is a <u>self-equilibrated stress field</u> on R if $\underset{\sim}{\sigma} \in \mathcal{C}^1(R$

if for every closed regular surface $S \subset R$,

$$\int_S \underset{\sim}{s}\, dA = \underset{\sim}{0}\ , \quad \int_S \underset{\sim}{x} \wedge \underset{\sim}{s}\, dA = \underset{\sim}{0} \qquad (**)$$

where $\underset{\sim}{s} = \underset{\sim}{\sigma}\, \underset{\sim}{n}$ on S and $\underset{\sim}{n}$ is the unit outer normal of

Remark.
Clearly ('div. thm) $\underset{\sim}{\sigma}$ self-equil. on $R \Rightarrow \underset{\sim}{\sigma}$ equil. on R.
But the converse is not true: explain counter examples

Thm. 8.3 (Incompleteness of Beltrami's solution). Under the hypotheses of Thm. 8.1, $\underset{\sim}{\sigma}$ is necessarily self-equilibrated on R.

Exercise 25. Use Thm. 8.2 to prove Thm. 8.3

Remark. One can show that Beltrami's solution of (8.2) is complete for sufficiently smooth self-equil. stress fields if R is a bounded regular region and ∂R is suitably smooth. See Gurtin (Encyclopedia)

Thm. 8.4 (Uniqueness of generating Beltrami stress function). Let the hypotheses of Thm. 8.1 hold and suppose $\underset{\sim}{v} \in C^4(R)$ is a vector field. Define

$$\underset{\sim}{\phi}' = \underset{\sim}{\phi} + \underset{\sim}{\varphi}\ , \quad \underset{\sim}{\varphi} = \text{sym}\, \nabla \underset{\sim}{v} \text{ on } R$$

Then $\underset{\sim}{\phi}' \in C^3(R)$ and

$$\underset{\sim}{\sigma} = \nabla \wedge \nabla \wedge \underset{\sim}{\phi}' \text{ on } R$$

Proof. Immediate from proof of Thm 2.13 (compatibility): identify $\underset{\sim}{v}$ with $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ with $\underset{\sim}{\gamma}$. State "converse" for simply connected R.

Thm. 8.5 (Generalized Beltrami solution). Let R be an open region. Let $\underset{\sim}{\phi} \in C^3(R)$ be a two-tensor field with $\underset{\sim}{\phi} = \underset{\sim}{\phi}^T$ and let $\underset{\sim}{h}$ be a harmonic vector field on R. Let

$$\underset{\sim}{\sigma} = \nabla \wedge \nabla \wedge \underset{\sim}{\phi} + 2\,\text{sym}\, \nabla \underset{\sim}{h} - \underset{\sim}{1}\, \nabla \cdot \underset{\sim}{h} \text{ on } R \qquad (8.4)$$

Then, $\underset{\sim}{\sigma} \in C^1(R)$ and $\underset{\sim}{\sigma}$ satisfies (8.2) on R.

Proof. By hyp. $\underset{\sim}{h} \in C^2(R)$, $\nabla^2 \underset{\sim}{h} = \underset{\sim}{0}$ on R, which impl $\underset{\sim}{h} \in C^\infty(R)$. Hence $\underset{\sim}{\sigma} \in C^1(R)$.

Set

$$\underset{\sim}{\Psi} = 2\,\mathrm{sym}\,\nabla\underset{\sim}{h} - \underset{\sim}{1}\,\nabla\cdot\underset{\sim}{h} \Leftrightarrow \Psi_{ij} = h_{i,j} + h_{j,i} - \delta_{ij} h_{k,k} \text{ on } R$$

In view of Thm. 8.1 it suffices to show that $\underset{\sim}{\Psi} = \underset{\sim}{\Psi}^T$ (obv.) and $\nabla\cdot\underset{\sim}{\Psi} = \underset{\sim}{0}$ on R. But,

$$\nabla\cdot\underset{\sim}{\Psi} = \Psi_{ij,j} = h_{i,jj} + h_{j,ij} - h_{k,ki} = h_{i,jj} + \cancel{h_{k,ki}} - \cancel{h_{k,ki}} =$$

since $\nabla^2 h_i = 0$ on R. qed.

In preparation for a proof of the completeness of the generalized Beltrami solution we cite the following theorem on Newtonian potentials, which will be essent for other purposes as well.

<u>Thm.</u> 8.6 (Properties of Newtonian potentials). Let R be a bounded regular region. Suppose $F \in \mathcal{C}^N(R) \cap \mathcal{C}(\overline{R})$ $(N \geq 1)$ be a scalar field and define

$$U(\underset{\sim}{x}) = -\frac{1}{4\pi} \int_R \frac{F(\underset{\sim}{\xi})}{|\underset{\sim}{x} - \underset{\sim}{\xi}|}\, dV_{\underset{\sim}{\xi}} \quad \forall\, \underset{\sim}{x} \in E. \tag{8.5}$$

Then $U \in \mathcal{C}^1(E) \cap \mathcal{C}^{N+1}(R) \cap \mathcal{C}^\infty(E - \overline{R})$ and

$$\nabla^2 U = F \text{ on } R, \quad \nabla^2 U = 0 \text{ on } E - \overline{R}. \tag{8.6}$$

A rigorous proof valid for $N=1$ and extendable to $N>1$ is given in Kellogg's book. See also Jeffrey & Jeffreys.

Remarks

Relate (8.5) to Newtonian pot. of a single mass-p[illegible] located at $\underset{\sim}{x}^{*}$. Note that the integral in (8.5) is improper for $\underset{\sim}{x} \in \bar{R}$ but not for $\underset{\sim}{x} \in E-\bar{R}$. Explain why second of (8.6) is trivial. Observe that while $U_{,i}$ is cont. across ∂R, the same cannot be true of all second derivatives. Clearly, conclusions hold if F is [illegible] tensor field of arbitrary order.

Thm. 8.7 (Completeness of generalized Beltrami sol.) Let R be a bounded regular region. Let $\underset{\sim}{\sigma} \in \mathcal{C}^2(R) \cap$ be a symmetric two-tensor field $\ni$

$$\nabla \cdot \underset{\sim}{\sigma} = \underset{\sim}{0} \text{ on } R. \tag{*}$$

Then $\exists$ a symmetric two-tensor field $\underset{\sim}{\phi} \in \mathcal{C}^3(R)$ an[d] a harmonic vector field $\underset{\sim}{h}$ on $R \ni$ (8.4) holds true.

<u>Proof</u>. Define a two-tensor field χ by means of

$$\chi(x) = \frac{1}{4\pi}\int_R \frac{\sigma(\xi)}{|x-\xi|}\, dV_\xi \quad \forall\ x \in \bar{R} \tag{1}$$

(1), Thm. 8.6 $\Rightarrow$ χ is a two-tensor field $\ni$

$$\chi \in C^3(R) \cap C^1(R),\ \chi = \chi^T,\ \sigma = -\nabla^2\chi \text{ on } R \tag{2}$$

One easily proves that $\forall$ symm. two-tensor field $\chi \in C^2$

$$-\nabla^2\chi = \nabla\wedge\nabla\wedge\underbrace{(\chi - \mathbf{1}\,\mathrm{tr}\,\chi)}_{\phi} - 2\,\mathrm{sym}\,\nabla\underbrace{\nabla\cdot\chi}_{-h} + \mathbf{1}\,\nabla\cdot\nabla\cdot\chi$$

Define

$$\phi = \chi - \mathbf{1}\,\mathrm{tr}\,\chi,\quad h = -\nabla\cdot\chi \text{ on } R \tag{4}$$

(2), (4) $\Rightarrow$

$$\phi \in C^3(R),\ \phi = \phi^T,\ h \in C^2(R) \tag{5}$$

(3), (4) $\Rightarrow$

$$\sigma = \nabla\wedge\nabla\wedge\phi + 2\,\mathrm{sym}\,\nabla h - \mathbf{1}\,\nabla\cdot h \text{ on } R \tag{6}$$

which is (8.4). We have yet to show that $\nabla^2 h = 0$ on R. To this end note from (6), Thm. 8.1, (*) that

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[1] Note that (3) is a tensorial analogue of

$$-\nabla^2 v = \nabla\wedge\nabla\wedge v - \nabla\nabla\cdot v$$

$$\nabla\cdot\underset{\sim}{\sigma} = 2\nabla\cdot(\mathrm{sym}\,\nabla\underset{\sim}{h}) - \nabla\cdot(\underset{\sim}{1}\,\nabla\cdot\underset{\sim}{h}) = \underset{\sim}{0} \text{ on } R \text{ or}$$

$$(h_{i,j}+h_{j,i})_{,j} - (\delta_{ij}\,h_{k,k})_{,j} = \nabla^2 h_i + \cancel{h_{k,ki}} - \cancel{h_{k,ki}} = 0 \text{ on } R$$

$$\text{or } \nabla^2\underset{\sim}{h} = \underset{\sim}{0} \text{ on } R. \qquad \text{qed.}$$

Remarks.
Above proof due to Gurtin (Encyclopedia).
In view of Thm. 8.3, one has as an immediate corollary of Thm. 8.7: Every $\underset{\sim}{\sigma}$ satisfying the hyp. in Thm. 8.7 admits a resolution into the sum of a self-equil. and a harmonic stress field.

Thm. 8.8 (Particular solution of (8.1)). Let R be an open region and $\underset{\sim}{\varphi} \in C^2(R)$ be a vector field. Let

$$\underset{\sim}{\sigma} = 2\,\mathrm{sym}\,\nabla\underset{\sim}{\varphi} - \underset{\sim}{1}\,\nabla\cdot\underset{\sim}{\varphi}\,,\quad \nabla^2\underset{\sim}{\varphi} = -\underset{\sim}{f} \text{ on } R. \tag{8.7}$$

Then $\underset{\sim}{\sigma} \in C^1(R)$ and $\underset{\sim}{\sigma}$ satisfies (8.1) on R.

Remarks. Proof trivial (see last part in proof of Thm
Thm. 8.8 reduces task of finding a particular sol. of (8.1) to that of solving the Poisson equation $\nabla^2\underset{\sim}{\varphi} = -\underset{\sim}{f}$

on R. If R is bounded & regular and $f \in C^1(R) \cap C(\bar{R})$ such a sol. is furnished by the Newtonian pot. of f

<u>Closing remark</u>. Usefulness of foregoing general sols. of (8.2) in connection with stress formulation of Prob. BII in linear elastostatics is limited since the stress eq. of compatibility leads to unwieldy partial diff. eqs. on the generating stress potentials. Of far greater importance, as far as sol. of boundary-value probs. in linear elastostatics for homog. & isotropic media are concerned, are general sols. of the displacement eqs. of equil. to which we turn now.

General solutions of the displacement eqs. of equil. for homogeneous, isotropic elastic solids.

Assume μ, ν constant in what follows. Recall displacement formulation of Problem BIII. Thus, if R is a regular region, from (6.20′), (6.21′):

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \nabla\nabla\cdot\underset{\sim}{u} + \frac{\underset{\sim}{f}}{\mu} = \underset{\sim}{0} \quad \text{on } R \tag{8.8}$$

$$\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}} \ \text{on } \partial_1 R, \quad 2\mu\left[\frac{\nu}{1-2\nu}\underset{\sim}{1}\,\nabla\cdot\underset{\sim}{u} + \operatorname{sym}\nabla\underset{\sim}{u}\right]\underset{\sim}{n} = \overset{*}{\underset{\sim}{s}} \ \text{on } \partial_2 R \tag{8.9}$$

Once (8.8) has been solved subject to the boundary conditions (8.9), one computes $\underset{\sim}{\gamma}$ and $\underset{\sim}{\sigma}$ from (6.1), (6.

$$\underset{\sim}{\gamma} = \operatorname{sym}\nabla\underset{\sim}{u}, \quad \underset{\sim}{\sigma} = 2\mu\left[\frac{\nu}{1-2\nu}\underset{\sim}{1}\,\operatorname{tr}\underset{\sim}{\gamma} + \underset{\sim}{\gamma}\right] \ \text{on } R. \tag{8.10}$$

<u>Objective</u>. Simplification of field eq. (8.8) through change of dependent variable $\underset{\sim}{u}$ by means of displacement potentials.

<u>Thm</u>. 8.9 (<u>Papkovich-Neuber</u> solution of (8.8)). Let R be an open region. Let $\varphi \in C^3(R)$ and $\underset{\sim}{\psi} \in C^3(R)$ be a scalar field and a vector field, respectively. Suppose

$$\left.\begin{aligned} \underset{\sim}{u} &= \frac{1}{2\mu}\left[\nabla(\varphi + \underset{\sim}{x}\cdot\underset{\sim}{\psi}) - 4(1-\nu)\underset{\sim}{\psi}\right] \ \text{on } R \\ \text{or} \quad u_i &= \frac{1}{2\mu}\left[(\varphi + x_j\psi_j)_{,i} - 4(1-\nu)\psi_i\right] \ \text{on } R \end{aligned}\right\} \tag{8.11}$$

$$\left.\begin{aligned} \nabla^2\varphi &= -\frac{\underset{\sim}{x}\cdot\underset{\sim}{f}}{2(1-\nu)}, \quad \nabla^2\underset{\sim}{\psi} = \frac{\underset{\sim}{f}}{2(1-\nu)} \ \text{on } R \\ \text{or} \quad \varphi_{,ii} &= -\frac{x_j f_j}{2(1-\nu)}, \quad \psi_{i,jj} = \frac{f_i}{2(1-\nu)} \ \text{on } R \end{aligned}\right\} \tag{8.12}$$

Then $\underset{\sim}{u} \in C^2(R)$ and satisfies (8.8). Moreover, the strains and stresses associated with $\underset{\sim}{u}$ in the sense of (8.10) are given by

$$\left.\begin{aligned} \gamma_{ij} &= \frac{1}{2\mu}\left[\varphi_{,ij} - 2(1-2\nu)\psi_{(i,j)} + x_k\psi_{k,ij}\right] \text{ on } R \\ \vartheta &= \gamma_{ii} = \nabla\cdot\underset{\sim}{u} = -\frac{1-2\nu}{\mu}\psi_{i,i} = -\frac{1-2\nu}{\mu}\nabla\cdot\underset{\sim}{\psi} \text{ on } R \\ \sigma_{ij} &= \varphi_{,ij} - 2\nu\,\delta_{ij}\,\psi_{k,k} - 2(1-2\nu)\psi_{(i,j)} + x_k\psi_{k,ij} \text{ on } R \end{aligned}\right\} \text{(8.}$$

<u>Proof.</u> Substitute from vectorial form of (8.11) into (8.8) and use vectorial form of (8.12) as well as the trivial identities:

$$\left.\begin{aligned} \nabla^2\nabla g &= \nabla\nabla^2 g,\ \nabla\cdot\nabla g = \nabla^2 g \text{ on } R \text{ if } g \in C^3(R) \\ \nabla^2(\underset{\sim}{x}\cdot\underset{\sim}{v}) &= \underset{\sim}{x}\cdot\nabla^2\underset{\sim}{v} + 2\nabla\cdot\underset{\sim}{v} \text{ on } R \text{ if } \underset{\sim}{v} \in C^2(R) \end{aligned}\right\} \text{(8.}$$

where g is a scalar field & $\underset{\sim}{v}$ a vector field. Once (8.8) has been confirmed, (8.13) follow by direct computation.

<u>Exercise</u> 22. Carry out the proof of Thm. 8.9.

Proof. Define $\underset{\sim}{U}$ on R through

$$\underset{\sim}{U}(\underset{\sim}{x}) = -\frac{1}{4\pi}\int_R \frac{\underset{\sim}{v}(\underset{\sim}{\xi})}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\, dV_{\underset{\sim}{\xi}} \quad \forall\, \underset{\sim}{x} \in R \tag{1}$$

Then, by Thm. 8.6,

$$\underset{\sim}{U} \in \mathcal{C}^{N+1}(R), \quad \nabla^2 \underset{\sim}{U} = \underset{\sim}{v} \text{ on } R \tag{2}$$

Recall from (1.44) that since $\underset{\sim}{U} \in \mathcal{C}^2(R)$ one has

$$\nabla^2 \underset{\sim}{U} = \nabla\nabla\cdot\underset{\sim}{U} - \nabla\wedge\nabla\wedge\underset{\sim}{U} \quad \text{on } R \tag{3}$$

Define

$$\Theta = \nabla\cdot\underset{\sim}{U}, \quad \underset{\sim}{\Omega} = -\nabla\wedge\underset{\sim}{U} \quad \text{on } R. \tag{4}$$

Clearly, $\Theta \in \mathcal{C}^N(R)$, $\underset{\sim}{\Omega} \in \mathcal{C}^N(R)$ and (2),(3),(4) $\Rightarrow$ (*)

Remark. Clearly, Θ and $\underset{\sim}{\Omega}$ (scalar & vector pot. of $\underset{\sim}{v}$) are not uniquely determined by $\underset{\sim}{v}$. Explain.

Corollary. Let R and $\underset{\sim}{v}$ satisfy the hypotheses in Thm. 8.10 with $N \geq 2$. Then $\exists\ \underset{\sim}{v}_1 \in \mathcal{C}^{N-1}(R)$, $\underset{\sim}{v}_2 \in \mathcal{C}^{N-1}(R)$

$$\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2, \quad \nabla\wedge\underset{\sim}{v}_1 = \underset{\sim}{0}, \quad \nabla\cdot\underset{\sim}{v}_2 = 0 \text{ on } R$$

Proof. Set $\underset{\sim}{v}_1 = \nabla\Theta$, $\underset{\sim}{v}_2 = \nabla\wedge\underset{\sim}{\Omega}$ in (*).

Remarks

(a) The origins of above sol. go back to last century (Kelvin, Boussinesq). In the form cited here, the sol. was apparently first published by Papkovich (1932) and independently by Neuber (1934)

(b) Thm. 8.9 enables one to generate solutions of (8.8) by means of sols. of the Poisson equation*.

(c) Thm. 8.9 lacks motivation which is supplied by the appropriate completeness proof. As a prerequisit for the latter we require the following theorem from vector calculus.

<INSERT Thm. 8.6,

Lemma 2

Thm. 8.10 (Helmholtz resolution of a vector field). Let R be a bounded open region and let $\underset{\sim}{v} \in C^N(R) \cap C$ $(N \geq 1)$ be a vector field. Then $\exists$ a scalar field $\Theta \in C^N($ and a vector field $\underset{\sim}{\Omega} \in C^N(R) \ni$

$$\underset{\sim}{v} = \nabla\Theta + \nabla\wedge\underset{\sim}{\Omega}\,, \quad \nabla\cdot\underset{\sim}{\Omega} = 0 \text{ on } R \qquad (*)$$

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* A particular sol. of the Poisson eq. may be generated by means of Newtonian pot. See Thm. 8.6.

Thm. 8.11 (Completeness of Papkovich-Neuber solution
Let R be a bounded open region. Let $\underset{\sim}{u} \in \mathcal{C}^{N+1}(R) \cap ?$ $(N \geq 2)$ be a vector field satisfying (8.8). Then $\exists$ a scalar field $\varphi \in \mathcal{C}^{N}(R)$ and vector field $\underset{\sim}{\psi} \in \mathcal{C}^{N}($ $\ni$ (8.11), (8.12) hold.

Proof. By hyp. & Thm. 8.10 $\exists$ $\Theta \in \mathcal{C}^{N+1}(R)$, $\underset{\sim}{\Omega} \in \mathcal{C}^{N+1}(R) \ni$

$$2\mu \underset{\sim}{u} = \nabla \Theta + \underset{\sim}{v}, \quad \underset{\sim}{v} = \nabla \wedge \underset{\sim}{\Omega} \quad \text{on } R \tag{1}$$

By (1) and since $\underset{\sim}{u}$ satisfies (8.8),

$$\nabla^2 \left[\frac{1}{2(1-2\nu)} \nabla \Theta + \frac{\underset{\sim}{v}}{4(1-\nu)} \right] = -\frac{\underset{\sim}{f}}{2(1-\nu)} \tag{2}$$

Define $\underset{\sim}{\psi}$ through

$$\underset{\sim}{\psi} = -\frac{\nabla \Theta}{2(1-2\nu)} - \frac{\underset{\sim}{v}}{4(1-\nu)} \quad \text{on } R \tag{3}$$

(2), (3) $\Rightarrow$

$$\underset{\sim}{\psi} \in \mathcal{C}^{N}(R), \quad \nabla^2 \underset{\sim}{\psi} = \frac{\underset{\sim}{f}}{2(1-\nu)} \quad \text{on } R \tag{4}$$

Operate on (3) with $\nabla \cdot$ and use second of (1) to get

$$\nabla \cdot \underset{\sim}{\psi} = -\frac{\nabla^2 \Theta}{2(1-2\nu)} \quad \text{on } R \tag{5}$$

Next, from the second of the identities (8.14),

$$\nabla^2(\underset{\sim}{x}\cdot\underset{\sim}{\psi}) = \underset{\sim}{x}\cdot\nabla^2\underset{\sim}{\psi} + 2\,\nabla\cdot\underset{\sim}{\psi} \tag{6}$$

Subst. from (4),(5) into (6) to see that

$$\nabla^2\left[\frac{\Theta}{1-2\nu} + \underset{\sim}{x}\cdot\underset{\sim}{\psi}\right] = \frac{\underset{\sim}{x}\cdot\underset{\sim}{f}}{2(1-\nu)} \quad \text{on } R \tag{7}$$

Define φ through

$$\varphi = -\left[\frac{\Theta}{1-2\nu} + \underset{\sim}{x}\cdot\underset{\sim}{\psi}\right] \quad \text{on } R \tag{8}$$

(7), (8) & the first of (4) $\Rightarrow$

$$\varphi \in C^N(R), \quad \nabla^2\varphi = -\frac{\underset{\sim}{x}\cdot\underset{\sim}{f}}{2(1-\nu)} \quad \text{on } R \tag{9}$$

(8) $\Rightarrow$

$$\nabla\Theta = -(1-2\nu)\,\nabla(\varphi + \underset{\sim}{x}\cdot\underset{\sim}{\psi}) \tag{10}$$

(3) $\Rightarrow$

$$\underset{\sim}{v} = -4(1-\nu)\,\underset{\sim}{\psi} - \frac{2(1-\nu)}{1-2\nu}\nabla\Theta \tag{11}$$

Finally, (1), (10), (11) $\Rightarrow$

$$2\mu\,\underset{\sim}{u} = \nabla(\varphi + \underset{\sim}{x}\cdot\underset{\sim}{\psi}) - 4(1-\nu)\,\underset{\sim}{\psi} \quad \text{on } R \tag{12}$$

qed.

Remarks

(a) Above completeness proof is essentially due to Mindlin (1936). Earlier "proof" given by Neuber is nonsense. Note that Mindlin's proof supplies motivation for PN-solution.

(b) Introduction of PN displacement potentials $\varphi, \underline{\psi}$ reduces integration of (8.8) to the integration of Poisson (Laplace) equation. Explain. However, this simplification is achieved at expense of increase complexity of bdy. conditions. By (8.11), (8.12), (8.13) Prob. BIII when cast in terms of φ, ψ_i becomes:

$$\nabla^2\varphi = -\frac{x_j f_j}{2(1-\nu)}\ , \quad \nabla^2\psi_i = \frac{f_i}{2(1-\nu)} \quad \text{on } R$$

$$\frac{1}{2\mu}\left[(\varphi + x_j\psi_j)_{,i} - 4(1-\nu)\psi_i\right] = \overset{*}{u}_i \quad \text{on } \partial_1 R,$$

$$\sigma_{ij}n_j = [\varphi_{,ij} - 2\nu\,\delta_{ij}\,\psi_{k,k} - 2(1-2\nu)\psi_{(i,j)} + x_k\,\psi_{k,ij}]n_j = \overset{*}{s}_i \quad \text{on } \partial_2$$

Compare (8.15) with indicial form of (8.8), (8.9). Note relative complexity of bdy-value pb. for φ, ψ_i compared to mixed pb. of potential theory!

(c) One finds that $\varphi, \underline{\psi}$ generating a given sol. $\underline{u}$ of (8.6) are not unique. Also note that the original three scalar unknowns u_i have been relinquished in favor of the four new scalar unknowns φ, ψ_i. This suggests question whether the number of scal potentials involved in PN solution can be reduced from four to three without loss in completeness. Mention bewildering history of this issue [Papkovich's "Solution générale ... par trois fonctions harmoniques", Neuber's "Drei-Funktionen-Ansätz"]. The next thm gives sufficient conds. for the reducibility of the number of scalar potentials in the PN-solution.

Thm. 8.12 (Reducibility of PN solution). Let R and $\underset{\sim}{u}$ satisfy the hyp. of Thm. 8.11. Assume $\underset{\sim}{f} = \underset{\sim}{0}$ on R.

(a) Suppose R is star-shaped with respect to the origin $\underset{\sim}{x} = \underset{\sim}{0}$, $-1 < \nu < 1/2$ and $\nu \neq 1/4$. Then the PN solution remains complete if one takes $\varphi = 0$ on R

(b) Suppose R is convex and $\exists$ a choice of the fram $X = \{0; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} \ni$ the orthogonal projection of ev point $\underset{\sim}{x} \in R$ onto the plane $x_1 = 0$ $\{x_2 = 0, x_3 = 0\}$ lies wholly in R. For such a choice of X the PN solution remains complete if one takes $\psi_1 = 0$ $\{\psi_2 = 0, \psi_3 = 0\}$ on R.

For a proof see Eubanks & Co., J.R.M.A. 5 (1956). In the same paper it is shown that the conclusion under (a) is false if the hyp. $\nu \neq 1/4$ is omitted.

Thm. 8.13 (Somigliana-Galerkin solution of (8.8)). Let R be an open region. Let $\underset{\sim}{\chi} \in C^4(R)$ be a vector field and supp

$$\left.\begin{aligned} \underset{\sim}{u} &= \frac{1}{2\mu}[2(1-\nu)\nabla^2\underset{\sim}{\chi} - \nabla\nabla\cdot\underset{\sim}{\chi}] \text{ on } R \\ \text{or} \quad u_i &= \frac{1}{2\mu}[2(1-\nu)\chi_{i,jj} - \chi_{j,ji}] \text{ on } R \end{aligned}\right\} \quad (8.16)$$

$$\nabla^4\underset{\sim}{\chi} = -\frac{\underset{\sim}{f}}{1-\nu} \quad \text{or} \quad \chi_{i,jjkk} = -\frac{f_i}{1-\nu} \text{ on } R \qquad (8.17)$$

Then $\underset{\sim}{u} \in C^2(R)$ and satisfies (8.8). Further, the associated stress field is given by

$$\sigma_{ij} = \nu\,\delta_{ij}\,\chi_{k,kll} + 2(1-\nu)\chi_{(i,j)kk} - \chi_{k,kij} \text{ on } R \quad (8.1$$

Moreover, if one defines $\varphi, \underset{\sim}{\psi}$ through

$$\varphi = -\nabla\cdot\chi + \tfrac{1}{2}\,\mathbf{x}\cdot\nabla^2\chi \;,\quad \boldsymbol{\psi} = -\tfrac{1}{2}\nabla^2\chi \text{ on } R, \tag{8.19}$$

then $\varphi\in\mathcal{C}^2(R)$, $\boldsymbol{\psi}\in\mathcal{C}^2(R)$ and (8.11), (8.12) hold, i.e. (8.19) reduce the S.G. solution to the PN solution of (8.8

<u>Exercise</u> 27. Prove Thm. 8.13.

<u>Remarks</u>. The SG solution is actually due to Boussinesq (1885); it is the equil. version of a general sol. to the displt. eqs. of motion originated by Somigliana (1899) and was independently rediscovered by Galerkin (1930), to whom it is usually credited.

<u>Thm</u>. 8.14 (Joint completeness proof for the SG & PN sols.) Let R be a bounded regular region. Let $\mathbf{u}\in\mathcal{C}^4(R)\cap\mathcal{C}(\bar{R})$ satisfy (8.8). Then $\exists\ \chi\in\mathcal{C}^4(R) \ni$ (8.16), (8.17) hold; furthe $\exists\ \varphi\in\mathcal{C}^2(R)$, $\boldsymbol{\psi}\in\mathcal{C}^2(R)\ni$ (8.11), (8.12) hold.

<u>Proof.</u> (8.16) is evidently equivalent to

$$\nabla^2\chi + \frac{1}{1-2\hat{\nu}}\nabla\nabla\cdot\chi + \frac{\hat{\mathbf{f}}}{\hat{\mu}} = \mathbf{0} \text{ on } R, \tag{1}$$

provided

$$\hat{f} = -\frac{u}{1-\nu} \text{ on } R, \ \hat{\mu} = \frac{1}{\mu}, \ \hat{\nu} = \frac{3}{2} - \nu \qquad (2)$$

Note that (1) becomes identical with (8.8) if one replaces $\chi, \hat{f}, \hat{\mu}, \hat{\nu}$ by u, f, μ, ν, respectively. Thus the task of constructing $\chi \in \mathcal{C}^4(R) \ni$ (8.16) hold for given $u \in \mathcal{C}^4(R) \cap \mathcal{C}$ is reduced to finding a $u \in \mathcal{C}^4(R) \ni$ (8.8) hold for a give body-force density $f \in \mathcal{C}^4(R) \cap \mathcal{C}(\bar{R})$. The latter task is easily accomplished by means of Thm. 8.9. Indeed, let

$$\hat{\varphi}(x) = \frac{1}{8\pi(1-\hat{\nu})} \int_R \frac{\xi \cdot \hat{f}(\xi)}{|x-\xi|} dV_\xi, \ \hat{\psi}(x) = \frac{-1}{8\pi(1-\hat{\nu})} \int_R \frac{\hat{f}(\xi)}{|x-\xi|} dV_\xi \ \forall x \in \qquad (3)$$

Then, by Thm. 8.6 on Newtonian potentials,

$$\hat{\varphi} \in \mathcal{C}^5(R), \ \hat{\psi} \in \mathcal{C}^5(R), \ \nabla^2 \hat{\varphi} = -\frac{x \cdot \hat{f}}{2(1-\hat{\nu})}, \ \nabla^2 \hat{\psi} = \frac{\hat{f}}{2(1-\hat{\nu})} \text{ on } R \qquad (4)$$

Define

$$\chi = \frac{1}{2\hat{\mu}} [\nabla(\hat{\varphi} + x \cdot \hat{\psi}) - 4(1-\hat{\nu})\hat{\psi}] \text{ on } R \qquad (5)$$

Then $\chi \in \mathcal{C}^4(R)$ and (4), (5), Thm. 8.9 $\Rightarrow \chi$ satisfies (1) and hence (8.16). Also, one verifies by subst. from (8.16) into (8.8) that χ satisfies (8.17). The completeness of the PN solution now follows from Thm. 8.13.

qed.

<u>Definition</u>: Let R be an open region and $\mathfrak{X}=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$.
Let (ρ,ϕ,z) be circular cylindrical coordinates defined by

$$x_1=\rho\cos\phi\,,\;x_2=\rho\sin\phi\,,\;x_3=z\quad(0\le\rho<\infty,\;0\le\phi<2\pi,\;-\infty<z<\infty)$$

A vector field $\underset{\sim}{v}$ defined on R is said to be <u>axisymmetric about the x_3-axis</u> if its cylindrical components obey

$$v_\rho=v_\rho(\rho,z)\;,\;v_\phi=0\,,\;v_z=v_z(\rho,z)\;\;\forall\,\underset{\sim}{x}\in R$$

<u>Thm</u>. 8.15 (Boussinesq's & Love's axisymm. sols. of (8.8)).
Let R be an open region and let $\underset{\sim}{f}=\underset{\sim}{0}$ on R.

(a) If φ and $\underset{\sim}{\psi}$ satisfy the hypotheses of Thm 8.9 and if

$$\varphi=\varphi(\rho,z)\,,\;\psi_3=\psi(\rho,z)\,,\;\psi_1=\psi_2=0\text{ on }R,$$

then $\underset{\sim}{u}\in\mathcal{C}^2(R)$ is a solution of (8.8) that is axisymmetric about the x_3-axis (Boussinesq's solution);

(b) If $\underset{\sim}{\chi}$ satisfies the hypotheses of Thm. 8.13 and if

$$\chi_3=\chi(\rho,z)\,,\;\chi_1=\chi_2=0\text{ on }R,$$

then the conclusion in (a) follows again (Love's solution).

Further, Love's solution reduces to Boussinesq's solution if one sets

$$\varphi = -\frac{\partial \chi}{\partial z} + \frac{z}{2}\nabla^2\chi,\ \psi = -\frac{1}{2}\nabla^2\chi \quad \text{on } R. \tag{8.20}$$

Proof. Trivial.

Remarks

Thus Boussinesq's & Love's solutions are the axisymm versions of the PN & GS solutions of (8.8) with $\underline{f}=\underline{0}$ on R. They are easily generalized to accommodate axisymm. body forces.

Both of the above axisymm. sols. are complete if R is a suitably restricted bounded regular region and ∂R is a surface of revolution. For completeness proofs see Eubank & Co. J.R.M.A, 5 (1956) & Love's treatise, p. 274.

Discussion. In view of the multitude of alternative complete solutions of (8.8) in terms of displacement potentials, it is essential to consider criteria of usefulness:

(a) the number of scalar potentials & the order and complexity of the eqs. to be satisfied by the generating potentials.

(b) the order and complexity of the underlying dis-placement transformation

(c) the amenability of sol. to transformation into curvilinear coords.

Apply above to a comparison of PN with GG sol. to infer that PN is preferable. Similarly, Boussinesq's axisymm. sol. deserves preference over Love's sol.

Remark on general sols. of displt. eqs. of equi[illegible] for <u>anisotropic</u> elastic solids. For the case of transverse isotropy, see Lekhnitsky's book as we[illegible] as Eubanks & Co. J.R.M.A., <u>3</u> (1954).

General solutions of the displacement equations of moti[on] for homogeneous, isotropic elastic solids

Assume ρ, μ, ν constant in what follows. Also to save time we assume absence of body forces, which can be accommodated by elementary generalizations of the subsequent results.

Thm. 8.16 (Some properties of elastodynamic fields). Let R be an open region and $\mathcal{T}=[t_0,t_1]$. Suppose

$$\mathcal{S}=[\underset{\sim}{u},\underset{\sim}{\varepsilon},\underset{\sim}{\sigma}]\in\mathcal{E}(\rho,\mu,\nu;R\times\mathcal{T}),\quad \vartheta=\nabla\cdot\underset{\sim}{u},\ \underset{\sim}{w}=\tfrac{1}{2}\nabla\wedge\underset{\sim}{u},\ \Theta=\operatorname{tr}\underset{\sim}{\sigma}$$

with ρ, μ, ν constant. Then,

$$\nabla^2\underset{\sim}{u}+\frac{1}{1-2\nu}\nabla\nabla\cdot\underset{\sim}{u}=\frac{\rho}{\mu}\ddot{\underset{\sim}{u}} \text{ on } R\times\mathring{\mathcal{T}} \qquad (8.2$$

and

$$\left.\begin{aligned} &\underset{\sim}{w}=\underset{\sim}{0} \text{ on } R\times\mathring{\mathcal{T}} \Rightarrow \square_1^2\underset{\sim}{u}=\underset{\sim}{0} \text{ on } R\times\mathring{\mathcal{T}},\\ &\vartheta=0 \text{ on } R\times\mathring{\mathcal{T}} \Rightarrow \square_2^2\underset{\sim}{u}=\underset{\sim}{0} \text{ on } R\times\mathring{\mathcal{T}}, \end{aligned}\right\} (8.22$$

where $\square_\alpha^2$ are the wave operators defined by

$$\left.\begin{aligned} &\square_\alpha^2=\nabla^2-\frac{1}{c_\alpha^2}\frac{\partial^2}{\partial t^2} \quad (\alpha=1,2)\\ &c_1^2=\frac{2(1-\nu)}{1-2\nu}\frac{\mu}{\rho}=\frac{\lambda+2\mu}{\rho},\quad c_2^2=\frac{\mu}{\rho} \end{aligned}\right\} (8.23$$

Further, if $\underset{\sim}{u} \in \mathcal{C}^3(R \times \mathring{\mathfrak{J}})$, then

$$\Box_1^2 \vartheta = 0\,, \quad \Box_1^2 \Theta = 0\,, \quad \Box_2^2 \underset{\sim}{w} = \underset{\sim}{0} \quad \text{on } R \times \mathring{\mathfrak{J}} \tag{8.24}$$

Finally, if $\underset{\sim}{u} \in \mathcal{C}^5(R \times \mathring{\mathfrak{J}})$, then

$$\Box_1^2 \Box_2^2 \underset{\sim}{u} = \underset{\sim}{0}\,, \quad \Box_1^2 \Box_2^2 \underset{\sim}{\gamma} = \underset{\sim}{0}\,, \quad \Box_1^2 \Box_2^2 \underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on } R \times \mathring{\mathfrak{J}}. \tag{8.24'}$$

<u>Proof.</u>

(8.21) follows from hyp. & Thm. 6.7. Next, by (8.21),

$$\vartheta = 0 \Rightarrow \nabla^2 \underset{\sim}{u} = \frac{\rho}{\mu} \ddot{\underset{\sim}{u}} \quad \text{or} \quad \Box_2^2 \underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } R \times \mathring{\mathfrak{J}}$$

Recall

$$\nabla^2 \underset{\sim}{u} = \nabla \nabla \cdot \underset{\sim}{u} - \nabla \wedge \nabla \wedge \underset{\sim}{u} = \nabla \nabla \cdot \underset{\sim}{u} - 2 \nabla \wedge \underset{\sim}{w}$$

whence $\underset{\sim}{w} = \underset{\sim}{0} \Rightarrow \nabla^2 \underset{\sim}{u} = \nabla \nabla \cdot \underset{\sim}{u}$ and thus (8.21) gives

$$\frac{2(1-\nu)}{1-2\nu} \nabla^2 \underset{\sim}{u} = \frac{\rho}{\mu} \ddot{\underset{\sim}{u}} \quad \text{or} \quad \Box_1^2 \underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } R \times \mathring{\mathfrak{J}}$$

So (8.22) hold. Now let $\underset{\sim}{u} \in \mathcal{C}^3(R \times \mathring{\mathfrak{J}})$ & operate on (8.21) with $\nabla \cdot$ to obtain

$$\frac{2(1-\nu)}{1-2\nu} \nabla^2 \nabla \cdot \underset{\sim}{u} = \frac{\rho}{\mu} \nabla \cdot \ddot{\underset{\sim}{u}} \quad \text{or} \quad \Box_1^2 \vartheta = 0 \quad \text{on } R \times \mathring{\mathfrak{J}}$$

Next, operate on (8.21) with $\frac{1}{2} \nabla \wedge$ to arrive at

$$\nabla^2 \underset{\sim}{w} = \frac{\rho}{\mu} \ddot{\underset{\sim}{w}} \quad \text{or} \quad \Box_2^2 \underset{\sim}{w} = \underset{\sim}{0} \quad \text{on } R \times \mathring{\mathfrak{J}}.$$

Hence (8.24) hold since $\Theta = \operatorname{tr} \underset{\sim}{\sigma} = (3\lambda + 2\mu) \operatorname{tr} \underset{\sim}{\gamma}$.

Let $\underset{\sim}{u} \in \mathcal{C}^5(R\times\mathring{\mathcal{T}})$. Operate on (8.21) with $\Box_1^2$ and use the first of (8.24) to obtain $\Box_1^2\Box_2^2\underset{\sim}{u}=\underset{\sim}{0}$ on $R\times\mathring{\mathcal{T}}$. The remaining two eqs. in (8.25) follow from the first and (6.1), (6.2).

q.e.d.

Remarks. Because of (8.22) one calls

c_1 ... speed of irrotational waves (waves of dilation)

c_2 ... speed of equivoluminal waves (shear waves)*

If $\underset{\sim}{u}$ is time-independent, one recovers from Thm 8.16 properties of elastostatic fields that follow from Thm. 6.10.

Thm. 8.17 (Green-Lamé's solution of (8.21)). Let R be an open region and $\mathcal{T}=[t_0,t_1]$. Suppose $\Phi\in\mathcal{C}^3(R\times\mathring{\mathcal{T}})$ is a scalar field, $\underset{\sim}{\Psi}\in\mathcal{C}^3(R\times\mathring{\mathcal{T}})$ is a vector field, and

$$\underset{\sim}{u}=\nabla\Phi+\nabla\wedge\underset{\sim}{\Psi} \quad \text{or} \quad u_i=\Phi_{,i}+\varepsilon_{ijk}\Psi_{k,j} \quad \text{on } R\times\mathring{\mathcal{T}} \qquad (8.25)$$

$$\Box_1^2\Phi=0\,,\ \Box_2^2\underset{\sim}{\Psi}=\underset{\sim}{0} \quad \text{or} \quad \Phi_{,ii}-\frac{1}{c_1^2}\ddot{\Phi}=0,\ \Psi_{i,jj}-\frac{1}{c_2^2}\ddot{\Psi}_i=0 \text{ on } R\times\mathring{\mathcal{T}} \quad (8$$

where $\Box_\alpha^2$ $(\alpha=1,2)$ are the wave operators defined in (8.23). Then $\underset{\sim}{u}\in\mathcal{C}^2(R\times\mathring{\mathcal{T}})$ and satisfies (8.21). Further,

* If $\nu=1/3$, then $c_1=2c_2$. In seismology c_1 and c_2 are called the speeds of P-waves (primary) & S-waves (secondary).

the associated strains and stresses are given by

$$\left.\begin{aligned} \gamma_{ij} &= \Phi_{,ij} + \tfrac{1}{2}(\varepsilon_{ipq}\Psi_{q,pj} + \varepsilon_{jpq}\Psi_{q,pi}), \quad \vartheta = \gamma_{kk} = \nabla^2\Phi \\ \sigma_{ij} &= \lambda\,\delta_{ij}\nabla^2\Phi + \mu(2\Phi_{,ij} + \varepsilon_{ipq}\Psi_{q,pj} + \varepsilon_{jpq}\Psi_{q,pi}) \end{aligned}\right\} \quad (8.2$$

Proof: By substitution and direct computation.

Exercise 28. Prove Thm. 8.17

Discuss implications of Thm. 8.17, raise completeness issu

Thm 8.18 (Completeness of Lamé's solution) Let R be a bounded regular region and $\mathcal{T} = [t_0, t_1]$. Let $\underset{\sim}{u} \in \mathcal{C}^3(R$ be a vector field $\ni \underset{\sim}{u}(\cdot, t_0) \in \mathcal{C}(\bar{R})$, $\dot{\underset{\sim}{u}}(\cdot, t_0) \in \mathcal{C}^3(R) \cap \mathcal{C}(\bar{R})$. Suppose $\underset{\sim}{u}$ satisfies (8.21). Then $\exists$ a scalar field $\Phi \in \mathcal{C}^2($ and a vector field $\underset{\sim}{\Psi} \in \mathcal{C}^2(R \times \mathring{\mathcal{T}}) \ni \nabla \cdot \underset{\sim}{\Psi} = \underset{\sim}{0}$ and (8.25), (8.26) hold on $R \times \mathring{\mathcal{T}}$.

Proof. In view of (8.23) and since $\nabla^2 \underset{\sim}{u} = \nabla\nabla\cdot\underset{\sim}{u} - \nabla\wedge\nabla\wedge$ one may write (8.21) as

$$c_1^2 \nabla\nabla\cdot\underset{\sim}{u} - c_2^2 \nabla\wedge\nabla\wedge\underset{\sim}{u} = \ddot{\underset{\sim}{u}} \text{ on } R\times\mathring{\mathcal{T}} \qquad (8.27)$$

Two successive time integrations applied to (8.27) yi

$$\int_{t_0}^{t}\int_{t_0}^{\tau} \ddot{\underset{\sim}{u}}(\underset{\sim}{x},\tau')\,d\tau'\,d\tau = \int_{t_0}^{t}\int_{t_0}^{\tau} \{c_1^2 \nabla\nabla\cdot\underset{\sim}{u}(\underset{\sim}{x},\tau') - c_2^2 \nabla\wedge\nabla\wedge\underset{\sim}{u}(\underset{\sim}{x},\tau')\}\,d\tau'\,d$$

whence

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \nabla\int_{t_0}^{t}\int_{t_0}^{\tau} c_1^2\,\nabla\cdot\underset{\sim}{u}(\underset{\sim}{x},\tau')\,d\tau'd\tau - \nabla\wedge\int_{t_0}^{t}\int_{t_0}^{\tau} c_2^2\,\nabla\wedge\underset{\sim}{u}(\underset{\sim}{x},\tau')\,d\tau'd\tau$$

$$+\ \underset{\sim}{u}(\underset{\sim}{x},t_0) + (t-t_0)\,\dot{\underset{\sim}{u}}(\underset{\sim}{x},t_0)\quad \forall\ (\underset{\sim}{x},t)\in R\times\mathring{J} \qquad (1)$$

By hyp. $\underset{\sim}{u}(\cdot,t_0)\in\mathcal{C}^3(R)\cap\mathcal{C}(\bar{R})$, $\dot{\underset{\sim}{u}}(\cdot,t_0)\in\mathcal{C}^3(R)\cap\mathcal{C}(\bar{R})$, whence by Thm. 8.10 (Helmholtz) $\exists\ \Theta_\alpha\in\mathcal{C}^3(R)$, $\underset{\sim}{\Omega}_\alpha\in\mathcal{C}^3$ $(\alpha=1,2)\ \ni$

$$\left.\begin{array}{l}\underset{\sim}{u}(\underset{\sim}{x},t_0) = \nabla\Theta_1(\underset{\sim}{x}) + \nabla\wedge\underset{\sim}{\Omega}_1(\underset{\sim}{x})\ \ \forall\ \underset{\sim}{x}\in R,\ \nabla\cdot\underset{\sim}{\Omega}_1 = 0 \text{ on } R\\ \dot{\underset{\sim}{u}}(\underset{\sim}{x},t_0) = \nabla\Theta_2(\underset{\sim}{x}) + \nabla\wedge\underset{\sim}{\Omega}_2(\underset{\sim}{x})\ \ \forall\ \underset{\sim}{x}\in R,\ \nabla\cdot\underset{\sim}{\Omega}_2 = 0 \text{ on } R\end{array}\right\}\ (2)$$

Define Φ, $\underset{\sim}{\Psi}$ through

$$\left.\begin{array}{l}\Phi(\underset{\sim}{x},t) = \displaystyle\int_{t_0}^{t}\int_{t_0}^{\tau} c_1^2\,\nabla\cdot\underset{\sim}{u}(\underset{\sim}{x},\tau')\,d\tau'd\tau + \Theta_1(\underset{\sim}{x}) + (t-t_0)\,\Theta_2(\underset{\sim}{x})\\ \underset{\sim}{\Psi}(\underset{\sim}{x},t) = -\displaystyle\int_{t_0}^{t}\int_{t_0}^{\tau} c_2^2\,\nabla\wedge\underset{\sim}{u}(\underset{\sim}{x},\tau')\,d\tau'd\tau + \underset{\sim}{\Omega}_1(\underset{\sim}{x}) + (t-t_0)\,\underset{\sim}{\Omega}_2(\underset{\sim}{x})\\ \hfill \forall\ (\underset{\sim}{x},t)\in R\times\bar{J}\end{array}\right\}\ (3)$$

Clearly, (1), (2), (3) $\Rightarrow$

$$\Phi\in\mathcal{C}^2(R\times\mathring{J}),\ \underset{\sim}{\Psi}\in\mathcal{C}^2(R\times\mathring{J}),\ \underset{\sim}{u} = \nabla\Phi + \nabla\wedge\underset{\sim}{\Psi} \text{ on } R\times\mathring{J} \qquad (4)$$

Also, (3), (2) $\Rightarrow$

$$\ddot{\Phi} = c_1^2 \nabla\cdot\underset{\sim}{u} \;,\; \ddot{\underset{\sim}{\Psi}} = -c_2^2 \nabla\wedge\underset{\sim}{u} \;,\; \nabla\cdot\underset{\sim}{\Psi} = 0 \text{ on } R\times\mathring{\mathcal{J}} \tag{5}$$

Further, (4), (1.44), and the last of (5) $\Rightarrow$

$$\nabla\cdot\underset{\sim}{u} = \nabla^2\Phi \;,\; \nabla\wedge\underset{\sim}{u} = \nabla\wedge\nabla\wedge\underset{\sim}{\Psi} = \nabla\underbrace{\nabla\cdot\underset{\sim}{\Psi}}_{\text{zero}} - \nabla^2\underset{\sim}{\Psi} = -\nabla^2\underset{\sim}{\Psi} \tag{6}$$

Finally, (5), (6), (8.23) $\Rightarrow$

$$\Box_1^2\Phi = 0 \;,\; \Box_2^2\underset{\sim}{\Psi} = \underset{\sim}{0} \text{ on } R\times\mathring{\mathcal{J}} \tag{7}$$

qed.

Remarks. The above completeness proof is due to Somigliana (1892); it was preceded and followed by numerous fallacious proofs.

Corollary. Under the hypotheses of Thm. 8.18 $\exists$ $\underset{\sim}{u}_\alpha \in C^1(R\times\mathring{\mathcal{J}})$ $(\alpha = 1,2) \ni \underset{\sim}{u}_\alpha$ satisfies (8.21) and

$$\underset{\sim}{u} = \underset{\sim}{u}_1 + \underset{\sim}{u}_2 \;,\; \nabla\wedge\underset{\sim}{u}_1 = \underset{\sim}{0} \;,\; \nabla\cdot\underset{\sim}{u}_2 = 0 \text{ on } R\times\mathring{\mathcal{J}}$$

Proof. Set $\underset{\sim}{u}_1 = \nabla\Phi$, $\underset{\sim}{u}_2 = \nabla\wedge\underset{\sim}{\Psi}$ in (8.25). Invoke Thm

Thus every sufficiently regular motion is the superposition of an irrotational & an equivoluminal motion.

<u>Remark</u>. It is clear from Thm. 8.18 that the introduction of Lamé's displacement potentials reduces Problem AIII to the problem of finding solutions of the wave equations (8.26) ∋ the displacement field (8.25) & the associated field of stress obey the appropriate initial & boundary conds. Note increased complexity of these conds.

<u>Thm</u>. 8.19 (Somigliana's solution of (8.21)). Let R be an open region, $\mathcal{T}=[t_0,t_1]$. Suppose $\underset{\sim}{\chi}\in\mathcal{C}^4(R\times\mathring{\mathcal{T}})$ is a vector field and

$$\underset{\sim}{u}=\frac{1}{2\mu}\left[2(1-\nu)\,\Box_1^2\underset{\sim}{\chi}-\nabla\nabla\cdot\underset{\sim}{\chi}\right]\ \text{on}\ R\times\mathring{\mathcal{T}} \tag{8.28}$$

$$\Box_1^2\,\Box_2^2\,\underset{\sim}{\chi}=\underset{\sim}{0}\ \text{on}\ R\times\mathring{\mathcal{T}}, \tag{8.29}$$

where $\Box_\alpha^2$ $(\alpha=1,2)$ are the operators defined in (8.23). Then $\underset{\sim}{u}\in\mathcal{C}^2(R\times\mathring{\mathcal{T}})$ and satisfies (8.21).

Further, if in particular $\underset{\sim}{\chi}$ is <u>time-independent</u> then (8.28), (8.29) reduce to the SG-solution (8.16) (8.17) of the <u>equil</u>. equation (8.8) with $\underset{\sim}{f}=\underset{\sim}{0}$ on R

Proof. Substitute from (8.28) into (8.21) and use (8.29). The remaining claim is obviously true.

Remarks. For a completeness proof appropriate to the Somigliana sol. (parallel to the proof of Thm. 8.14) see Gurtin & Co., Proc., 4th. U.S. Nat. Cong. Appl. Mech., 196
For the reasons cited in comparing the SG-sol. with the PN-sol., Lamé's sol. deserves preference over Somigliana's.

The equilibrium paradox of Lamé's solution

As we have seen, Somigliana's complete sol. (8.28), (8.29) of (8.21) reduces to the complete SG solutions of (8.8) (in case $\underset{\sim}{f} = \underset{\sim}{0}$ on R), when $\underset{\sim}{\chi}$ is time-independent.

According to Thm. 8.18, Lamé's sol. with $\nabla\cdot\underset{\sim}{\Psi}$ on $R \times \mathring{T}$ is complete also in the equil. case since $\underset{\sim}{\ddot{u}} = \underset{\sim}{0}$ on $R \times \mathring{T}$ was not excluded in the proof of Thm. 8.18.

On the other hand, <u>if</u> Φ and $\underset{\sim}{\Psi}$ are time-independ in (8.25), (8.26), and $\nabla\cdot\underset{\sim}{\Psi}=0$ on R, one has

$$\underset{\sim}{u}=\nabla\Phi+\nabla\wedge\underset{\sim}{\Psi},\ \nabla^2\Phi=0,\ \nabla^2\underset{\sim}{\Psi}=\underset{\sim}{0},\ \nabla\cdot\underset{\sim}{\Psi}=0 \text{ on } R \ (*$$

$(*) \Rightarrow$

$$\nabla\cdot\underset{\sim}{u}=\nabla^2\Phi=0,\ \nabla\wedge\underset{\sim}{\overset{u}{\omega}}=\nabla\wedge\nabla\wedge\underset{\sim}{\Psi}=\underbrace{\nabla\nabla\cdot\underset{\sim}{\Psi}}_{\text{zero}}-\nabla^2\underset{\sim}{\Psi}=\underset{\sim}{0} \text{ on } R$$

Thus if (*) were a complete sol. of (8.8) with $\underset{\sim}{f}=\underset{\sim}{0}$ on R, it would follow that <u>every</u> sufficiently smooth elastostatic displacement field — in the abs of body forces — is both equi-voluminal and irro ational, which is contradicted by innumerable counter-examples.

<u>Resolution of paradox</u> (ARMA, $\underline{6}$, 1, 1960)

Φ and $\underset{\sim}{\Psi}$ remain time-dependent also in the equil. case, but here $\nabla\Phi+\nabla\wedge\underset{\sim}{\Psi}$ is time-independen Thus, in view of Thm. 8.18 (see its Corollary), every suff. regular elasto<u>static</u> displacement field (if $\underset{\sim}{f}=\underset{\sim}{0}$ on R is the superposition of the displacement fields associate with an irrotational & an equivoluminal motion:

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underbrace{\underset{\sim}{u}_1(\underset{\sim}{x},t)}_{\nabla\Phi(\underset{\sim}{x},t)} + \underbrace{\underset{\sim}{u}_2(\underset{\sim}{x},t)}_{\nabla\wedge\underset{\sim}{\Psi}(\underset{\sim}{x},t)}, \quad \nabla\wedge\underset{\sim}{u}_1 = 0,\ \nabla\cdot\underset{\sim}{u}_2 = 0 \ \text{ on } R\times\mathring{\mathcal{T}}$$

For the explicit nature of the time dependence of Φ and Ψ in the equil. case, see the above paper & the exercise below:

Exercise 29. Let R be a domain, $\mathcal{T}=[t_0,t_1]$ and let $\varphi, \underset{\sim}{\psi}$ be harmonic PN-potentials that generate an elast static displ. field $\underset{\sim}{u}$ on R through (8.11). Suppose $\underset{\sim}{\beta} \in C^2(R)$ satisfies

$$\nabla^2\underset{\sim}{\beta} = 4(1-\nu)\underset{\sim}{\psi}, \quad \nabla\cdot\underset{\sim}{\beta} = 2(1-\nu)\underset{\sim}{x}\cdot\underset{\sim}{\psi} \ \text{ on } R \quad (*)$$

and define $\Phi, \underset{\sim}{\Psi}$ through

$$2\mu\,\Phi(\underset{\sim}{x},t) = \varphi(\underset{\sim}{x}) - 2(1-\nu)\frac{\mu t^2}{\rho}\nabla\cdot\underset{\sim}{\psi}(\underset{\sim}{x}) - (1-2\nu)\underset{\sim}{x}\cdot\underset{\sim}{\psi}(\underset{\sim}{x}),$$

$$2\mu\,\underset{\sim}{\Psi}(\underset{\sim}{x},t) = 2(1-\nu)\frac{\mu t^2}{\rho}\nabla\wedge\underset{\sim}{\psi}(\underset{\sim}{x}) + \nabla\wedge\underset{\sim}{\beta}(\underset{\sim}{x}) \quad \forall\, (\underset{\sim}{x},t)\in R\times\mathring{\mathcal{T}}.$$

Confirm that $\Phi, \underset{\sim}{\Psi}$ satisfy (8.26) and generate $\underset{\sim}{u}$ through (8.25).

(*) One can show that the required field $\underset{\sim}{\beta}$ exists.

Remarks. Note that Lamé's solution has no compa ably simple complete static counterpart: in this sense the degenerate equil. case is more involved. Cf. doub root degeneracy in the theory of ordinary differentia equations with constant coefficients. Mention useless extension of PN-solution to elastodynamics.

Plane waves in a homogeneous, isotropic elastic solid

Objective: Additional insight into the physical significance of the wave speeds c_1, c_2 encountered earlier.

Recall appropriate displacement eq. of motion in the absence of body forces:

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \nabla \nabla \cdot \underset{\sim}{u} = \frac{\rho}{\mu} \ddot{\underset{\sim}{u}} \quad (\mu > 0; -1 < \nu < \tfrac{1}{2}) \quad (**)$$

A plane wave motion in a body occupying E is a motion with a displacement field of the form

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{a}\, \varphi(\underset{\sim}{x} \cdot \underset{\sim}{m} - ct) \ \forall\ (\underset{\sim}{x}, t) \in E \times (-\infty, \infty) \quad (\S)$$

subject to the following requirements:

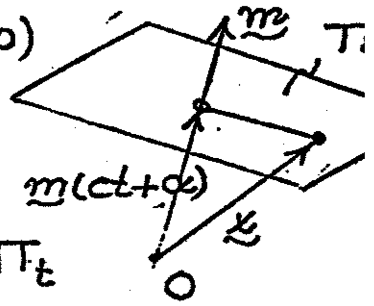
(i) $\varphi \in C^2((-\infty,\infty))$ is scalar-valued, $\varphi'' \not\equiv 0$ on $(-\infty,\infty)$

(ii) $c > 0$ is a constant (speed of propagation), $\underset{\sim}{a}$ and $\underset{\sim}{m}$ are constant unit vectors (called the direction of motion and the direction of propagation, respectively)

Emphasize purely kinematical nature of above conc
To account for the foregoing terminology note first tha

$\mathbf{u}$ is parallel to $\mathbf{a}$. Next let α be a given real constant and consider the family of planes

$$\Pi_t = \Pi_t(\alpha) = \{\mathbf{x} \mid \mathbf{x}\cdot\mathbf{m} - ct = \alpha\} \quad (-\infty < t < \infty)$$



Clearly, at any instant $t \in (-\infty, \infty)$,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}\,\varphi(\alpha) = \text{const.} \ \forall\, \mathbf{x} \in \Pi_t$$

Also, Π_t is $\perp \mathbf{m}$ $\forall\, t \in (\infty,\infty)$ and Π_t moves in the direction of $\mathbf{m}$ with the speed c.

(§) $\Leftrightarrow$ $u_i = a_i \varphi(x_l m_l - ct)$. Thus, using (i), (ii) one fin

$$u_{i,j} = a_i m_j \varphi'(x_l m_l - ct)$$
$$u_{i,jk} = a_i m_j m_k \varphi''(x_l m_l - ct)$$
$$\ddot{u}_i = a_i c^2 \varphi''(x_l m_l - ct)$$

With the aid of these formulas one easily confirm that (§), (i), (ii) $\Rightarrow$

$$\nabla\cdot\mathbf{u} = (\mathbf{m}\cdot\mathbf{a})\,\varphi'(\mathbf{x}\cdot\mathbf{m} - ct), \quad \nabla\wedge\mathbf{u} = (\mathbf{m}\wedge\mathbf{a})\,\varphi'(\mathbf{x}\cdot\mathbf{m} - ct)$$

$$\left.\begin{aligned} \nabla^2 \mathbf{u} = \mathbf{a}\,\varphi''(\mathbf{x}\cdot\mathbf{m}-ct)\,,\ \nabla\nabla\cdot\mathbf{u} = (\mathbf{a}\cdot\mathbf{m})\,\mathbf{m}\,\varphi''(\mathbf{x}\cdot\mathbf{m}-ct) \\ \ddot{\mathbf{u}} = c^2\mathbf{a}\,\varphi''(\mathbf{x}\cdot\mathbf{m}-ct) \end{aligned}\right\} \quad (2)$$

In view of (1), one infers that (§), (i), (ii) $\Rightarrow$

$\nabla\wedge\mathbf{u} = \mathbf{0} \iff \mathbf{m}\wedge\mathbf{a} = \mathbf{0}$ ($\mathbf{a}\parallel\mathbf{m}$, <u>longitud.</u> plane wave);

$\nabla\cdot\mathbf{u} = 0 \iff \mathbf{m}\cdot\mathbf{a} = 0$ ($\mathbf{a}\perp\mathbf{m}$, <u>transverse</u> plane wave).

<u>Questions</u>: Are plane waves possible in a homog., isotr elastic solid? If so, what are the possibilities as to $\varphi, c, \mathbf{a}, \mathbf{m}$
Substitute from (2) into (**) & use $\varphi''\not\equiv 0$ on $(-\infty,\infty)$ to obta

$$\mathbf{a} + \frac{1}{1-2\nu}(\mathbf{a}\cdot\mathbf{m})\,\mathbf{m} - \frac{\rho}{\mu}c^2\mathbf{a} = \mathbf{0} \quad \text{or}$$

$$\boxed{(\mu-\rho c^2)\,\mathbf{a} + \frac{\mu}{1-2\nu}(\mathbf{a}\cdot\mathbf{m})\,\mathbf{m} = \mathbf{0}} \quad (4)$$

Thus a displt field $\mathbf{u}$ obeying (§), (i), (ii) satisfies (**) $\iff$ $\mathbf{a}, \mathbf{m}, c$ satisfy (4). (No further restrictions on

<u>Case 1</u>: $\mathbf{a}\cdot\mathbf{m}\neq 0$

Here (4) $\Rightarrow \mathbf{a} = \pm\mathbf{m} \Rightarrow \mathbf{a}\cdot\mathbf{m} = \pm 1\,,\ \mathbf{m}\wedge\mathbf{a} = \mathbf{0}$

Regardless of sign alternative (4) reduces to

---

[1] Here we used the fact that $\varphi''\not\equiv 0 \Rightarrow \varphi'\not\equiv 0$ on $(-\infty,\infty)$
Note (3) is a purely kinematic result.

$$\left(\mu - \rho c^2 + \frac{\mu}{1-2\nu}\right)\underset{\sim}{a} = \underset{\sim}{0}, \text{ so that}$$

$$c = \sqrt{\frac{2(1-\nu)\mu}{(1-2\nu)\rho}} = \sqrt{\frac{\lambda + 2\mu}{\rho}} = c_1 > 0$$

Since $\underset{\sim}{m} \wedge \underset{\sim}{a} = \underset{\sim}{0}$, the plane wave in Case 1 is <u>longitudinal</u> and hence by (3) <u>irrotational</u>. Thus by a previous result $\Box_1^2 \underset{\sim}{u} = \underset{\sim}{0}$ on $E \times (-\infty, \infty)$ in this case. For reasons now clear c_1 is also called the speed of longitudinal plane waves.

<u>Case</u> 2. $\underset{\sim}{a} \cdot \underset{\sim}{m} = 0 \quad (\underset{\sim}{a} \perp \underset{\sim}{m})$

Here (4) holds $\Leftrightarrow c = \sqrt{\frac{\mu}{\rho}} = c_2 > 0$

The plane wave in Case 2 is <u>transverse</u> and hence <u>equivoluminal</u>. By a previous result $\Box_2^2 \underset{\sim}{u} = \underset{\sim}{0}$ in Case 2. Above explains why c_2 is also called the speed of transverse plane waves.

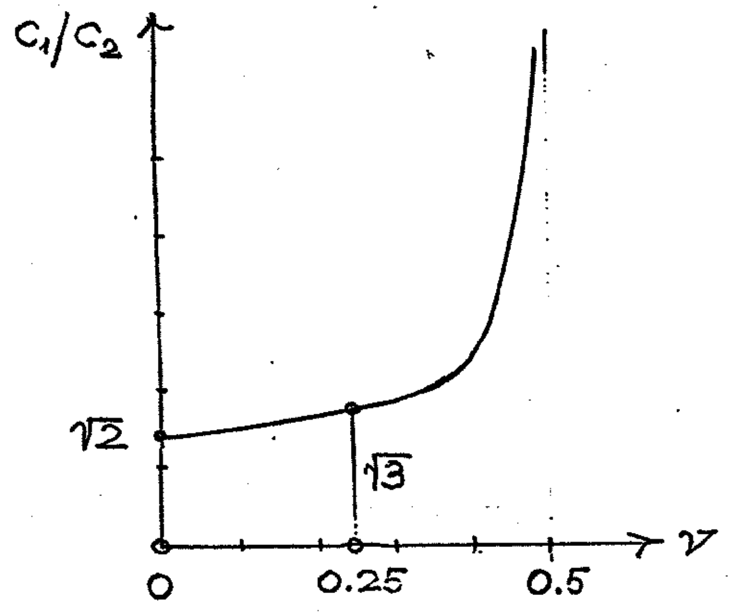
<u>Note</u>: Above classification is clearly exhaustive. There are no other types of plane waves possible in a homog. & isotropic elastic solid.

The following terminology is in common use:

c_1 ... speed of propagation of irrotational, dilatational, longitudinal, primary, pressure, or P-waves.

c_2 ... speed of propagation of solenoidal, rotational, equi-voluminal, transverse, secondary, shear or S-waves.

Note that $c_1/c_2 = \sqrt{\frac{2(1-\nu)}{1-2\nu}}$ $(-1<\nu<\frac{1}{2}) \Rightarrow c_1 > c_2$



steel: $c_1 \doteq 5900$ m/sec, $c_2 \doteq 3200$ m/sec

rubber: $c_1 \doteq 1040$ m/sec, $c_2 \doteq 27$ m/sec

Rayleigh surface waves in a homogeneous isotropic elastic solid

Let $\mathfrak{X}=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$ and let R be the halfspace

$R=\{\underset{\sim}{x}\,|\,-\infty<x_1<\infty,-\infty<x_2<\infty,\ x_3>0\}$, $\mathcal{T}=(-\infty,\infty)$

$\partial R\ldots$ plane $x_3=0$

We seek solutions of

$$\nabla^2\underset{\sim}{u}+\frac{1}{1-2\nu}\nabla\nabla\cdot\underset{\sim}{u}=\frac{\rho}{\mu}\ddot{\underset{\sim}{u}}\quad(\rho>0,\mu>0,-1<\nu<\tfrac{1}{2})\ (**)$$

(no body forces) with the following properties:

(i) $\underset{\sim}{u}\in\mathcal{C}^2(\bar{R}\times\mathcal{T})$, $u_2=0$, $u_{i,2}=0$ on $R\times\mathcal{T}$. Thus;

$$u_1=u_1(x_1,x_3,t),\ u_2=0,\ u_3=u_3(x_1,x_3,t)\ \forall(\underset{\sim}{x},t)\in R\times$$

(ii) $u_i(x_1,x_3,t)\to 0$ as $x_3\to\infty$ $\forall\ (x_1,t)\in(-\infty,\infty)\times(-\infty$

(iii) $\underset{\sim}{s}=\underset{\sim}{0}$ on $\partial R\times\mathcal{T}$ $\quad(\sigma_{3k}=0$ at $x_3=0\ \forall\ t\in\mathcal{T})$

<u>Remarks</u>. Above requirements do not characterize $\underset{\sim}{u}$ uniquely. Note that (i) $\Rightarrow \gamma_{2i}=0$ on $R\times\mathcal{T}$ (plane str

Recall Lamé's solution of (**):

$$\left.\begin{aligned}&\underset{\sim}{u}=\nabla\Phi+\nabla\wedge\underset{\sim}{\Psi}\ \text{ on } R\times\mathcal{T} && (a)\\ &\Box_1^2\Phi=0,\ \Box_2^2\underset{\sim}{\Psi}=\underset{\sim}{0}\ \text{ on } R\times\mathcal{T} && (b)\end{aligned}\right\}(1)$$

Guided by (i) assume

$$\Phi = \varphi(x_1, x_3, t)\ ,\ \Psi_2 = \psi(x_1, x_3, t)\ ,\ \Psi_1 = \Psi_3 = 0 \qquad (2)$$

(2), (1a) $\Rightarrow$

$$u_1 = \varphi_{,1} - \psi_{,3}\ ,\ u_2 = 0\ ,\ u_3 = \varphi_{,3} + \psi_{,1} \qquad (3)$$

Remark. One can show that (1), (2) is the complete solution of (**) consistent with (i).

In view of (2), the wave eqs. (1b) become

$$\varphi_{,11} + \varphi_{,33} - \frac{1}{c_1^2}\ddot{\varphi} = 0\ ,\ \psi_{,11} + \psi_{,33} - \frac{1}{c_2^2}\ddot{\psi} = 0 \qquad (4)$$

Rayleigh's Ansatz now restricts φ, ψ as follows:

$$\varphi(x_1, x_3, t) = f(x_3)\, e^{ik(x_1 - ct)}\ ,\ \psi(x_1, x_3, t) = g(x_3)\, e^{ik(x_1 - ct)} \qquad (5$$

where $k>0$, $c>0$ are as yet undetermined constants while f, g are still arbitrary complex-valued[1] fcs

Substitution from (5) into (4) yields

$$f'' - k^2\left(1 - \frac{c^2}{c_1^2}\right) f = 0\ ,\ g'' - k^2\left(1 - \frac{c^2}{c_2^2}\right) g = 0 \text{ on } (0, \infty) \quad ($$

Requirement (ii) demands that the coeff's of f and g in (6) both be negative. Thus we must have

[1] Explain admissibility of complex values

$$0 < c < c_2 \quad (c_2 < c_1) \tag{7}$$

Assume (7) holds. The only solutions of (6) consistent with (ii) are

$$\left.\begin{aligned} &f(x_3) = A e^{-\kappa_1 x_3}, \quad g(x_3) = B e^{-\kappa_2 x_3} \quad (0 < x_3 < \infty) \\ &\kappa_1 = k\sqrt{1 - \frac{c^2}{c_1^2}} > 0, \quad \kappa_2 = k\sqrt{1 - \frac{c^2}{c_2^2}} > 0 \end{aligned}\right\} \tag{8}$$

where A, B are arbitrary complex constants.

(8), (5), (3) $\Rightarrow$

$$u_1(x_1, x_3, t) = \left[ikA e^{-\kappa_1 x_3} + \kappa_2 B e^{-\kappa_2 x_3}\right] e^{ik(x_1 - ct)},$$

$$u_3(x_1, x_3, t) = \left[-\kappa_1 A e^{-\kappa_1 x_3} + ikB e^{-\kappa_2 x_3}\right] e^{ik(x_1 - ct)}, \quad u_2 = 0$$

Note that $\underset{\sim}{u}$ given by (9) satisfies (**) and conforms to (i), (ii) for every choice of $A, B, k > 0$, and $c \in (0, c_2)$.

<u>Question</u>: can we determine these four constants so that the boundary cond. (iii) holds as well?

Recall the appropriate stress-displt. relations

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i})$$

Hence, bearing (9) in mind, we have

$$\sigma_{31} = \mu(u_{1,3} + u_{3,1}) \,,\; \sigma_{32} = 0 \,,\; \sigma_{33} = \lambda(u_{1,1} + u_{3,3}) + 2\mu u_{3,3}$$

Compute σ_{31}, σ_{33} from (10), (9), set $\sigma_{3\alpha}(x_1, 0, t) = 0$, u $\underset{\alpha=1,3}{}$
$c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$. In this manner (iii) leads to

$$\left.\begin{aligned} &2ik\kappa_1 A + \left(2 - \frac{c^2}{c_2^2}\right)k^2 B = 0 \\ &\left(2 - \frac{c^2}{c_2^2}\right) ik A + 2\kappa_2 B = 0 \end{aligned}\right\} \quad (11)$$

(11) have a non-trivial solution for $A, B \Leftrightarrow$

$$\left(2 - \frac{c^2}{c_2^2}\right)^2 - 4\sqrt{1 - \frac{c^2}{c_1^2}}\sqrt{1 - \frac{c^2}{c_2^2}} = 0 \qquad (12)$$

Note that k does not enter (12). If one sets

$$\xi = \frac{c}{c_2} \,,\quad \beta = \frac{c_2}{c_1} = \sqrt{\frac{1-2\nu}{2(1-\nu)}} \quad (0<\beta<1) \qquad (13)$$

Rayleigh's eq. (12) becomes

$$(2-\xi^2)^2 - 4\sqrt{1-\beta^2\xi^2}\sqrt{1-\xi^2} = 0 \qquad (0<\beta<1), \qquad (14)$$

which can be shown to have one and only one real root $\xi \in (0,1)$. Note that $\xi = \xi(\nu)$, whence $c = \xi(\nu) c_2$. Also, from (11), A/B is now computable.

References on Rayleigh waves:

Achenbach, p. 187 (He mentions a <u>book</u> on this topic
Miklowitz, p. 146
Hayes & Rivlin, ZAMP, <u>13</u> (1962), 80 (Discussion of (1

Knowles, J. Geophys. Research, <u>71</u> (1966), 22, 5480)

ν	$\xi = c/c_2$
0	0.862
1/4	0.919
1/3	0.932
1/2	0.955