# A method to analyze electromechanical stability of dielectric elastomer actuators

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## Abstract

This letter describes a method to analyze electromechanical stability of dielectric elastomer actuators. We write the free energy of an actuator using stretches and nominal electric displacement as generalized coordinates, and pre-stresses and voltage as control parameters. When the Hessian of the free-energy function ceases to be positive-definite, the actuator thins down drastically, often resulting in electrical breakdown. Our calculation shows that stability of the actuator is markedly enhanced by pre-stresses.

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Dielectric elastomer actuators have been intensely studied in recent years.[1-6] Possible applications include medical devices, energy harvesters, and space robotics.[7-11] Figure 1 illustrates a planar actuator, consisting of a thin layer of dielectric elastomer sandwiched between two compliant electrodes. A battery applies a voltage between the electrodes, and two weights apply forces in the plane of the actuator. In response to the change in the voltage, the actuator is capable of rapid and large deformation. Such an actuator, however, is susceptible to an electromechanical instability. As the electric field increases, the elastomer thins down, so that the same voltage will induce an even higher electric field. The positive feedback may cause the elastomer to thin down drastically, resulting in electrical breakdown.

While this electromechanical instability has long been recognized in the electrical power industry as a failure mode of polymer insulators,[12] we are unaware of any detailed analysis of the instability beyond a heuristic model by Stark and Garton.[13] In particular, it is unclear how such a model may account for external mechanical loads, which are important for dielectric elastomer actuators. This Letter formulates a general method to analyze this electromechanical instability. We will show that the forces due to the weights can markedly enhance the stability of the actuator. This enhancement is known empirically to the researchers of dielectric elastomer actuators, but has so far not been understood theoretically.

Our analysis is based on a recent formulation of the nonlinear field theory of deformable dielectrics.[14-18] With the reference to Fig.1, the elastomer has the dimension $L_1L_2L_3$ in the undeformed state. Subject to the electric voltage $\Phi$ and mechanical forces

$P_1$ and $P_2$, the elastomer deforms to a homogeneous state with stretches $\lambda_1$, $\lambda_2$ and $\lambda_3$, as well as a magnitude of electric charge $Q$ on either electrode. The elastomer is taken to be incompressible, so that $\lambda_3 = 1/(\lambda_1 \lambda_2)$.

Define the nominal electric field by the voltage in the deformed state divided by the thickness of the elastomer in the undeformed state, $\tilde{E} = \Phi / L_3$, and define the nominal electric displacement as the charge on an electrode in the deformed state divided by the area of the electrode in the undeformed state, $\tilde{D} = Q/(L_1 L_2)$. By contrast, the true electric field is defined as the voltage divided by the thickness of the elastomer in the current state, $E = \Phi/(\lambda_3 L_3)$, and the true electric displacement is defined as the charge divided by the area of the electrode in the deformed state, $D = Q/(\lambda_1 L_1 \lambda_2 L_2)$. Denote the nominal stresses by $s_1 = P_1/(L_2 L_3)$ and $s_2 = P_2/(L_1 L_3)$.

The elastomer is taken to be an elastic dielectric, with the free-energy function $W(\lambda_1, \lambda_2, \tilde{D})$. The elastomer, the weights and the battery constitute a thermodynamic system, characterized by three generalized coordinates $\lambda_1, \lambda_2, \tilde{D}$, and three control parameters $P_1, P_2, \Phi$. The free energy of the system is

$$G = L_1 L_2 L_3 W(\lambda_1, \lambda_2, \tilde{D}) - P_1 \lambda_1 L_1 - P_2 \lambda_2 L_2 - \Phi Q. \qquad (1)$$

When the generalized coordinates vary by small amounts, $\delta\lambda_1, \delta\lambda_2, \delta\tilde{D}$, the free energy of the system varies by

$$\frac{\delta G}{L_1L_2L_3}=\left(\frac{\partial W}{\partial \lambda_1}-s_1\right)\delta\lambda_1+\left(\frac{\partial W}{\partial \lambda_2}-s_2\right)\delta\lambda_2+\left(\frac{\partial W}{\partial \tilde{D}}-\tilde{E}\right)\delta\tilde{D}$$
$$+\frac{1}{2}\frac{\partial^2 W}{\partial \lambda_1^2}\delta\lambda_1^2+\frac{1}{2}\frac{\partial^2 W}{\partial \lambda_2^2}\delta\lambda_2^2+\frac{1}{2}\frac{\partial^2 W}{\partial \tilde{D}^2}\delta\tilde{D}^2 \tag{2}$$
$$+\frac{\partial^2 W}{\partial \lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2+\frac{\partial^2 W}{\partial \lambda_1\partial\tilde{D}}\delta\lambda_1\delta\tilde{D}+\frac{\partial^2 W}{\partial \lambda_2\partial\tilde{D}}\delta\lambda_2\delta\tilde{D}$$

Thermodynamics dictates that a stable equilibrium state should minimize *G*. In equilibrium, the coefficients of the first-order variations vanish:

$$s_1=\frac{\partial W}{\partial \lambda_1},\quad s_2=\frac{\partial W}{\partial \lambda_2},\quad \tilde{E}=\frac{\partial W}{\partial \tilde{D}}. \tag{3}$$

To ensure that this equilibrium state minimizes *G*, the sum of the second-order variations must be positive for arbitrary combination of $\delta\lambda_1,\delta\lambda_2,\delta\tilde{D}$; that is, the Hessian of the free energy,

$$\mathbf{H}=\begin{bmatrix}\frac{\partial^2 W}{\partial \lambda_1^2} & \frac{\partial^2 W}{\partial \lambda_1\partial\lambda_2} & \frac{\partial^2 W}{\partial \lambda_1\partial\tilde{D}}\\ \frac{\partial^2 W}{\partial \lambda_1\partial\lambda_2} & \frac{\partial^2 W}{\partial \lambda_2^2} & \frac{\partial^2 W}{\partial \lambda_2\partial\tilde{D}}\\ \frac{\partial^2 W}{\partial \lambda_1\partial\tilde{D}} & \frac{\partial^2 W}{\partial \lambda_2\partial\tilde{D}} & \frac{\partial^2 W}{\partial \tilde{D}^2}\end{bmatrix}, \tag{4}$$

must be positive-definite at the equilibrium state.

An elastic dielectric is characterized by a free-energy function $W(\lambda_1,\lambda_2,\tilde{D})$. For a given set of control parameters, $P_1,P_2,\Phi$, Equation (3) is a set of nonlinear algebraic equations that determine the equilibrium values of the generalized coordinates $\lambda_1,\lambda_2,\tilde{D}$. We now fix the forces $P_1,P_2$ but vary the voltage $\Phi$. When the voltage is small, the Hessian is positive-definite. When the voltage reaches a critical value, $\Phi^c$, the Hessian ceases to be positive-definite, and $\det(\mathbf{H})=0$. The condition $\det(\mathbf{H})=0$, along with the equilibrium equations (3), determine the critical values $\tilde{E}^c$, $\lambda_1^c$, $\lambda_2^c$ and $\tilde{D}^c$ for any given pre-stresses $s_1$ and $s_2$.

To illustrate the method, consider a model material, called the ideal dielectric elastomer, which has the free-energy function:[15]

$$W\left(\lambda_1, \lambda_2, \tilde{D}\right) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3\right) + \frac{\tilde{D}^2}{2\varepsilon}\lambda_1^{-2}\lambda_2^{-2}. \tag{5}$$

The first term is the elastic energy, with $\mu$ being the small-strain shear modulus. The second term is the dielectric energy, with $\varepsilon$ being the permittivity. When more realistic material models are used, more complicated behavior is expected.[15,16]

Using (5), the equilibrium equations (3) become

$$s_1 = \mu\left(\lambda_1 - \lambda_1^{-3}\lambda_2^{-2}\right) - \frac{\tilde{D}^2}{\varepsilon}\lambda_1^{-3}\lambda_2^{-2}, \tag{6a}$$

$$s_2 = \mu\left(\lambda_2 - \lambda_2^{-3}\lambda_1^{-2}\right) - \frac{\tilde{D}^2}{\varepsilon}\lambda_2^{-3}\lambda_1^{-2}, \tag{6b}$$

$$\tilde{E} = \frac{\tilde{D}}{\varepsilon}\lambda_1^{-2}\lambda_2^{-2}, \tag{6c}$$

and the Hessian (4) becomes

$$\mathbf{H} = \begin{bmatrix} \mu\left(1 + 3\lambda_1^{-4}\lambda_2^{-2}\right) + \frac{3\tilde{D}^2}{\varepsilon}\lambda_1^{-4}\lambda_2^{-2} & 2\mu\lambda_1^{-3}\lambda_2^{-3} + \frac{2\tilde{D}^2}{\varepsilon}\lambda_1^{-3}\lambda_2^{-3} & -\frac{2\tilde{D}}{\varepsilon}\lambda_1^{-3}\lambda_2^{-2} \\ 2\mu\lambda_1^{-3}\lambda_2^{-3} + \frac{2\tilde{D}^2}{\varepsilon}\lambda_1^{-3}\lambda_2^{-3} & \mu\left(1 + 3\lambda_2^{-4}\lambda_1^{-2}\right) + \frac{3\tilde{D}^2}{\varepsilon}\lambda_2^{-4}\lambda_1^{-2} & -\frac{2\tilde{D}}{\varepsilon}\lambda_2^{-3}\lambda_1^{-2} \\ -\frac{2\tilde{D}}{\varepsilon}\lambda_1^{-3}\lambda_2^{-2} & -\frac{2\tilde{D}}{\varepsilon}\lambda_2^{-3}\lambda_1^{-2} & \frac{1}{\varepsilon}\lambda_1^{-2}\lambda_2^{-2} \end{bmatrix}. \tag{7}$$

In the special case when the elastomer is under equal biaxial stresses, $s_1 = s_2 = s$, the stretches are also equal biaxial, $\lambda_1 = \lambda_2 = \lambda$. The equilibrium condition (6) becomes

$$\frac{\tilde{D}}{\sqrt{\varepsilon\mu}} = \sqrt{\lambda^6 - 1 - \frac{s}{\mu}\lambda^5}, \quad \frac{\tilde{E}}{\sqrt{\mu/\varepsilon}} = \sqrt{\lambda^{-2} - \lambda^{-8} - \frac{s}{\mu}\lambda^{-3}}. \tag{8}$$

For a prescribed mechanical load, $s/\mu$, this pair of equations provides the equilibrium relation between the normalized voltage $\tilde{E}/\sqrt{\mu/\varepsilon}$ and the normalized charge $\tilde{D}/\sqrt{\mu\varepsilon}$, using the stretch $\lambda$ as a parameter.

Figure 2 shows the effects of the equal biaxial pre-stress. At a fixed level of the pre-stress $s_1/\mu$, the function $\tilde{E}(\tilde{D})$ has a peak (Fig. 2a). The left-hand side of each curve corresponds to a positive-definite Hessian, the right-hand side corresponds to a non-positive-definite Hessian, and the peak is determined by $\det(\mathbf{H}) = 0$. By contrast, the true electric field is a monotonic function of $\tilde{D}$ (Fig. 2b). As the pre-stress increases, the critical nominal electric field decreases while the critical true electric field increases. The actuation stretch is defined as $\lambda_1/\lambda_1^p$, where $\lambda_1^p$ is the pre-stretch due to the weights in the absence of the voltage. The bi-axial pre-stress increases the critical actuation stretch (Fig. 2c).

In the absence of the pre-stress, maximizing $\tilde{E}$ in (8), we obtain the critical stretch $\lambda^c \approx 1.26$, which corresponds to reduction in the thickness by ~37%, and is consistent with the maximum thickness strain of ~40% observed experimentally.[1] The critical nominal electric field is $\tilde{E}^c \approx 0.69\sqrt{\mu/\varepsilon}$, which is high when the elastomer is stiff or when the permittivity is low. For representative values $\mu = 10^6\,\mathrm{N/m^2}$ and $\varepsilon = 4\times10^{-11}\,\mathrm{F/m}$, the critical nominal electric field is $\tilde{E}^c \approx 10^8\,\mathrm{V/m}$, which is on the same order of magnitude of the reported breakdown fields.[6]

Figure 3 shows the effects of unequal biaxial pre-stresses, with $s_2/s_1 = 1$ corresponding to equal biaxial pre-stresses, and $s_2/s_1 = 0$ corresponding to uniaxial pre-stress. The critical nominal electric field $\tilde{E}^c$ decreases as $s_1/\mu$ increases or as $s_2/s_1$

decreases (Fig. 3a). The critical true electric field $E^c$ increases with $s_1/\mu$ if $s_2/s_1>0$; however, the uni-axial pre-stretch keeps $E^c$ at an almost constant level as $s_1/\mu$ changes. Figures 3c and d show the effects of pre-stresses on the actuation stretches $\lambda_1^c/\lambda_1^p$ and $\lambda_2^c/\lambda_2^p$. It is desirable for an actuator to work under a low voltage and a low true electric field, but generate a high actuation strain. In this connection, note that when the actuator is uniaxially pre-stressed, the critical true electric field is low, and the actuation stretch in the direction normal to the pre-stress is large. This trend agrees with experimental observations.[2]

In summary, we have formulated a method to analyze electromechanical stability of dielectric elastomer actuators. While the method is applicable to free-energy function of any form, we have applied the method to the ideal dielectric elastomer. We show that the pre-stress can markedly increase the actuation stretch. This method can be used to guide the design of actuator configurations, as well as the design of actuator materials.

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## Figure Captions

**Figure 1.** A layer of a dielectric elastomer coated with two compliant electrodes, and loaded by a battery of voltage $\Phi$ and by two weights $P_1$ and $P_2$. The loads deform the elastomer from lengths $L_1$, $L_2$ and $L_3$ to lengths $\lambda_1 L_1$, $\lambda_2 L_2$ and $\lambda_3 L_3$, as well as induce an electric charge of magnitude $Q$ on either electrode.

**Figure 2.** The behavior of a dielectric elastomer actuator under several levels of equal biaxial pre-stresses: (a) nominal electric field vs. nominal electric displacement, (b) true electric field vs. nominal electric displacement, and (c) nominal electric field vs. actuation stretch. The critical points for instability are marked as crosses.

**Figure 3.** The effects of unequal biaxial pre-stresses on the critical nominal electric field (a), the critical true electric field (b), and the critical actuation stretches (c, d).

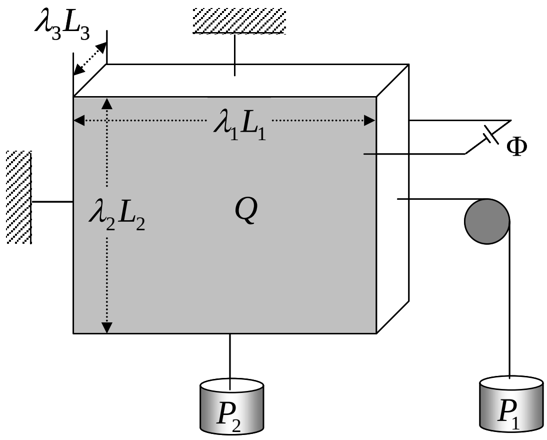


**Figure 1**

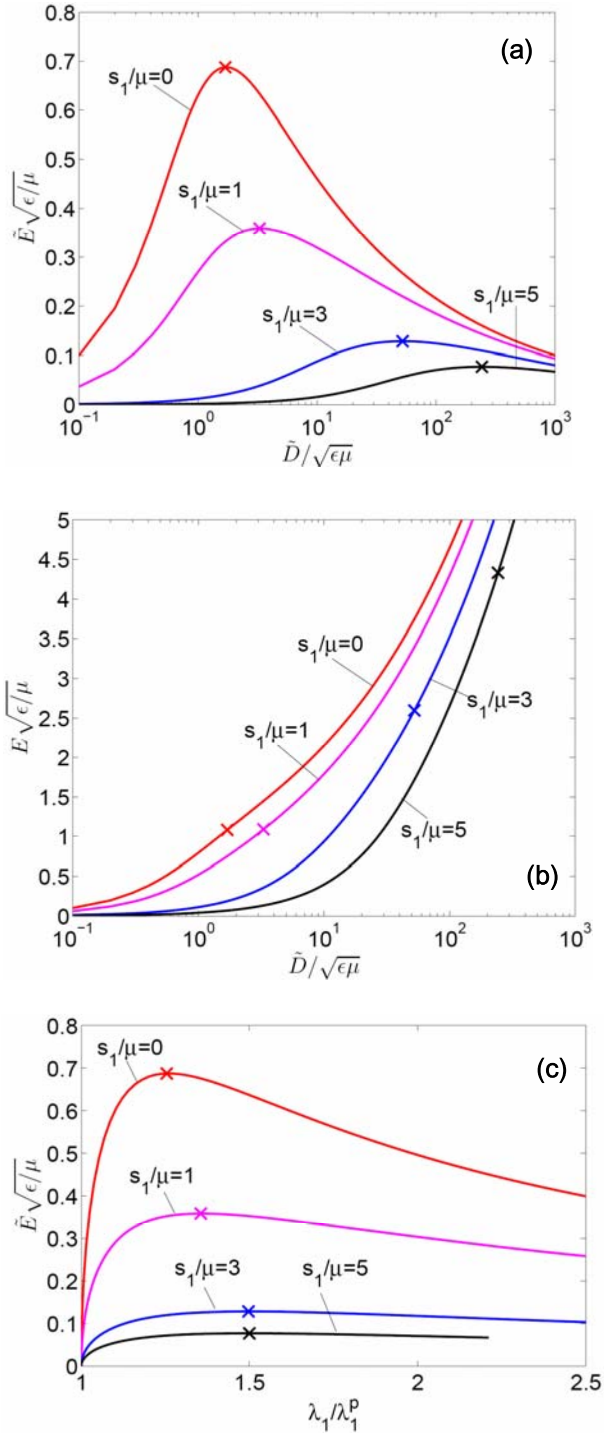
(a)
s1/μ=0
s1/μ=1
s1/μ=3
s1/μ=5
(b)
s1/μ=0
s1/μ=1
s1/μ=3
s1/μ=5
(c)
s1/μ=0
s1/μ=1
s1/μ=3
s1/μ=5

**Figure 2**

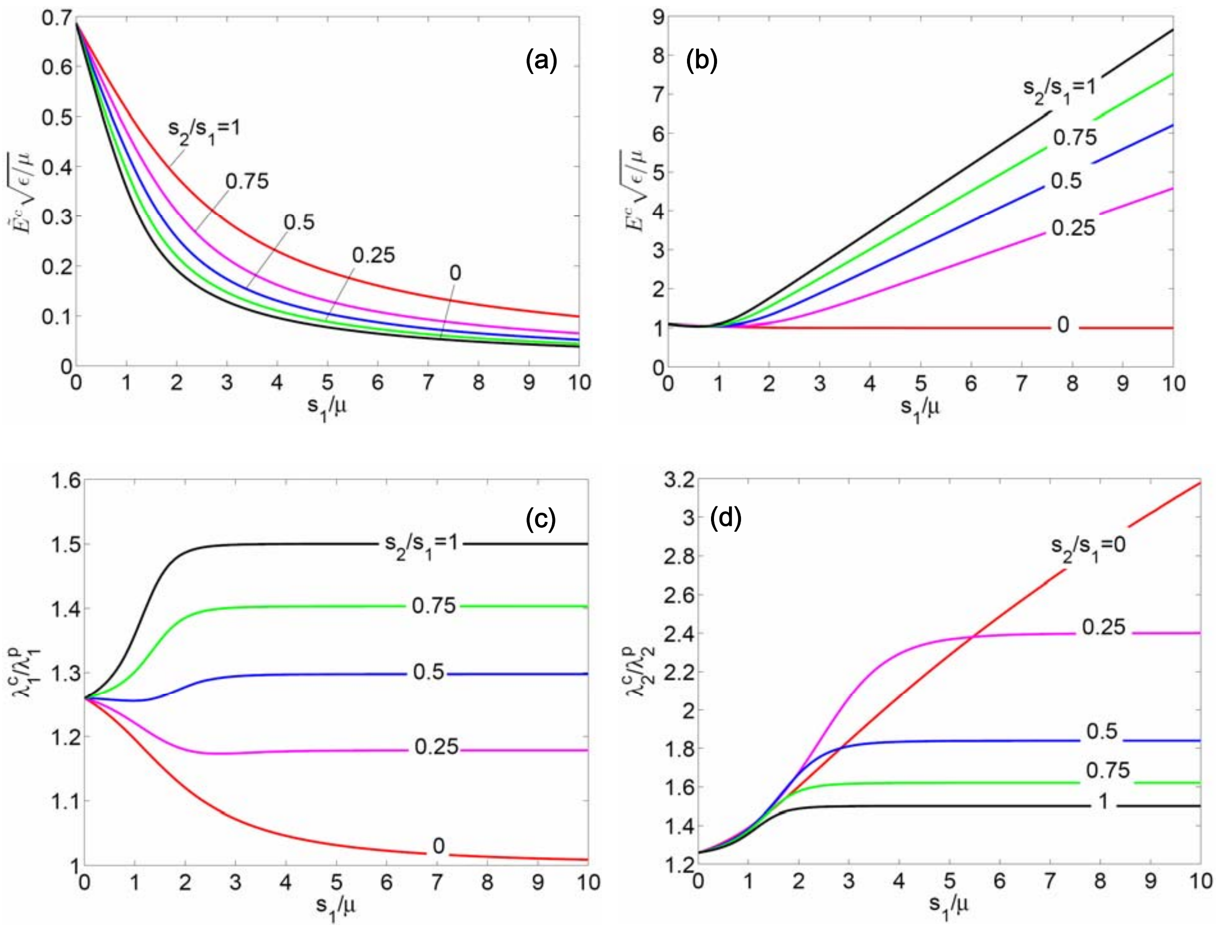
(a)
s_2/s_1=1
0.75
0.5
0.25
0
s_1/μ
(b)
s_2/s_1=1
0.75
0.5
0.25
0
s_1/μ
(c)
s_2/s_1=1
0.75
0.5
0.25
0
s_1/μ
λ_1^c/λ_1^p
(d)
s_2/s_1=0
0.25
0.5
0.75
1
s_1/μ
λ_2^c/λ_2^p

**Figure 3**