

# A precise method for solving wave propagation problems in layered anisotropic media

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Received 15 April 2002; received in revised form 11 July 2003; accepted 19 September 2003

Available online 28 May 2004

## Abstract

A precise integration algorithm is used in combination with the extended Wittrick–Williams algorithm to solve the differential equations governing stationary random wave propagation in layered anisotropic material. The natural frequencies and corresponding modes of vibration are calculated to arbitrary precision. Four examples are given, two of which highlight a phenomenon corresponding to beats.

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**Keywords:** Wittrick–Williams algorithm; Layered material; Precise integration; Wave propagation

## 1. Introduction

The ability to solve wave propagation problems has improved significantly from straightforward applications involving homogeneous material through progressively more demanding situations that necessitate solutions involving layered media. Such development is adequately reflected in [1–6]. Today, the efficient solution of wave propagation problems in layered media has increased importance since it is applicable to a wide range of physical problems ranging from ultrasonic wave propagation in fibre reinforced composite materials to seismic disturbance of stratified soil. Within the latter category, the response of stratified soil overlying rock substrate that is being randomly excited is currently of particular interest.

The technique that is usually adopted for the solution of wave propagation problems is to apply the spectral analysis method and transform the resulting equations to the frequency–wave number domain. As a consequence it becomes necessary to solve only ordinary differential equations with the frequency and wave number regarded as parameters. However, analytical solution is still intractable and resort is made to a matrix approach [7–10] that is similar to the conventional finite element technique. The propagating waves correspond to the eigensolutions of the ordinary differential equations derived from spectral analysis and their efficient calculation lies at the heart of the solution process. The purpose of this paper is therefore to use an efficient and precise integration method [11] in combination with the extended Wittrick–Williams algorithm [12–14] to solve the equations that are derived from

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semi-analytical methods and result in the solution of a two point boundary value problem. The method can also be used to find the eigensolutions to any desired accuracy, subject only to the accuracy of the host computer.

## 2. Basic theory

Fig. 1 shows a layered medium overlying a base layer. The coordinate axes form the right-handed set  $x, y, z$  with the  $z$ -axis positive downwards and  $z = 0$  at the surface. The  $l$  layers are separated by the horizontal planes  $z = z_i$  ( $i = 0, 1, \dots, l$ ), where  $z_i < z_j$ , for  $j > i$  and  $z = z_l$  is the lower boundary with the base layer. The following assumptions are made:

1. Waves are only propagated in the  $x$  direction.
2. The base layer is subjected to random excitation in the  $x$ – $z$  plane alone.
3. The material of each layer is assumed to be anisotropic, homogeneous and of constant depth, but these properties can differ in value from layer to layer.

These assumptions imply a generalised plane strain problem, due to the necessary displacement of the layers in the transverse ( $y$ ) direction following from the assumption of anisotropy. Furthermore, the geometrical and material properties of the layers, as well as the forces to which they are subjected, are independent of the transverse coordinate. Hence the stress, strain and displacement are functions of  $x$  and  $z$  alone.

Consider the problem formulated in the frequency domain, with  $\omega$  denoting the wave frequency. Then, assuming all quantities are multiplied by  $\exp(i\omega t)$ , where  $i = \sqrt{-1}$ , the total displacements  $\hat{u}, \hat{v}, \hat{w}$  along the respective inertial coordinates  $x, y, z$  are given by

$$\hat{u} = \hat{u}(x, z, \omega) + \hat{u}_g(x, \omega), \quad \hat{v} = \hat{v}(x, z, \omega) + \hat{v}_g(x, \omega), \quad \hat{w} = \hat{w}(x, z, \omega) + \hat{w}_g(x, \omega), \quad (2.1)$$

where  $\hat{u}, \hat{v}, \hat{w}$  are the layer displacements and  $\hat{u}_g, \hat{v}_g, \hat{w}_g(x, t)$  are the displacements at the lower boundary emanating from the random excitation of the base layer. The stress–strain and strain–displacement relationships for the generalised plane problem are then given by

$$\begin{Bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_z \\ \hat{\tau}_{xy} \\ \hat{\tau}_{xz} \\ \hat{\tau}_{yz} \end{Bmatrix} = \mathbf{D} \begin{Bmatrix} \hat{\epsilon}_x \\ \hat{\epsilon}_z \\ \hat{\gamma}_{xy} \\ \hat{\gamma}_{xz} \\ \hat{\gamma}_{yz} \end{Bmatrix}, \quad \mathbf{D} = \begin{bmatrix} D_{11} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{31} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}, \quad (2.2)$$

$$\hat{\epsilon}_x = \frac{\partial \hat{u}}{\partial x}, \quad \hat{\epsilon}_y = 0, \quad \hat{\epsilon}_z = \frac{\partial \hat{w}}{\partial z}, \quad \hat{\gamma}_{xy} = \frac{\partial \hat{v}}{\partial x}, \quad \hat{\gamma}_{xz} = \frac{\partial \hat{w}}{\partial x} + \frac{\partial \hat{u}}{\partial z}, \quad \hat{\gamma}_{yz} = \frac{\partial \hat{v}}{\partial z},$$

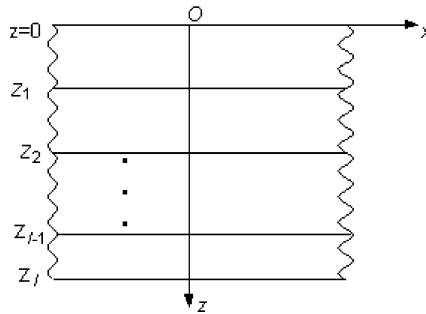


Fig. 1. Schematic diagram of layered medium.

where  $\mathbf{D}$  is the material rigidity matrix, which differs from layer to layer. The total strain, analogous to the total displacements of Eq. (2.1), is then given by

$$\hat{\varepsilon}_x = \varepsilon_x + \hat{\varepsilon}_{xg}, \quad \hat{\varepsilon}_{xg} = \frac{\partial \hat{u}_g}{\partial x}, \quad \hat{\gamma}_{xy} = \gamma_{xy} + \hat{\gamma}_{xyg}, \quad \hat{\gamma}_{xyg} = \frac{\partial \hat{v}_g}{\partial x}, \quad \hat{\gamma}_{xz} = \gamma_{xz} + \hat{\gamma}_{xzg}, \quad \hat{\gamma}_{xzg} = \frac{\partial \hat{w}_g}{\partial x}, \quad (2.3)$$

where the base layer quantities are calculated by inserting appropriate properties into Eq. (2.2). The total stress follows directly as  $\hat{\sigma}_x = \sigma_x + \hat{\sigma}_{xg}$ , etc. The dynamic equations are then given by

$$\begin{aligned} \frac{\partial \hat{\sigma}_x}{\partial x} + \frac{\partial \hat{\tau}_{xz}}{\partial z} + \rho \omega^2 \hat{u} + \hat{f}_x &= 0, & \hat{f}_x &= \rho \omega^2 \hat{u}_g + \frac{\partial \hat{\sigma}_{xg}}{\partial x} + \frac{\partial \hat{\tau}_{xzg}}{\partial z}, \\ \frac{\partial \hat{\tau}_{yz}}{\partial z} + \frac{\partial \hat{\tau}_{yx}}{\partial x} + \rho \omega^2 \hat{v} + \hat{f}_y &= 0, & \hat{f}_y &= \rho \omega^2 \hat{v}_g + \frac{\partial \hat{\tau}_{yzg}}{\partial z} + \frac{\partial \hat{\tau}_{yxg}}{\partial x}, \\ \frac{\partial \hat{\sigma}_z}{\partial z} + \frac{\partial \hat{\tau}_{zx}}{\partial x} + \rho \omega^2 \hat{w} + \hat{f}_z &= 0, & \hat{f}_z &= \rho \omega^2 \hat{w}_g + \frac{\partial \hat{\sigma}_{zg}}{\partial z} + \frac{\partial \hat{\tau}_{zxg}}{\partial x}, \end{aligned} \quad (2.4)$$

where  $\rho$  is the density. The surface boundary conditions are

$$\hat{\sigma}_z + \hat{\sigma}_{zg} = 0, \quad \hat{\tau}_{xz} + \hat{\tau}_{xzg} = 0, \quad \hat{\tau}_{yz} + \hat{\tau}_{yzg} = 0, \quad \text{when } z = 0 \quad (2.5)$$

while the interface boundary conditions are

$$\hat{u}, \hat{v}, \hat{w} \text{ and } \hat{\sigma}_z + \hat{\sigma}_{zg}, \hat{\tau}_{xz} + \hat{\tau}_{xzg}, \hat{\tau}_{yz} + \hat{\tau}_{yzg} \text{ are continuous at } z_i \text{ (} 0 < i < l \text{)} \quad (2.6)$$

and the boundary conditions at the base layer are

$$\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{when } z = z_l \quad (2.7)$$

If now  $\kappa$  is defined as the wave number in the  $x$ -direction, all variables can be expressed as

$$\begin{aligned} \hat{u} &= u(z) \exp(i\kappa x), & \hat{v} &= v(z) \exp(i\kappa x), & \hat{w} &= w(z) \exp(i\kappa x), & \hat{\varepsilon}_x &= \varepsilon_x(z) \exp(i\kappa x), \\ \hat{\sigma}_x &= \sigma_x(z) \exp(i\kappa x), & \hat{\varepsilon}_{xg} &= \varepsilon_{xg} \exp(i\kappa x), & \hat{u}_g &= u_g \exp(i\kappa x), & \hat{v}_g &= v_g \exp(i\kappa x), \\ \hat{w}_g &= w_g \exp(i\kappa x), \dots, \end{aligned} \quad (2.8)$$

where  $u, v, w, \sigma_x, \varepsilon_x$  are functions of  $z$  and  $u_g, v_g, w_g, \varepsilon_{xg}$  are independent of  $z$ . For eigensolutions the parameters  $\omega$  and  $\kappa$  are not mutually independent, so that a value must be assigned to one of them in order to calculate the other. It is easier to calculate  $\omega$  with  $\kappa$  known than vice versa.

### 3. The dual equations

Substituting Eq. (2.8) into Eqs. (2.2) and (2.3) gives

$$\begin{aligned} \hat{\varepsilon}_x &= i\kappa u, & \hat{\varepsilon}_z &= \frac{\partial w}{\partial z}, & \hat{\gamma}_{xy} &= i\kappa v, & \hat{\gamma}_{xz} &= \frac{\partial u}{\partial z} + i\kappa w, & \hat{\gamma}_{yz} &= \frac{\partial v}{\partial z}, & \hat{\varepsilon}_{xg} &= i\kappa u_g, & \hat{\varepsilon}_{zg} &= 0, \\ \hat{\gamma}_{xyg} &= i\kappa v_g, & \hat{\gamma}_{xzg} &= i\kappa w_g, & \hat{\gamma}_{yzg} &= 0. \end{aligned} \quad (3.1)$$

If we now let

$$\boldsymbol{\sigma} = \{ \hat{\sigma}_x \quad \hat{\sigma}_z \quad \hat{\tau}_{xy} \quad \hat{\tau}_{xz} \quad \hat{\tau}_{yz} \}^T, \quad \boldsymbol{\varepsilon} = \{ \hat{\varepsilon}_x \quad \hat{\varepsilon}_z \quad \hat{\gamma}_{xy} \quad \hat{\gamma}_{xz} \quad \hat{\gamma}_{yz} \}^T \quad (3.2)$$

then

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \quad \text{and} \quad \boldsymbol{\sigma}_g = \mathbf{D}\boldsymbol{\varepsilon}_g. \quad (2.2')$$

Substituting Eq. (3.2) into the dynamics equations (2.4) and writing the result in matrix form gives

$$\mathbf{K}_{22}\ddot{\mathbf{q}} + (\mathbf{K}_{21} - \mathbf{K}_{12})\dot{\mathbf{q}} - \mathbf{K}_{11}\mathbf{q} + \mathbf{p}_g = \mathbf{0}, \quad (3.3)$$

where  $\mathbf{q}$  is the displacement vector, the dot notation represents differentiation with respect to  $z$ , it should be noted that  $\dot{\sigma}_{zg} = 0$  etc. and

$$\mathbf{q} = \{u, v, w\}^T, \quad (3.4a)$$

$$\mathbf{p}_g = -\mathbf{K}_{11}\{u_g, v_g, w_g\}^T, \quad (3.4b)$$

$$\begin{aligned} \mathbf{K}_{22} &= \begin{bmatrix} D_{55} & D_{56} & D_{53} \\ D_{65} & D_{66} & D_{63} \\ D_{35} & D_{36} & D_{33} \end{bmatrix}, & \mathbf{K}_{21} &= i\kappa \begin{bmatrix} D_{51} & D_{54} & D_{55} \\ D_{61} & D_{64} & D_{65} \\ D_{31} & D_{34} & D_{35} \end{bmatrix}, \\ \mathbf{K}_{12} &= -i\kappa \begin{bmatrix} D_{51} & D_{54} & D_{55} \\ D_{61} & D_{64} & D_{65} \\ D_{31} & D_{34} & D_{35} \end{bmatrix}^T, & \mathbf{K}_{11} &= \kappa^2 \begin{bmatrix} D_{11} & D_{14} & D_{15} \\ D_{41} & D_{44} & D_{45} \\ D_{51} & D_{54} & D_{55} \end{bmatrix} - \rho\omega^2 \mathbf{I}_3 \end{aligned} \quad (3.5)$$

which are all functions of  $\kappa$  and  $\omega$ . It can be seen that  $\mathbf{K}_{22} = \mathbf{K}_{22}^H$ ,  $\mathbf{K}_{11} = \mathbf{K}_{11}^H$ ,  $\mathbf{K}_{21}^H = \mathbf{K}_{12}$ , where the superscript H means Hermitian transpose.

For ease of solution, the second-order differential equation (3.3) can be transformed into dual form by introducing the dual vector

$$\mathbf{p}' = -(\mathbf{K}_{22}\dot{\mathbf{q}} + \mathbf{K}_{21}\mathbf{q}) \quad (3.6)$$

from which the dual differential equations are derived as

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{D}\mathbf{p}', \quad (3.7a)$$

$$\dot{\mathbf{p}}' = \mathbf{B}\mathbf{q} + \mathbf{C}\mathbf{p}' + \mathbf{p}_g, \quad (3.7b)$$

where

$$\mathbf{A} = -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}, \quad \mathbf{B} = -\mathbf{K}_{11} + \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}, \quad \mathbf{C} = \mathbf{K}_{12}\mathbf{K}_{22}^{-1} \quad \text{and} \quad \mathbf{D} = -\mathbf{K}_{22}^{-1}. \quad (3.8)$$

Clearly  $\mathbf{B}$  and  $\mathbf{D}$  are Hermitian matrices and  $\mathbf{C} = -\mathbf{A}^H$ .

The boundary conditions are

$$\mathbf{q} = \{0, 0, 0\}^T \quad \text{when } z = z_l, \quad (3.9a)$$

and

$$\mathbf{p}' = \mathbf{p}_0 = \{\tau_{xzg}, \tau_{y zg}, \sigma_{zg}\}^T \quad \text{when } z = 0, \quad (3.9b)$$

where  $\tau_{xzg}$ ,  $\tau_{y zg}$  and  $\sigma_{zg}$  can be obtained from Eqs. (3.1) and (2.2').

If we now let  $\mathbf{p}' = \mathbf{p} + \mathbf{p}_0$  then

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{D}\mathbf{p} + \mathbf{D}\mathbf{p}_0, \quad (3.3a')$$

$$\dot{\mathbf{p}} = \mathbf{B}\mathbf{q} + \mathbf{C}\mathbf{p} + (\mathbf{p}_g + \mathbf{C}\mathbf{p}_0) \quad (3.3b')$$

and the boundary conditions become

$$\mathbf{q} = \{0, 0, 0\}^T \quad \text{when } z = z_l \quad (3.9a')$$

and

$$\mathbf{p} = \{0, 0, 0\}^T \quad \text{when } z = 0 \quad (3.9b')$$

and  $\mathbf{q}, \mathbf{p}$  are continuous at an interface.

The dual equation can now be written in state space form as

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v} + \mathbf{s}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \mathbf{s}_q \\ \mathbf{s}_p \end{bmatrix} = \begin{bmatrix} \mathbf{D}\mathbf{p}_0 \\ \mathbf{C}\mathbf{p}_0 + \mathbf{p}_g \end{bmatrix}, \quad (3.10)$$

where  $\mathbf{s}$  is the force vector.

It should be noted that the eigensolutions of the system are independent of the force, but it will be retained for use in the next section, where it will be convenient when dealing with response due to random excitation of the base layer.

#### 4. The interval formulation and matrix differential equations

Let the two stations  $z_a$  and  $z_b$  ( $z_b > z_a$ ) required by the interval formulation of the precise integration method be selected arbitrarily within a layer to form the interval  $[z_a, z_b]$ . If vectors  $\mathbf{q}_a$  and  $\mathbf{p}_b$  are given at the two ends, respectively, and vector  $\mathbf{s}$  is given on the interval, the solution  $\mathbf{q}, \mathbf{p}$  on the interval  $[z_a, z_b]$  is known.

Because the system is linear, the relationship is

$$\mathbf{q}_b = \mathbf{F}\mathbf{q}_a - \mathbf{G}\mathbf{p}_b + \bar{\mathbf{q}}, \quad (4.1a)$$

$$\mathbf{p}_a = \mathbf{Q}\mathbf{q}_a + \mathbf{E}\mathbf{p}_b + \bar{\mathbf{p}} \quad (4.1b)$$

where  $\mathbf{F}, \mathbf{Q}, \mathbf{G}, \mathbf{E}$  are complex matrices to be determined,  $\bar{\mathbf{q}}$  the displacement vector at end  $z_b$ ,  $\bar{\mathbf{p}}$  the force vector at end  $z_a$  when the interval is subjected to force  $\mathbf{s}$ , end  $z_a$  is clamped and end  $z_b$  is free. Differentiating equation (4.1a) and (4.1b) with respect to  $z$  and assuming  $\mathbf{q}_a, \mathbf{p}_b$  are given yields

$$\dot{\mathbf{q}}_b = \dot{\mathbf{F}}\mathbf{q}_a - \dot{\mathbf{G}}\mathbf{p}_b - \mathbf{G}\dot{\mathbf{p}}_b + \dot{\bar{\mathbf{q}}} \quad (4.2a)$$

$$\mathbf{0} = \dot{\mathbf{Q}}\mathbf{q}_a + \dot{\mathbf{E}}\mathbf{p}_b + \mathbf{E}\dot{\mathbf{p}}_b + \dot{\bar{\mathbf{p}}}. \quad (4.2b)$$

The dual equation can then be written as

$$\dot{\mathbf{q}}_b = \mathbf{A}\mathbf{q}_b + \mathbf{D}\mathbf{p}_b + \mathbf{s}_q, \quad (4.3a)$$

$$\dot{\mathbf{p}}_b = \mathbf{B}\mathbf{q}_b + \mathbf{C}\mathbf{p}_b + \mathbf{s}_p. \quad (4.3b)$$

Combining Eqs. (4.2a) and (4.2b) and Eqs. (4.3a) and (4.3b) gives

$$(\dot{\mathbf{F}} - \mathbf{A}\mathbf{F} - \mathbf{G}\mathbf{B}\mathbf{F})\mathbf{q}_a + (-\dot{\mathbf{G}} - \mathbf{D} - \mathbf{G}\mathbf{C} + \mathbf{A}\mathbf{G} + \mathbf{G}\mathbf{B}\mathbf{G})\mathbf{p}_b + (-\mathbf{A} - \mathbf{G}\mathbf{B})\bar{\mathbf{q}} + \dot{\bar{\mathbf{q}}} - \mathbf{G}\mathbf{s}_p - \mathbf{s}_q = \mathbf{0}, \quad (4.4a)$$

$$(\mathbf{E}\mathbf{B}\mathbf{F} + \dot{\mathbf{Q}})\mathbf{q}_a + (-\mathbf{E}\mathbf{B}\mathbf{G} + \dot{\mathbf{E}} + \mathbf{E}\mathbf{C})\mathbf{p}_b + \mathbf{E}\mathbf{B}\bar{\mathbf{q}} + \dot{\bar{\mathbf{p}}} + \mathbf{E}\mathbf{s}_p = \mathbf{0}. \quad (4.4b)$$

Noting that the vectors  $\mathbf{q}_a, \mathbf{p}_b$  are mutually independent yields the equations

$$\dot{\mathbf{F}} = (\mathbf{A} + \mathbf{G}\mathbf{B})\mathbf{F}, \quad (4.5a)$$

$$\dot{\mathbf{E}} = \mathbf{E}(\mathbf{B}\mathbf{G} - \mathbf{C}), \quad (4.5b)$$

$$\dot{\mathbf{G}} = \mathbf{A}\mathbf{G} - \mathbf{G}\mathbf{C} - \mathbf{D} + \mathbf{G}\mathbf{B}\mathbf{G}, \quad (4.5c)$$

$$\dot{\mathbf{Q}} = -\mathbf{E}\mathbf{B}\mathbf{F}, \quad (4.5d)$$

$$\dot{\bar{\mathbf{q}}} = (\mathbf{A} + \mathbf{GB})\bar{\mathbf{q}} + \mathbf{Gs}_p + \mathbf{s}_q, \quad (4.5e)$$

$$\dot{\bar{\mathbf{p}}} = -\mathbf{EB}\bar{\mathbf{q}} - \mathbf{Es}_p. \quad (4.5f)$$

By going to the limit as  $z_b \rightarrow z_a$  the boundary conditions for these equations are

$$\begin{aligned} \mathbf{G}(z_a, z_b) = \mathbf{Q}(z_a, z_b) = \mathbf{0}, \quad \mathbf{F}(z_a, z_b) = \mathbf{E}(z_a, z_b) = \mathbf{I}, \\ \bar{\mathbf{q}}(z_a, z_b) = \bar{\mathbf{p}}(z_a, z_b) = \mathbf{0} \quad \text{when } z_a \rightarrow z_b. \end{aligned} \quad (4.6)$$

## 5. The combination of adjacent intervals

Two adjacent intervals  $[z_a, z_b]$  and  $[z_b, z_c]$ , for which the interval matrices/vectors are  $(\mathbf{F}_1, \mathbf{Q}_1, \mathbf{G}_1, \mathbf{E}_1, \bar{\mathbf{p}}_1, \bar{\mathbf{q}}_1)$  and  $(\mathbf{F}_2, \mathbf{Q}_2, \mathbf{G}_2, \mathbf{E}_2, \bar{\mathbf{p}}_2, \bar{\mathbf{q}}_2)$ , respectively, can be combined to form a longer interval  $[z_a, z_c]$  with interval matrices/vectors  $(\mathbf{F}_c, \mathbf{Q}_c, \mathbf{G}_c, \mathbf{E}_c, \bar{\mathbf{p}}_c, \bar{\mathbf{q}}_c)$ .

Applying Eqs. (4.1a) and (4.1b) to intervals 1 and 2 gives

$$\mathbf{q}_b = \mathbf{F}_1 \mathbf{q}_a - \mathbf{G}_1 \mathbf{p}_b + \bar{\mathbf{q}}_1 \quad \text{for } [z_a, z_b], \quad (5.1a)$$

$$\mathbf{p}_a = \mathbf{Q}_1 \mathbf{q}_a + \mathbf{E}_1 \mathbf{p}_b + \bar{\mathbf{p}}_1 \quad \text{for } [z_a, z_b], \quad (5.1b)$$

$$\mathbf{q}_c = \mathbf{F}_2 \mathbf{q}_b - \mathbf{G}_2 \mathbf{p}_c + \bar{\mathbf{q}}_2 \quad \text{for } [z_b, z_c], \quad (5.2a)$$

$$\mathbf{p}_b = \mathbf{Q}_2 \mathbf{q}_b + \mathbf{E}_2 \mathbf{p}_c + \bar{\mathbf{p}}_2 \quad \text{for } [z_b, z_c] \quad (5.2b)$$

and the requirement is to find the equations of the combined interval in the form

$$\mathbf{q}_c = \mathbf{F}_c \mathbf{q}_a - \mathbf{G}_c \mathbf{p}_c + \bar{\mathbf{q}}_c \quad \text{for } [z_a, z_c], \quad (5.3a)$$

$$\mathbf{p}_a = \mathbf{Q}_c \mathbf{q}_a + \mathbf{E}_c \mathbf{p}_c + \bar{\mathbf{p}}_c \quad \text{for } [z_a, z_c] \quad (5.3b)$$

Solving for the vectors  $\mathbf{q}_b, \mathbf{p}_b$  by using Eqs. (5.1a) and (5.2b) gives

$$\mathbf{q}_b = (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} \mathbf{F}_1 \mathbf{q}_a - (\mathbf{G}_1^{-1} + \mathbf{Q}_2)^{-1} \mathbf{E}_2 \mathbf{p}_c + (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} \bar{\mathbf{q}}_1 - (\mathbf{G}_1^{-1} + \mathbf{Q}_2)^{-1} \bar{\mathbf{p}}_2, \quad (5.4a)$$

$$\mathbf{p}_b = (\mathbf{Q}_2^{-1} + \mathbf{G}_1)^{-1} \mathbf{F}_1 \mathbf{q}_a + (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} \mathbf{E}_2 \mathbf{p}_c + (\mathbf{Q}_2^{-1} + \mathbf{G}_1)^{-1} \bar{\mathbf{q}}_1 + (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} \bar{\mathbf{p}}_2. \quad (5.4b)$$

Substituting these back into Eqs. (5.1b) and (5.2a) gives

$$\mathbf{q}_c = \mathbf{F}_2 (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} \mathbf{F}_1 \mathbf{q}_a - [\mathbf{G}_2 + \mathbf{F}_2 (\mathbf{G}_1^{-1} + \mathbf{Q}_2)^{-1} \mathbf{E}_2] \mathbf{p}_c + \mathbf{F}_2 (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} (\bar{\mathbf{q}}_1 - \mathbf{G}_1 \bar{\mathbf{p}}_2) + \bar{\mathbf{q}}_2, \quad (5.5a)$$

$$\mathbf{p}_a = [\mathbf{Q}_1 + \mathbf{E}_1 (\mathbf{Q}_2^{-1} + \mathbf{G}_1)^{-1} \mathbf{F}_1] \mathbf{q}_a + \mathbf{E}_1 (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} \mathbf{E}_2 \mathbf{p}_c + \mathbf{E}_1 (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} (\mathbf{Q}_2 \bar{\mathbf{q}}_1 + \bar{\mathbf{p}}_2) + \bar{\mathbf{p}}_1. \quad (5.5b)$$

Comparing Eqs. (5.5a) and (5.5b) with Eqs. (5.3a) and (5.3b) gives

$$\mathbf{F}_c = \mathbf{F}_2 (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} \mathbf{F}_1, \quad (5.6a)$$

$$\mathbf{G}_c = \mathbf{G}_2 + \mathbf{F}_2 (\mathbf{G}_1^{-1} + \mathbf{Q}_2)^{-1} \mathbf{E}_2, \quad (5.6b)$$

$$\mathbf{Q}_c = \mathbf{Q}_1 + \mathbf{E}_1 (\mathbf{Q}_2^{-1} + \mathbf{G}_1)^{-1} \mathbf{F}_1, \quad (5.6c)$$

$$\mathbf{E}_c = \mathbf{E}_1 (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} \mathbf{E}_2, \quad (5.6d)$$

$$\bar{\mathbf{q}}_c = \bar{\mathbf{q}}_2 + \mathbf{F}_2 (\mathbf{I} + \mathbf{G}_1 \mathbf{Q}_2)^{-1} (\bar{\mathbf{q}}_1 - \mathbf{G}_1 \bar{\mathbf{p}}_2), \quad (5.6e)$$

$$\bar{\mathbf{p}}_c = \bar{\mathbf{p}}_1 + \mathbf{E}_1 (\mathbf{I} + \mathbf{Q}_2 \mathbf{G}_1)^{-1} (\mathbf{Q}_2 \bar{\mathbf{q}}_1 + \bar{\mathbf{p}}_2). \quad (5.6f)$$

These equations are important for eigenvalue problems or the solution of ordinary differential equations.

## 6. Initialisation of interval matrices

Eqs. (5.6a)–(5.6f) have shown how interval matrices/vectors operate, but so far no interval matrices/vectors have been derived, so that only the system matrices **A**, **B**, **C**, **D** and force vector **s** are available. Therefore, it is now necessary to generate a set of interval matrices/vectors from **A**, **B**, **C**, **D** and **s**.

The present case constitutes a system which is independent of the coordinate  $z$  within each layer of material, which can therefore be considered as an interval. It is then possible to subdivide this interval into any number of equal length subintervals. For convenience, the procedure adopted is a two-step process. The layer thickness  $Dz_i = (z_i - z_{i-1})$ , can be first divided into 64 identical sublayers, so that each one has thickness sub  $Dz_i = Dz_i/64$ . This provides the points at which the elements of subsequent eigenvectors will be calculated if required. Each of these sublayers can then be further divided into  $2^N$  sublayers, each with the extremely small thickness:

$$\tau = \frac{\text{sub } Dz_i}{2^N} = \frac{\text{sub } Dz_i}{1\,048\,576} \quad \text{for } N = 20.$$

For this interval  $\tau$ , the interval matrices/vectors **F**, **Q**, **G**, **E**,  $\bar{\mathbf{p}}$ ,  $\bar{\mathbf{q}}$  can be found as follows. Based on the differential equation (4.5) and using the interval boundary conditions of Eq. (4.6), the Taylor series expansion can be expressed as

$$\mathbf{Q}(\tau) = \boldsymbol{\theta}_1 \tau + \boldsymbol{\theta}_2 \tau^2 + \boldsymbol{\theta}_3 \tau^3 + \boldsymbol{\theta}_4 \tau^4, \quad (6.1a)$$

$$\mathbf{G}(\tau) = \boldsymbol{\gamma}_1 \tau + \boldsymbol{\gamma}_2 \tau^2 + \boldsymbol{\gamma}_3 \tau^3 + \boldsymbol{\gamma}_4 \tau^4, \quad (6.1b)$$

$$\mathbf{F}'(\tau) = \boldsymbol{\varphi}_1 \tau + \boldsymbol{\varphi}_2 \tau^2 + \boldsymbol{\varphi}_3 \tau^3 + \boldsymbol{\varphi}_4 \tau^4, \quad \mathbf{F}(\tau) = \mathbf{I} + \mathbf{F}'(\tau), \quad (6.1c)$$

$$\mathbf{E}'(\tau) = \boldsymbol{\psi}_1 \tau + \boldsymbol{\psi}_2 \tau^2 + \boldsymbol{\psi}_3 \tau^3 + \boldsymbol{\psi}_4 \tau^4, \quad \mathbf{E}(\tau) = \mathbf{I} + \mathbf{E}'(\tau), \quad (6.1d)$$

$$\bar{\mathbf{q}}(\tau) = \mathbf{a}_1 \tau + \mathbf{a}_2 \tau^2 + \mathbf{a}_3 \tau^3 + \mathbf{a}_4 \tau^4, \quad (6.1e)$$

$$\bar{\mathbf{p}}(\tau) = \mathbf{b}_1 \tau + \mathbf{b}_2 \tau^2 + \mathbf{b}_3 \tau^3 + \mathbf{b}_4 \tau^4, \quad (6.1f)$$

where  $\boldsymbol{\theta}_i$ ,  $\boldsymbol{\varphi}_i$ ,  $\boldsymbol{\psi}_i$ ,  $\boldsymbol{\gamma}_i$  ( $i = 1, 2, 3, 4$ ) are all  $2 \times 2$  coefficient matrices and  $\mathbf{a}_i$ ,  $\mathbf{b}_i$  ( $i = 1, 2, 3, 4$ ) are all  $2 \times 1$  coefficient vectors which have yet to be determined. Substituting Eq. (6.1) into Eq. (4.5), and comparing coefficients of various powers of  $\tau$  gives the following equations:

$$\boldsymbol{\theta}_1 = -\mathbf{B}, \quad \boldsymbol{\gamma}_1 = -\mathbf{D}, \quad \boldsymbol{\varphi}_1 = \mathbf{A}, \quad \boldsymbol{\psi}_1 = -\mathbf{C}, \quad \mathbf{a}_1 = \mathbf{s}_q, \quad \mathbf{b}_1 = -\mathbf{s}_p, \quad (6.2)$$

$$\begin{aligned} \boldsymbol{\theta}_2 &= -\frac{1}{2}(\boldsymbol{\psi}_1 \mathbf{B} + \mathbf{B} \boldsymbol{\varphi}_1), & \boldsymbol{\gamma}_2 &= \frac{1}{2}(\mathbf{A} \boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_1 \mathbf{C}), & \boldsymbol{\varphi}_2 &= \frac{1}{2}(\mathbf{A} \boldsymbol{\varphi}_1 + \boldsymbol{\gamma}_1 \mathbf{B}), & \boldsymbol{\psi}_2 &= \frac{1}{2}(\mathbf{B} \boldsymbol{\gamma}_1 - \boldsymbol{\psi}_1 \mathbf{C}), \\ \mathbf{a}_2 &= \frac{1}{2}(\mathbf{A} \mathbf{a}_1 + \boldsymbol{\gamma}_1 \mathbf{s}_p), & \mathbf{b}_2 &= -\frac{1}{2}(\mathbf{B} \mathbf{a}_1 + \boldsymbol{\psi}_1 \mathbf{s}_p), \end{aligned} \quad (6.3)$$

$$\begin{aligned} \boldsymbol{\theta}_3 &= -\frac{1}{3}(\boldsymbol{\psi}_2 \mathbf{B} + \mathbf{B} \boldsymbol{\varphi}_2 + \boldsymbol{\psi}_1 \mathbf{B} \boldsymbol{\varphi}_1), & \boldsymbol{\gamma}_3 &= \frac{1}{3}(\mathbf{A} \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_2 \mathbf{C} + \boldsymbol{\gamma}_1 \mathbf{B} \boldsymbol{\gamma}_1), & \boldsymbol{\varphi}_3 &= \frac{1}{3}(\mathbf{A} \boldsymbol{\varphi}_2 + \boldsymbol{\gamma}_2 \mathbf{B} + \boldsymbol{\gamma}_1 \mathbf{B} \boldsymbol{\varphi}_1), \\ \boldsymbol{\psi}_3 &= \frac{1}{3}(\mathbf{B} \boldsymbol{\gamma}_2 + \boldsymbol{\psi}_1 \mathbf{B} \boldsymbol{\gamma}_1 - \boldsymbol{\psi}_2 \mathbf{C}), & \mathbf{a}_3 &= \frac{1}{3}(\mathbf{A} \mathbf{a}_2 + \boldsymbol{\gamma}_1 \mathbf{B} \mathbf{a}_1 + \boldsymbol{\gamma}_2 \mathbf{s}_p), & \mathbf{b}_3 &= -\frac{1}{3}(\mathbf{B} \mathbf{a}_2 + \boldsymbol{\psi}_1 \mathbf{B} \mathbf{a}_1 + \boldsymbol{\psi}_2 \mathbf{s}_p), \end{aligned} \quad (6.4)$$

$$\begin{aligned} \boldsymbol{\theta}_4 &= -\frac{1}{4}(\boldsymbol{\psi}_3 \mathbf{B} + \mathbf{B} \boldsymbol{\varphi}_3 + \boldsymbol{\psi}_2 \mathbf{B} \boldsymbol{\varphi}_1 + \boldsymbol{\psi}_1 \mathbf{B} \boldsymbol{\varphi}_2), & \boldsymbol{\gamma}_4 &= \frac{1}{4}(\mathbf{A} \boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_3 \mathbf{C} + \boldsymbol{\gamma}_2 \mathbf{B} \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_1 \mathbf{B} \boldsymbol{\gamma}_2), \\ \boldsymbol{\varphi}_4 &= \frac{1}{4}(\mathbf{A} \boldsymbol{\varphi}_3 + \boldsymbol{\gamma}_3 \mathbf{B} + \boldsymbol{\gamma}_2 \mathbf{B} \boldsymbol{\varphi}_1 + \boldsymbol{\gamma}_1 \mathbf{B} \boldsymbol{\varphi}_2), & \boldsymbol{\psi}_4 &= \frac{1}{4}(\mathbf{B} \boldsymbol{\gamma}_3 + \boldsymbol{\psi}_1 \mathbf{B} \boldsymbol{\gamma}_2 + \boldsymbol{\psi}_2 \mathbf{B} \boldsymbol{\gamma}_1 - \boldsymbol{\psi}_3 \mathbf{C}), \\ \mathbf{a}_4 &= \frac{1}{4}(\mathbf{A} \mathbf{a}_3 + \boldsymbol{\gamma}_1 \mathbf{B} \mathbf{a}_2 + \boldsymbol{\gamma}_2 \mathbf{B} \mathbf{a}_1 + \boldsymbol{\gamma}_3 \mathbf{s}_p), & \mathbf{b}_4 &= -\frac{1}{4}(\mathbf{B} \mathbf{a}_3 + \boldsymbol{\psi}_1 \mathbf{B} \mathbf{a}_2 + \boldsymbol{\psi}_2 \mathbf{B} \mathbf{a}_1 + \boldsymbol{\psi}_3 \mathbf{s}_p). \end{aligned} \quad (6.5)$$

These matrices/vectors can be computed successively without iteration. However, the precision must be considered carefully as follows. The Taylor series expansions are truncated after the  $\tau^4$  terms and the first term is of order  $\tau$ .

Therefore, the relative order of the neglected terms is  $\tau^4$ . However, if  $N = 20$ ,  $\tau$  has been divided by rather more than  $10^6$ . Thus,  $\tau^4$  will be of the order of  $10^{-24}$ , which is well beyond the precision of  $10^{-16}$  of a real\*8 word.

All derivations in the previous sections are exact, in the sense that the sole approximation made, namely the truncation of the Taylor series expansion of Eqs. (6.1), has just been shown to cause a numerical error which is less than the round-off error of double precision computation. Therefore, the method is exact in the sense that any method can be exact, i.e. it is as exact as the computer precision permits. However, because  $\mathbf{F} = \mathbf{I} + \mathbf{F}'$  and  $\mathbf{F}'$  is a matrix of small quantities, the addition must not be executed when the interval is very small, as otherwise unnecessary numerical errors would be induced and the exactness lost. Therefore, it is vitally important that only  $\mathbf{F}'$  is generated and stored in the computer memory, and never  $\mathbf{F}$ . Hence it is necessary to replace Eqs. (5.6) by

$$\mathbf{F}'_c = (\mathbf{F}' - \frac{1}{2}\mathbf{GQ})(\mathbf{I} + \mathbf{GQ})^{-1} + (\mathbf{I} + \mathbf{GQ})^{-1}(\mathbf{F}' - \frac{1}{2}\mathbf{GQ}) + \mathbf{F}'(\mathbf{I} + \mathbf{GQ})^{-1}\mathbf{F}', \quad (5.6a')$$

$$\mathbf{G}_c = \mathbf{G} + (\mathbf{I} + \mathbf{F}')(\mathbf{G}^{-1} + \mathbf{Q})^{-1}(\mathbf{I} + \mathbf{E}'), \quad (5.6b')$$

$$\mathbf{Q}_c = \mathbf{Q} + (\mathbf{I} + \mathbf{E}')(\mathbf{Q}^{-1} + \mathbf{G})^{-1}(\mathbf{I} + \mathbf{F}'), \quad (5.6c')$$

$$\mathbf{E}'_c = (\mathbf{F}' - \frac{1}{2}\mathbf{GQ})(\mathbf{I} + \mathbf{GQ})^{-1} + (\mathbf{I} + \mathbf{GQ})^{-1}(\mathbf{F}' - \frac{1}{2}\mathbf{GQ}) + \mathbf{F}'(\mathbf{I} + \mathbf{GQ})^{-1}\mathbf{F}', \quad (5.6d')$$

$$\bar{\mathbf{q}}_c = \bar{\mathbf{q}}_2 + (\mathbf{I} + \mathbf{F}')(\mathbf{I} + \mathbf{GQ})^{-1}(\bar{\mathbf{q}}_1 - \mathbf{G}\bar{\mathbf{p}}_2), \quad (5.6e')$$

$$\bar{\mathbf{p}}_c = \bar{\mathbf{p}}_1 + (\mathbf{I} + \mathbf{E}')(\mathbf{I} + \mathbf{QG})^{-1}(\mathbf{Q}\bar{\mathbf{q}}_1 + \bar{\mathbf{p}}_2). \quad (5.6f')$$

## 7. The $2^N$ algorithm

After generation of the interval matrices/vectors  $\mathbf{F}(\tau)$ ,  $\mathbf{Q}(\tau)$ ,  $\mathbf{G}(\tau)$ ,  $\bar{\mathbf{p}}(\tau)$ ,  $\bar{\mathbf{q}}(\tau)$  of length  $\tau$ , combination must be used to obtain the matrices/vectors  $\mathbf{F}'(\text{sub } Dz_i)$ ,  $\mathbf{Q}(\text{sub } Dz_i)$ ,  $\mathbf{G}(\text{sub } Dz_i)$ ,  $\bar{\mathbf{p}}(\text{sub } Dz_i)$ ,  $\bar{\mathbf{q}}(\text{sub } Dz_i)$  of the given interval  $\text{sub } Dz_i$ . There are  $2^N = 1\,048\,576$  for  $N = 20$ , small  $\tau$  intervals. Since all these small intervals are identical, the interval combination of Eqs. (5.6) can be applied with  $\mathbf{Q}_1 = \mathbf{Q}_2$ , etc., and the combination of the  $2^N$  identical intervals requires  $N$  ‘doubling up’ executions of Eqs. (5.6) as given by the instructions

$$\begin{aligned} &\{\mathbf{F}'(\tau), \mathbf{Q}(\tau), \mathbf{G}(\tau), \bar{\mathbf{p}}(\tau), \bar{\mathbf{q}}(\tau)\} \text{ generated by Eq. (6.1)} \\ &\text{for (itera = 0; itera < N; itera++) \{ } \\ &\quad \{\text{The execution of Eqs. (5.6a'–f')}\} \\ &\quad \mathbf{Q} = \mathbf{Q}_c; \mathbf{G} = \mathbf{G}_c; \mathbf{F}' = \mathbf{F}'_c; \mathbf{E}' = \mathbf{E}'_c; \bar{\mathbf{p}}_1 = \bar{\mathbf{p}}_c; \bar{\mathbf{q}}_2 = \bar{\mathbf{q}}_c \\ &\quad \} \\ &\quad \{\mathbf{Q}(\text{sub } Dz_i) = \mathbf{Q}_c; \mathbf{G}(\text{sub } Dz_i) = \mathbf{G}_c; \mathbf{F}(\text{sub } Dz_i) = \mathbf{I} + \mathbf{F}'_c; \\ &\quad \mathbf{E}(\text{sub } Dz_i) = \mathbf{I} + \mathbf{E}'_c; \bar{\mathbf{p}}_1(\text{sub } Dz_i) = \bar{\mathbf{p}}_c; \bar{\mathbf{q}}_2(\text{sub } Dz_i) = \bar{\mathbf{q}}_c\} \end{aligned} \quad (7.1)$$

The algorithm (7.1) gives the computation for a sublayer of thickness  $\text{sub } Dz_i$ . However, it is required further to find the layer interval matrices/vectors  $\mathbf{Q}(Dz_i)$ ;  $\mathbf{G}(Dz_i)$ ;  $\mathbf{F}(Dz_i)$ ;  $\mathbf{E}(Dz_i)$ ;  $\bar{\mathbf{p}}(Dz_i)$ ;  $\bar{\mathbf{q}}(Dz_i)$ . The reason for setting up a sublayer of thickness  $\text{sub } Dz_i$  is that if the thickness of say, the  $i$ th layer,  $Dz_i = (z_i - z_{i-1})$ , is too large, a wave could possibly occur within it. The division into 64 small sublayers ensures that internal waves are impossible, i.e. for each such sublayer its eigenvalue count  $J_m(\omega) = 0$ . Subsequent combinations of these layers involves keeping track of the eigenvalue count by using the equation

$$J_{mc}(\omega) = J_{m1}(\omega) + J_{m2}(\omega) - s\{\mathbf{G}_1\} + s\{(\mathbf{G}_1^{-1} + \mathbf{Q}_2)\}, \quad (7.2)$$

where  $s\{\cdots\}$  is the sign count of the matrix within the brackets. Eq. (7.2) is a necessary complement to Eqs. (5.6).



The following instructions give the computation of layer interval matrices/vectors

$$\begin{aligned}
 & \mathbf{Q}(\mathbf{D}z_i); \mathbf{G}(\mathbf{D}z_i); \mathbf{F}(\mathbf{D}z_i); \mathbf{E}(\mathbf{D}z_i); \bar{\mathbf{p}}_1(\mathbf{D}z_i); \bar{\mathbf{q}}_2(\mathbf{D}z_i) \\
 & \{\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}(\text{sub } \mathbf{D}z); \quad \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}(\text{sub } \mathbf{D}z); \quad \mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}(\text{sub } \mathbf{D}z) \\
 & \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{E}(\text{sub } \mathbf{D}z); \quad \bar{\mathbf{p}}_1 = \bar{\mathbf{p}}_1(\text{sub } \mathbf{D}z); \quad \bar{\mathbf{q}}_2 = \bar{\mathbf{q}}_2(\text{sub } \mathbf{D}z); \quad J_{m1} = J_{m2} = 0\} \\
 & \text{for (itera = 0; itera < 6; itera ++)} \{ \\
 & \quad \{\text{The execution of Eqs. (5.6) and (7.2)}\} \\
 & \quad \mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}_c; \mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}_c; \mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_c; \\
 & \quad \mathbf{E}_1 = \mathbf{E}_2 = \mathbf{E}_c; \bar{\mathbf{p}}_1 = \bar{\mathbf{p}}_c; \bar{\mathbf{q}}_2 = \bar{\mathbf{q}}_c; J_{m1} = J_{m2} = J_{mc} \\
 & \} \\
 & \{\mathbf{Q}(\mathbf{D}z) = \mathbf{Q}_c; \mathbf{G}(\mathbf{D}z) = \mathbf{G}_c; \mathbf{F}(\mathbf{D}z) = \mathbf{F}_c; \\
 & \quad \mathbf{E}(\mathbf{D}z) = \mathbf{E}_c; \bar{\mathbf{p}}_1(\mathbf{D}z) = \bar{\mathbf{p}}_c; \bar{\mathbf{q}}_2(\mathbf{D}z) = \bar{\mathbf{q}}_c; J_m = J_{mc}\}
 \end{aligned} \tag{7.3}$$

The algorithm for combining the matrices/vectors of all the intervals into one overall interval is similar. Let  $z = 0$  be treated as end  $a$ , then the algorithm is

$$\begin{aligned}
 & \text{for (layer = 1; layer } \leq \text{layers; layer ++)} \{ \\
 & \quad \text{if (layer == 1)} \{ \\
 & \quad \quad \mathbf{Q}_c = \mathbf{Q}(\text{layer}); \mathbf{G}_c = \mathbf{G}(\text{layer}); \mathbf{F}_c = \mathbf{F}(\text{layer}); \\
 & \quad \quad \mathbf{E}_c = \mathbf{E}(\text{layer}); \bar{\mathbf{p}}_c = \bar{\mathbf{p}}(\text{layer}); \bar{\mathbf{q}}_c = \bar{\mathbf{q}}(\text{layer}); J_{mc} = J_{m1} \\
 & \quad \} \\
 & \quad \text{else } \{\mathbf{Q}_1 = \mathbf{Q}(\text{layer}); \mathbf{G}_1 = \mathbf{G}(\text{layer}); \mathbf{F}_1 = \mathbf{F}(\text{layer}); \\
 & \quad \quad \mathbf{E}_1 = \mathbf{E}(\text{layer}); \bar{\mathbf{p}}_1 = \bar{\mathbf{p}}(\text{layer}); \bar{\mathbf{q}}_1 = \bar{\mathbf{q}}(\text{layer}); J_{m1} = J_{m1} \\
 & \quad \quad \{\mathbf{Q}_2 = \mathbf{Q}_c; \mathbf{G}_2 = \mathbf{G}_c; \mathbf{F}_2 = \mathbf{F}_c; \\
 & \quad \quad \mathbf{E}_2 = \mathbf{E}_c; \bar{\mathbf{p}}_2 = \bar{\mathbf{p}}_c; \bar{\mathbf{q}}_2 = \bar{\mathbf{q}}_c; J_{m2} = J_{mc}; \} \\
 & \quad \quad \{\text{The execution of Eqs. (5.6) and (7.2); } \\
 & \quad \} \\
 & \}
 \end{aligned} \tag{7.4}$$

So far we have  $\mathbf{Q}_c, \mathbf{G}_c, \mathbf{F}_c, \mathbf{E}_c, \bar{\mathbf{p}}_c, \bar{\mathbf{q}}_c, J_{mc}$ , the overall interval matrices/vectors and EC, where EC stands for the eigenvalue count and  $J_{m1}$  means the EC for the layer. From the EC computation in combination with a binary search, one can find all the eigenvalues without ever missing any.

In order to maintain the mode shape, consider the dual equation (3.10). Its homogeneous equations are

$$\dot{\mathbf{v}}_r = \mathbf{H}_r \mathbf{v}_r, \quad \text{where } \mathbf{H}_r = \begin{bmatrix} \mathbf{A}_r & \mathbf{D}_r \\ \mathbf{B}_r & \mathbf{C}_r \end{bmatrix}, \quad \mathbf{v}_r = \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{p}_r \end{Bmatrix} \tag{7.5}$$

and whose solution is

$$\mathbf{v}_r = \exp(\mathbf{H}_r \cdot z) \mathbf{v}_{0,r-1} \quad z \in [z_{i-1}, z_i], \tag{7.6}$$

where  $\mathbf{v}_r$  is the state variable defined for the  $r$ th layer and  $\mathbf{v}_{0,r-1}$  is the value of  $\mathbf{v}_r$  at  $z = z_{r-1}$ .

For given  $\omega_l^2$  and making use of Eqs. (7.5) and (7.6), the state variable  $\mathbf{v}_{0,l}^i$  at  $z = l$  can be expressed in state variable form  $\mathbf{v}_{0,0}^i$  at  $z = 0$  as

$$\begin{aligned}
 \mathbf{v}_{0,l} &= \exp(\mathbf{H}_l \cdot h_l) \times \exp(\mathbf{H}_{l-1} \cdot h_{l-1}) \times \cdots \times \exp(\mathbf{H}_1 \cdot h_1) \mathbf{v}_{0,0} = \bar{\mathbf{H}} \mathbf{v}_{0,0} \\
 \bar{\mathbf{H}} &= \exp(\mathbf{H}_l \cdot h_l) \times \exp(\mathbf{H}_{l-1} \cdot h_{l-1}) \times \cdots \times \exp(\mathbf{H}_1 \cdot h_1),
 \end{aligned} \tag{7.7}$$

where  $h_r$  ( $r = 1, 2, \dots, l$ ) is the thickness of the  $r$ th layer and  $i$  represents the  $i$ th eigensolution. The matrix  $\bar{\mathbf{H}}$  can be obtained using the precise integration method. Let  $\bar{\mathbf{H}}$  be the terms that lie in the first three rows and first three columns of  $\bar{\mathbf{H}}$ , then considering the boundary to be clamped at  $z = z_l$  and free at  $z = 0$  gives

$$\bar{\mathbf{H}}\mathbf{q}_{i,0} = \mathbf{0}, \quad (7.8)$$

where  $\mathbf{q}_{i,0}$  is the value of the  $i$ th mode shape  $\mathbf{q}_i$  at  $z = 0$ . From Eq. (7.8) and noting that  $\mathbf{p}_{i,0} = \mathbf{0}$ , we can obtain the state vector  $\mathbf{v}_{0,0}^i$  at  $z = 0$  when the eigenvalue is  $\omega_i^2$  and the analytical mode shape follows from Eq. (5.5).

Let  $\mathbf{H}_r = \mathbf{M}_r \text{diag}(\lambda_r) \mathbf{M}_r^{-1}$  and  $\mathbf{M}_{qr}$  = the first three rows of  $\mathbf{M}_r$ , then

$$\mathbf{v}_r^i = \mathbf{M}_r^i \times \text{diag}[\exp(\lambda_r^i \cdot z)] \times (\mathbf{M}_r^i)^{-1} \mathbf{v}_{0,r-1}^i \quad (7.9)$$

and

$$\mathbf{q}_{r,i} = \mathbf{M}_{q,r}^i \times \text{diag}[\exp(\lambda_r^i \cdot z)] \times (\mathbf{M}_r^i)^{-1} \mathbf{v}_{0,r-1}^i. \quad (7.10)$$

Thus if we know the state vector  $\mathbf{v}_{0,0}^i$  at  $z = 0$ , we can use Eqs. (7.9) and (7.10) to express the mode shape analytically. In this paper we will present the  $\omega_i^2$  and  $\mathbf{v}_{0,0}^i$ .

For stationary excitations one can use the vectors  $\bar{\mathbf{p}}_c, \bar{\mathbf{q}}_c$  to obtain the power spectral density of the ground; for non-stationary excitations it is necessary to solve time-dependent partial differential equations, after which the eigensolutions obtained from this paper can be used in conjunction with the mode shape superposition method to solve for the power spectral density of the ground. However, these problems are beyond the scope of the current paper.

## 8. Numerical examples

Four examples are now given to illustrate the theory and computations. The first two are straightforward while the last two illustrate a phenomenon corresponding to beats.

The first example comprises two layers of material of equal thickness, while the second comprises three layers of variable thickness. In both these examples the data are given non-dimensionally and the wave number is  $\kappa = 0.02$ . Solutions are given according to the assumptions of anisotropic, orthotropic and isotropic material properties and the material rigidity matrix,  $\mathbf{D}$ , given for each case should be multiplied by  $10^7$ .

**Example 1** (Two layer problem). The data for anisotropic, orthotropic and isotropic material properties are given in Tables 1–3, respectively, in each case followed by the corresponding results.

Table 1  
Parameters for anisotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	1.0	0.2	0.6	2.0	1.2	0.2	0.6	1.4	0.4	0.2	0.8	0.6	2.0	200	2000
2	1.4	0.5	0.5	0.1	0.4	1.0	0.8	0.1	0.3	0.8	0.1	0.4	0.3	0.2	1.0	200	2000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned} \omega_1^2 &= 0.56771078, & \mathbf{v}_{0,0}^1 &= \{1.0000; -1.0189 + 1.0314i; 0.6441 - 1.5332i; 0; 0; 0\}^T, \\ \omega_2^2 &= 0.94576844, & \mathbf{v}_{0,0}^2 &= \{1.0000; -0.2411 + 0.4765i; 0.3328 - 1.1797i; 0; 0; 0\}^T, \\ \omega_3^2 &= 1.1943356, & \mathbf{v}_{0,0}^3 &= \{1.0000; -0.7515 - 0.1938i; 0.3487 - 0.9319i; 0; 0; 0\}^T, \\ \omega_4^2 &= 1.3675573, & \mathbf{v}_{0,0}^4 &= \{1.0000; -0.1124 - 0.7611i; -0.5845 - 0.6440i; 0; 0; 0\}^T, \\ \omega_5^2 &= 1.7488804, & \mathbf{v}_{0,0}^5 &= \{1.0000; -0.9999 - 0.7494i; -1.5415 - 0.4247i; 0; 0; 0\}^T. \end{aligned}$$

Table 2  
Parameters for orthotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	0.0	0.0	0.0	2.0	0.0	0.0	0.0	1.4	0.0	0.0	0.8	0.0	2.0	200	2000
2	1.4	0.5	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.8	0.0	0.0	0.3	0.0	1.0	200	2000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned}
 \omega_1^2 &= 1.0242032, & \mathbf{v}_{0,0}^1 &= \{1.0000; 0; 0 - 0.9666i; 0; 0; 0\}^T, \\
 \omega_2^2 &= 1.1415283, & \mathbf{v}_{0,0}^2 &= \{1.0000; 0; 0 - 1.3310i; 0; 0; 0\}^T, \\
 \omega_3^2 &= 1.6744677, & \mathbf{v}_{0,0}^3 &= \{1.0000; 0; 0 - 4.0391i; 0; 0; 0\}^T, \\
 \omega_4^2 &= 2.3072738, & \mathbf{v}_{0,0}^4 &= \{0; 1; 0; 0; 0; 0\}^T, \\
 \omega_5^2 &= 2.4597761, & \mathbf{v}_{0,0}^5 &= \{1.0000; 0; 0 + 3.5706i; 0; 0; 0\}^T.
 \end{aligned}$$

Table 3  
Parameters for isotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	0.0	0.0	0.0	2.0	0.0	0.0	0.0	0.5	0.0	0.0	0.5	0.0	0.5	200	2000
2	1.4	0.35	0.0	0.0	0.0	1.4	0.0	0.0	0.0	0.0935	0.0	0.0	0.0935	0.0	0.0935	200	2000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned}
 \omega_1^2 &= 1.0000000, & \mathbf{v}_{0,0}^1 &= \{1.0000; 0; 0 - 1.7181i; 0; 0; 0\}^T, \\
 \omega_2^2 &= 1.0482336, & \mathbf{v}_{0,0}^2 &= \{0; 1.0000; 0; 0; 0; 0\}^T, \\
 \omega_3^2 &= 1.3839699, & \mathbf{v}_{0,0}^3 &= \{0; 1.0000; 0; 0; 0; 0\}^T, \\
 \omega_4^2 &= 1.4111748, & \mathbf{v}_{0,0}^4 &= \{1.0000; 0; 0 - 2.7737i; 0; 0; 0\}^T, \\
 \omega_5^2 &= 2.0122216, & \mathbf{v}_{0,0}^5 &= \{0; 1.0000; 0; 0; 0; 0\}^T.
 \end{aligned}$$

**Example 2** (Three layer problem). The data for anisotropic, orthotropic and isotropic material properties are given in Tables 4–6, respectively, in each case followed by the corresponding results.

Table 4  
Parameters for anisotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	1.0	0.2	0.6	2.0	1.2	0.2	0.6	1.4	0.4	0.2	0.8	0.6	2.0	200	2000
2	1.4	0.5	0.5	0.1	0.4	1.0	0.8	0.1	0.3	0.8	0.1	0.4	0.3	0.2	1.0	200	2000
3	2.0	1.2	1.5	0.4	0.3	1.5	0.8	0.1	0.4	1.2	0.2	0.1	0.5	0.3	1.0	100	3000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned}
 \omega_1^2 &= 0.47829529, & \mathbf{v}_{0,0}^1 &= \{1.0000; -0.9262 + 1.3731i; 0.5452 - 1.6817i; 0; 0; 0\}^T, \\
 \omega_2^2 &= 0.56397667, & \mathbf{v}_{0,0}^2 &= \{1.0000; -1.0278 + 0.9654i; 0.6581 - 1.5101i; 0; 0; 0\}^T, \\
 \omega_3^2 &= 0.79006273, & \mathbf{v}_{0,0}^3 &= \{1.0000; -0.6321 + 1.1292i; 0.4757 - 1.4686i; 0; 0; 0\}^T, \\
 \omega_4^2 &= 1.0785011, & \mathbf{v}_{0,0}^4 &= \{1.0000; -0.4947 - 0.0668i; 0.3618 - 0.9770i; 0; 0; 0\}^T, \\
 \omega_5^2 &= 1.3237950, & \mathbf{v}_{0,0}^5 &= \{1.0000; -0.4453 - 0.3584i; -0.1105 - 0.8955i; 0; 0; 0\}^T.
 \end{aligned}$$

Table 5  
Parameters for orthotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	0.0	0.0	0.0	2.0	0.0	0.0	0.0	1.4	0.0	0.0	0.8	0.0	2.0	200	2000
2	1.4	0.5	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.8	0.0	0.0	0.3	0.0	1.0	200	2000
3	2.0	1.2	0.0	0.0	0.0	1.5	0.0	0.0	0.0	1.2	0.0	0.0	0.5	0.0	1.0	100	3000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned}
 \omega_1^2 &= 0.78744368, & \mathbf{v}_{0,0}^1 &= \{1.0000; 0; 0 - 0.3142i; 0; 0; 0\}^T, \\
 \omega_2^2 &= 1.0811937, & \mathbf{v}_{0,0}^2 &= \{1.0000; 0; 0 - 1.1405i; 0; 0; 0\}^T, \\
 \omega_3^2 &= 1.2549827, & \mathbf{v}_{0,0}^3 &= \{1.0000; 0; 0 - 1.7099i; 0; 0; 0\}^T, \\
 \omega_4^2 &= 1.6871181, & \mathbf{v}_{0,0}^4 &= \{1.0000; 0; 0 - 4.3072i; 0; 0; 0\}^T, \\
 \omega_5^2 &= 1.9694166, & \mathbf{v}_{0,0}^5 &= \{0; 1; 0; 0; 0; 0\}^T.
 \end{aligned}$$

Table 6  
Parameters for isotropic material

Layer	$D_{11}$	$D_{13}$	$D_{14}$	$D_{15}$	$D_{16}$	$D_{33}$	$D_{34}$	$D_{35}$	$D_{36}$	$D_{44}$	$D_{45}$	$D_{46}$	$D_{55}$	$D_{56}$	$D_{66}$	Thickness	$\rho$
1	2.0	1.0	0.0	0.0	0.0	2.0	0.0	0.0	0.0	0.5	0.0	0.0	0.5	0.0	0.5	200	2000
2	1.4	0.35	0.0	0.0	0.0	1.4	0.0	0.0	0.0	0.0935	0.0	0.0	0.0935	0.0	0.0935	200	2000
3	2.0	1.2	0.0	0.0	0.0	2.0	0.0	0.0	0.0	1.2	0.0	0.0	1.2	0.0	1.2	100	3000

The first five eigenvalues and mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$

$$\begin{aligned}
 \omega_1^2 &= 1.0000000, & \mathbf{v}_{0,0}^1 &= \{1.0000; 0; 0 - 1.7285i; 0; 0; 0\}^T, \\
 \omega_2^2 &= 1.0432632, & \mathbf{v}_{0,0}^2 &= \{0; 1.0000; 0; 0; 0; 0\}^T, \\
 \omega_3^2 &= 1.3161562, & \mathbf{v}_{0,0}^3 &= \{0; 1.0000; 0; 0; 0; 0\}^T, \\
 \omega_4^2 &= 1.3372344, & \mathbf{v}_{0,0}^4 &= \{1.0000; 0; 0 - 2.4862i; 0; 0; 0\}^T, \\
 \omega_5^2 &= 1.7694160, & \mathbf{v}_{0,0}^5 &= \{0; 1.0000; 0; 0; 0; 0\}^T.
 \end{aligned}$$

In the following two examples we will demonstrate that a relationship can exist between different eigensolutions that corresponds to ‘beats’ in vibration problems. The conditions for this to occur in structural dynamics would typically involve two weakly coupled systems vibrating with nearly equal frequencies.

In the present case, any displacement in the  $u, w$  plane will not be coupled to displacements in the  $v$ -direction when the material is isotropic or orthotropic. However, it will be shown in [Example 3](#) that even when a small level of anisotropy is present it is possible to develop the phenomenon corresponding to beats. [Example 4](#) goes on to show that this phenomenon can also be generated between the components of  $u$  and  $w$ , i.e. the energy in the  $u$  component of the mode can be transferred to the  $w$  component. This can be of particular importance since many slender structures are damaged primarily due to the vertical ground motion  $w$ .

It will be assumed in both of the following examples that a satisfactory solution can be obtained from the first three eigensolutions. Thus

$$\begin{aligned}
 \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} &= \mathbf{v}_{0,0}^1 \times [a_1 \exp(i\omega_1 t) + b_1 \exp(-i\omega_1 t)] + \mathbf{v}_{0,0}^2 \times [a_2 \exp(i\omega_2 t) + b_2 \exp(-i\omega_2 t)] \\
 &\quad + \mathbf{v}_{0,0}^3 \times [a_3 \exp(i\omega_3 t) + b_3 \exp(-i\omega_3 t)].
 \end{aligned}$$

If the initial conditions are

$$\{u; v; w\}^T = y_0; \{\dot{u}; \dot{v}; \dot{w}\}^T = \{0; 0; 0\}$$

then

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [a_1 \times \mathbf{v}_{0,0}^1 \times \cos(\omega_1 t) + a_2 \times \mathbf{v}_{0,0}^2 \times \cos(\omega_2 t) + a_3 \times \mathbf{v}_{0,0}^3 \times \cos(\omega_3 t)],$$

where

$$a_1 = \frac{(\mathbf{v}_{0,0}^1)^T y_0}{(\mathbf{v}_{0,0}^1)^T (\mathbf{v}_{0,0}^1)}, \quad a_2 = \frac{(\mathbf{v}_{0,0}^2)^T y_0}{(\mathbf{v}_{0,0}^2)^T (\mathbf{v}_{0,0}^2)}, \quad a_3 = \frac{(\mathbf{v}_{0,0}^3)^T y_0}{(\mathbf{v}_{0,0}^3)^T (\mathbf{v}_{0,0}^3)}.$$

**Example 3.** The parameters of the material are: wave number  $\kappa = 0.0 \text{ m}^{-1}$ , density  $\rho = 3000 \text{ kg/m}^3$  and the soil comprises a single layer of thickness  $100.0 \text{ m}$ . The resulting matrix,  $\mathbf{D}$ , is predominantly orthotropic, but exhibits weak anisotropy

$$\mathbf{D} = \begin{bmatrix} 0.7 & 0.5 & 0.006 & 0.012 & 0.006 \\ 0.5 & 0.8 & 0.018 & 0.006 & 0.006 \\ 0.006 & 0.018 & 0.4 & 0.006 & 0.012 \\ 0.012 & 0.006 & 0.006 & 0.3 & 0.006 \\ 0.006 & 0.006 & 0.012 & 0.006 & 0.3 \end{bmatrix} \times 10^9.$$

The first three eigenvalues and corresponding mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned} \omega^2 &= 24.18053; 25.15551; 65.80934, & \mathbf{v}_{0,0}^1 &= \{1.82574; -1.82574; 0.0\}^T \times 10^{-3}, \\ \mathbf{v}_{0,0}^2 &= \{1.82547; 1.82547; -0.04433\}^T \times 10^{-3}, & \mathbf{v}_{0,0}^3 &= \{3.13457; 3.13463; 258.16083\}^T \times 10^{-5}. \end{aligned}$$

The displacement components  $u$ ,  $v$  and  $w$  are plotted against time,  $t$ , in Figs. 2–7, assuming initial conditions  $y_0 = \{1; 0; 0\}^T$ . Figs. 2, 4 and 6 are the results when  $\mathbf{D}$  is anisotropic and Figs. 3, 5 and 7 are the results when  $\mathbf{D}$  is orthotropic. For the orthotropic case,  $u$  does not couple with  $v$  or  $w$ , which remain zero due to the initial conditions. However, when a small level of anisotropy is present, the components  $u$  and  $v$  couple to generate the beat-like phenomenon shown in Figs. 2 and 4. It can be argued physically that the reason for such coupling is due to internal

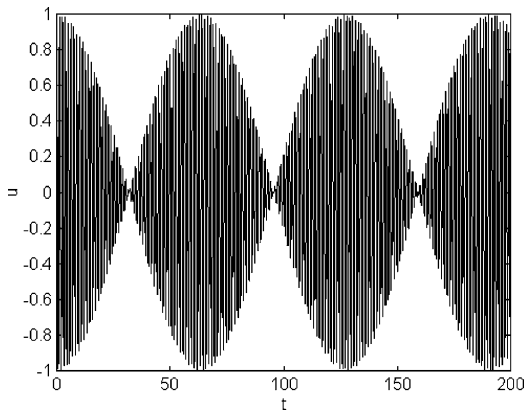


Fig. 2.

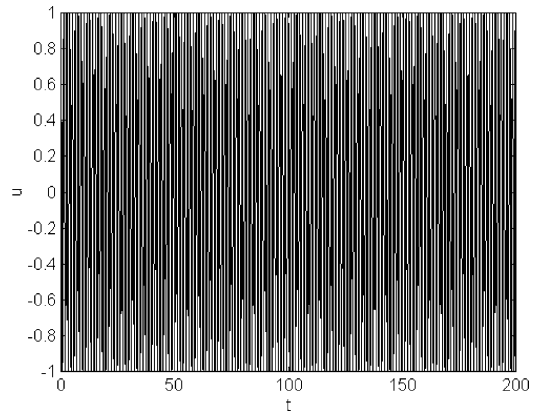


Fig. 3.

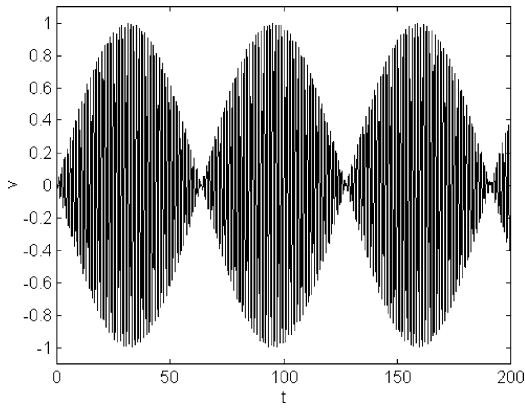


Fig. 4.

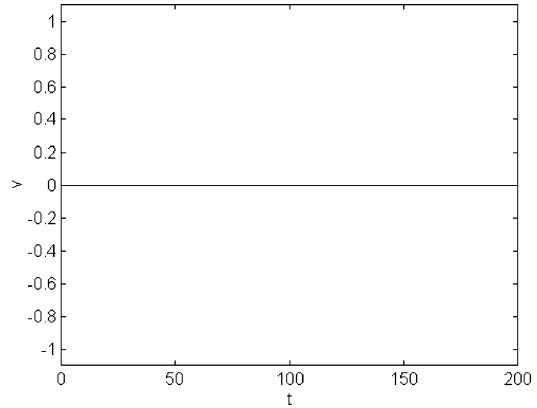


Fig. 5.

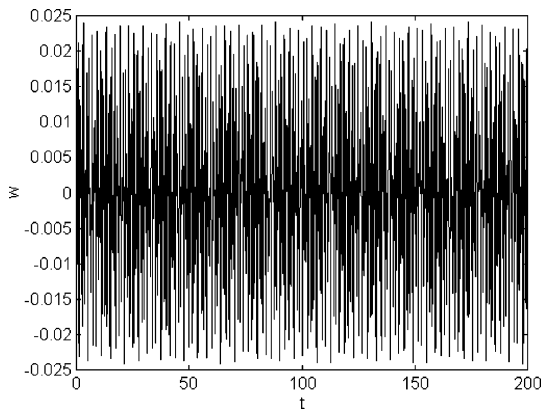


Fig. 6.

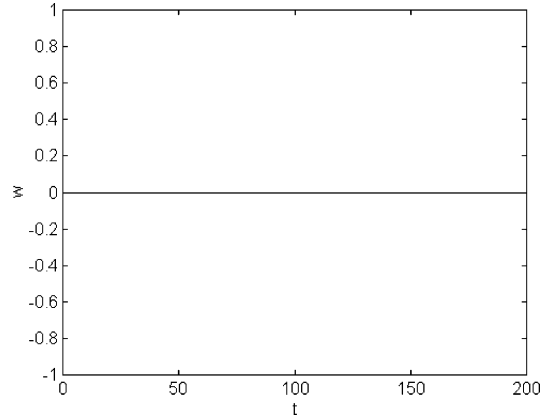


Fig. 7.

resonance between the modes caused by the frequencies  $\omega_1^2 = 24.18053$  and  $\omega_2^2 = 25.15551$  being so close to each other.

**Example 4.** The parameters of the material are: wave number  $\kappa = 0.02 \text{ m}^{-1}$ , density  $\rho = 3000 \text{ kg/m}^3$  and the soil comprises a single layer of thickness 100.0 m. The resulting matrix,  $\mathbf{D}$ , is orthotropic

$$\mathbf{D} = \begin{bmatrix} 0.3 & 0.3 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.2 \end{bmatrix} \times 10^{10}.$$

The first three eigenvalues and corresponding mode shapes  $\mathbf{v}_{0,0}^i$  at  $z = 0$  are

$$\begin{aligned} \omega^2 &= 406.99987; 410.57741; 564.49341, & \mathbf{v}_{0,0}^1 &= \{2.37360; 0.0; 0.87978i\}^T \times 10^{-3}, \\ \mathbf{v}_{0,0}^2 &= \{1.39893; 0.0; -2.25864i\}^T \times 10^{-3}, & \mathbf{v}_{0,0}^3 &= \{0.0; 2.58199; 0.0\}^T \times 10^{-3}. \end{aligned}$$

The results for  $u$ ,  $v$  and  $w$  are shown in Figs. 8–10 for  $y_0 = \{1; 1; 0\}^T$ . It can be seen from Figs. 8 and 10 that there has been an exchange of energy between the  $u$  and  $w$  components resulting once more in a phenomenon like

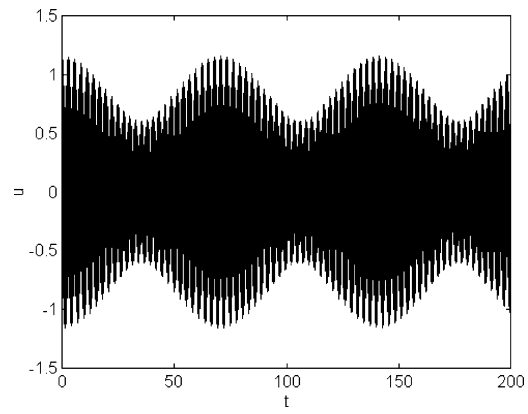


Fig. 8.

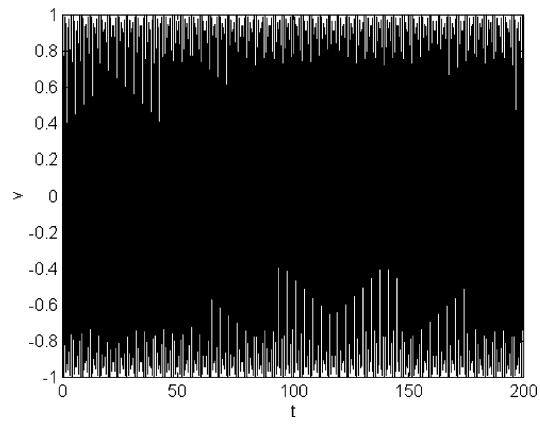


Fig. 9.

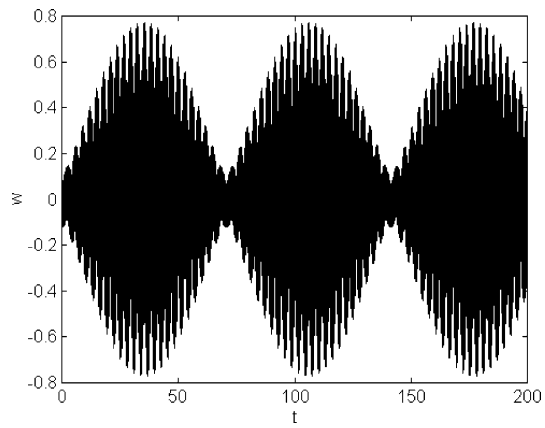


Fig. 10.

beating. It is important to note that  $w$  is zero in the initial condition, but  $w$  of the response is not zero and is not a small number.

## 9. Discussion and conclusions

The key element in the solution of wave propagation problems lies in the efficient determination of the eigen-solution. A few simple problems can be solved analytically, but for more complex problems there is no alternative to using numerical methods. When spectral analysis techniques are used it will normally necessitate the solution of a second order, ordinary differential equation in the form of a two-point boundary value problem, which will be posed in Lagrangian form. In this paper it has been shown that for the generalised plane strain problem considered, a very attractive solution procedure can be achieved by transforming the resulting Lagrangian to Hamiltonian form, which has the effect of changing the ordinary differential equations to first order, but in mixed form. These dual equations can then be solved using the precise integration method in combination with the extended Wittrick–Williams algorithm to yield the required eigenvalues to arbitrary precision with the certain knowledge that none have been missed. The corresponding eigenvectors follow directly and can be described analytically.

The proposed method is likely to find considerable application in those very important areas of earthquake engineering where it is required to establish an accurate estimation of the ground motion. For such problems, the physical model of the soil is usually considered to be multi-layered above the base rock, while the earthquake's hypocenter is located appropriately far away. The wave thus propagates along the base rock first and the upper layer wave is excited from the waves of the base rock. Such excitation is particularly complicated, with the nature of the wave best considered to be a random feature. However, the widely accepted Kanai–Tajimi spectra of ground excitation is based on the assumption that the upper soil is composed of a single layer and only one prominent frequency  $\omega_g$  of the soil layer is considered, together with its corresponding damping ratio  $\zeta_g$ , while the bottom of the soil, i.e. the base rock, is subjected to a stationary acceleration excitation with white spectrum  $S_0$ . The multi-layered anisotropic soil model presented thus offers the possibility of a substantially more detailed description of the physical model. Furthermore, the model lends itself to further extension through the recently developed pseudo-excitation method for structural random vibration [15–18] which will enable the solution of the layered soil responses to be calculated by a technique that will place the ground motion power spectral density estimation on a firm computational footing.

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