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A state space formalism for anisotropic elasticity. Part II: Cylindrical anisotropy

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Abstract

A state space formalism for thermoelastic analysis of a cylindrically anisotropic elastic body is developed. By proper grouping of the stress and taking $r\sigma_{ij}$ instead of σ_{ij} as the stress variables, the three-dimensional equations of elasticity in the cylindrical coordinates are concisely formulated into a state equation and an output equation using matrix notations. The general solution for the generalized plane problems is derived in a simple manner. Effects of extension, torsion, bending, temperature change and body force are accounted for systematically. The eigen relation arising from the solution process and the degenerate cases of repeated eigenvalues are examined. To demonstrate the power of the formalism, exact solutions to extension, torsion, bending and thermo-mechanical loading of a general cylindrically anisotropic circular tube or bar are obtained. The surface tractions and temperature field may vary in θ but not in z. Displacement as well as traction boundary conditions are considered. It appears that many problems of anisotropic elasticity are best viewed and treated in the state space framework.

Keywords: Anisotropic elasticity; Cylindrical anisotropy; Generalized plane strain; Torsion; State space formalism; Cylindrical coordinates

1. Introduction

This paper presents a state space formalism for thermoelastic analysis of a cylindrically anisotropic elastic body. In a cylindrically anisotropic material the elastic property at each point is characterized by the radial, tangential, and axial directions in the cylindrical coordinates. The material possesses different properties in each direction. Cylindrical anisotropy is not uncommon in the cylindrical body, for examples, it appears in bamboo, tree trunk and carbon fiber (Christensen, 1994). The metallic forming process, such as extrusion or drawing, may result in cylindrically anisotropic products. The filamentary wound composite is a cylindrically orthotropic material on the macroscopic scale. According to the class of elastic symmetry,

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the cylindrical body may exhibit special type of material anisotropy such as monoclinic anisotropy, cylindrical orthotropy, and transverse isotropy.

The formulation of anisotropic elasticity in the cylindrical coordinates could be very complicated, involving lengthy and cumbersome equations. Solutions were known only for axisymmetric problems (Ting, 1996, 1999; Chen et al., 2000), or for special anisotropy (Lekhnitskii, 1981; Pagano, 1972; Jolicoeur and Cardou, 1994; Kollár and Springer, 1992). It is known that the Lekhnitskii formalism is ineffective for displacement boundary value problem, whereas the Stroh formalism is intended for problems of plane deformation in the Cartesian coordinates, unsuitable for problems in the cylindrical coordinates. Recently, we have treated extension, torsion, bending, shearing and pressuring of laminated composite tubes and functionally graded cylinders (Tarn, 2001; Tarn and Wang, 2001) by a state space approach. The material was assumed to be cylindrically monoclinic anisotropic having elastic symmetry with respect to $r =$ constant. Attention then was paid to an explicit formulation and analysis of the specific problems. Herein we develop the state space formalism in the cylindrical coordinate system for an elastic body of a general cylindrically anisotropic material. It is noteworthy that Alshits and Kirchner (2001) has studied a class of problem of cylindrically anisotropic elastic materials by representing the basic equations in a system of differential equations of the first order.

One of the prominent features of the state space formalism is that a compact state equation and an output equation in terms of the displacement vector and two stress variables embrace in full the three-dimensional equations of anisotropic elasticity in the cylindrical coordinate system. The components of the displacement and stress and the thermoelastic constants of the material appear explicitly in the equations, yet the formulation is remarkably simple. This is achieved by grouping the stress into $[\sigma_r, \sigma_{r\theta}, \sigma_{rz}]$ and $[\sigma_\theta, \sigma_z, \sigma_{\theta z}]$ and expressing the basic equations in matrix forms. The reason for making this grouping is because the traction vector $\boldsymbol{\tau}_r$ on the cylindrical surfaces $r =$ constant is precisely $[\sigma_r, \sigma_{r\theta}, \sigma_{rz}]$; these stress components often appear as a group. Moreover, it is advantageous to take $r\boldsymbol{\tau}_r$ instead of $\boldsymbol{\tau}_r$ as the stress variable. The novel arrangements greatly simplify the formulation and result in concise matrix equations in terms of the displacement and $r\boldsymbol{\tau}_r$. The analysis is much more clear and easier in the state space setting.

The formalism enables us to determine a general solution for a circular cylindrical elastic body of a general cylindrically anisotropic material subjected to thermo-mechanical loads. The surface tractions and the temperature field may vary in θ-direction but not in z-direction. Effects of antiplane deformations, end loads, temperature change, and body force are accounted for through the particular solution. In determining the homogeneous solution to the state equation an eigen relation arises naturally. The eigenvalues and eigenvectors possess useful properties which will be explored along the way. In particular, the degenerate cases of repeated eigenvalues occurring when the material is cylindrically orthotropic, transverse isotropic or isotropic are studied. To demonstrate the power of the formalism, exact thermoelastic solutions for a circular tube or bar subjected to extension, torsion, bending, nonuniform surface tractions and temperature field are obtained directly from the general solution. Displacement and traction boundary conditions are treated in the same way. As far as the author is aware of, analytic solutions to these problems of a general cylindrically anisotropic material are not available in the literature. Various solutions obtained herein appear for the first time. The study shows that it is expedient to treat many problem of anisotropic elasticity in the cylindrical coordinates as well as in the Cartesian coordinates in the state space framework.

2. State space formulation

2.1. Basic equations in matrix forms

We consider a cylindrically anisotropic elastic material of the most general kind. Referred to the cylindrical coordinates, the thermoelastic constitutive equations of the material are

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ 2\varepsilon_{\theta z} \\ 2\varepsilon_{rz} \\ 2\varepsilon_{r\theta} \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} T, \tag{1}$$

where $\varepsilon_r, \varepsilon_\theta, \ldots, \sigma_r, \sigma_\theta, \ldots,$ are the strain and stress components, c_{ij} and β_i are the elastic constants and the thermal coefficients of the material, T is the temperature change.

The strain–displacement relations are

$$\begin{aligned} &\varepsilon_r = u_{r,r}, \qquad \varepsilon_\theta = r^{-1}(u_{\theta,\theta} + u_r), \qquad \varepsilon_z = u_{z,z}, \\ &2\varepsilon_{\theta z} = u_{\theta,z} + r^{-1}u_{z,\theta}, \qquad 2\varepsilon_{rz} = u_{z,r} + u_{r,z}, \qquad 2\varepsilon_{r\theta} = r^{-1}u_{r,\theta} + u_{\theta,r} - r^{-1}u_\theta, \end{aligned} \tag{2}$$

where a comma stands for differentiation with respect to the suffix variables.

The equilibrium equations in the cylindrical coordinates are

$$\partial_r\sigma_r + r^{-1}\partial_\theta\sigma_{r\theta} + r^{-1}(\sigma_r - \sigma_\theta) + \partial_z\sigma_{rz} + R = 0, \tag{3}$$

$$\partial_r\sigma_{r\theta} + r^{-1}\partial_\theta\sigma_\theta + 2r^{-1}\sigma_{r\theta} + \partial_z\sigma_{\theta z} + \Theta = 0, \tag{4}$$

$$\partial_r\sigma_{rz} + r^{-1}\partial_\theta\sigma_{\theta z} + r^{-1}\sigma_{rz} + \partial_z\sigma_z + Z = 0, \tag{5}$$

where ∂_r, ∂_θ, ∂_z denote partial differentiation with respect the suffix coordinate; R, Θ, Z denote the components of the body force.

In view of the fact that the tractions on the cylindrical surfaces $r =$ constant are expressed by the stress components σ_r, $\sigma_{r\theta}$ and σ_{rz}, it is expedient to group the stress into $[\sigma_r, \sigma_{r\theta}, \sigma_{rz}]$ and $[\sigma_\theta , \sigma_z, \sigma_{\theta z}]$, and rewrite Eq. (1) as

$$\begin{bmatrix} \boldsymbol{\tau}_r \\ \boldsymbol{\tau}_s \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{rr} & \mathbf{C}_{rs} \\ \mathbf{C}_{rs}^{\mathrm{T}} & \mathbf{C}_{ss} \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\gamma}_s \end{bmatrix} - \begin{bmatrix} \boldsymbol{\beta}_r \\ \boldsymbol{\beta}_s \end{bmatrix} T, \tag{6}$$

where

$$\begin{aligned} &\boldsymbol{\tau}_r = [\sigma_r \quad \sigma_{r\theta} \quad \sigma_{rz}]^{\mathrm{T}}, \qquad \boldsymbol{\tau}_s = [\sigma_\theta \quad \sigma_z \quad \sigma_{\theta z}]^{\mathrm{T}}, \\ &\boldsymbol{\gamma}_r = [\varepsilon_r \quad 2\varepsilon_{r\theta} \quad 2\varepsilon_{rz}]^{\mathrm{T}}, \qquad \boldsymbol{\gamma}_s = [\varepsilon_\theta \quad \varepsilon_z \quad 2\varepsilon_{\theta z}]^{\mathrm{T}}, \end{aligned}$$

$$\mathbf{C}_{rr} = \mathbf{C}_{rr}^{\mathrm{T}} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix}, \qquad \mathbf{C}_{ss} = \mathbf{C}_{ss}^{\mathrm{T}} = \begin{bmatrix} c_{22} & c_{23} & c_{24} \\ c_{23} & c_{33} & c_{34} \\ c_{24} & c_{34} & c_{44} \end{bmatrix},$$

$$\mathbf{C}_{rs} = \begin{bmatrix} c_{12} & c_{13} & c_{14} \\ c_{25} & c_{35} & c_{45} \\ c_{26} & c_{36} & c_{46} \end{bmatrix}, \qquad \boldsymbol{\beta}_r = \begin{bmatrix} \beta_1 \\ \beta_6 \\ \beta_5 \end{bmatrix}, \qquad \boldsymbol{\beta}_s = \begin{bmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}.$$

The strain–displacement relations can be expressed in a matrix form as

$$\begin{bmatrix} r\boldsymbol{\gamma}_r \\ r\boldsymbol{\gamma}_s \end{bmatrix} = r\partial_r \begin{bmatrix} \mathbf{u} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{L}_1\mathbf{u} \\ \mathbf{L}_2\mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_3\mathbf{u} \\ \mathbf{K}_6\mathbf{u} \end{bmatrix}, \tag{7}$$

where $\mathbf{I}$ is the identity matrix, and

$$\mathbf{L}_1 = \mathbf{K}_1\partial_\theta + \mathbf{K}_2 r\partial_z, \quad \mathbf{L}_2 = \mathbf{K}_4\partial_\theta + \mathbf{K}_5 r\partial_z,$$

$$\mathbf{u} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_3 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{K}_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using Eq. (7), we may rewrite Eq. (6) as

$$\begin{bmatrix} r\boldsymbol{\tau}_r \\ r\boldsymbol{\tau}_s \end{bmatrix} = r\partial_r \begin{bmatrix} \mathbf{C}_{rr}\mathbf{u} \\ \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{u} \end{bmatrix} + \begin{bmatrix} [\mathbf{C}_{rr}(\mathbf{L}_1 + \mathbf{K}_3) + \mathbf{C}_{rs}(\mathbf{L}_2 + \mathbf{K}_6)]\mathbf{u} \\ [\mathbf{C}_{rs}^{\mathrm{T}}(\mathbf{L}_1 + \mathbf{K}_3) + \mathbf{C}_{ss}(\mathbf{L}_2 + \mathbf{K}_6)]\mathbf{u} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\beta}_r \\ \boldsymbol{\beta}_s \end{bmatrix} Tr. \tag{8}$$

The equilibrium equations (3)–(5) may be cast into a single matrix equation as

$$r\partial_r(r\boldsymbol{\tau}_r) + (\mathbf{L}_1^{\mathrm{T}} - \mathbf{K}_3^{\mathrm{T}})(r\boldsymbol{\tau}_r) + (\mathbf{L}_2^{\mathrm{T}} - \mathbf{K}_6^{\mathrm{T}})(r\boldsymbol{\tau}_s) + \mathbf{F}r^2 = \mathbf{0}, \tag{9}$$

where $\mathbf{F} = [R \quad \Theta \quad Z]^{\mathrm{T}}$ is the body force vector.

By means of the $\mathbf{K}_i$ matrices and the differential operators $\mathbf{L}_1$ and $\mathbf{L}_2$ we are able to arrange the basic equations of anisotropic elasticity in the cylindrical coordinate system into Eqs. (8) and (9). The two matrix differential equations are remarkably simple and concise. Hereafter $\mathbf{u}$, $r\boldsymbol{\tau}_r$, $r\boldsymbol{\tau}_s$ and the four sub-matrices of the stiffness matrix play the principal roles; the displacement and stress components and the individual elastic constants are no longer in view.

2.2. Three-dimensional equations in the state space

Eq. $(8)_1$ may be expressed as

$$r\partial_r\mathbf{u} = \mathbf{C}_{rr}^{-1}\mathbf{D}_{11}\mathbf{u} + \mathbf{C}_{rr}^{-1}(r\boldsymbol{\tau}_r) + \mathbf{C}_{rr}^{-1}\boldsymbol{\beta}_r Tr, \tag{10}$$

where

$$\mathbf{D}_{11} = -[\mathbf{C}_{rr}(\mathbf{K}_3 + \mathbf{L}_1) + \mathbf{C}_{rs}(\mathbf{K}_6 + \mathbf{L}_2)].$$

Substituting Eq. (10) into Eq. $(8)_2$ and the resulting equation into Eq. (9) yields

$$r\boldsymbol{\tau}_s = \begin{bmatrix} \widetilde{\mathbf{C}}_{ss}(\mathbf{K}_6 + \mathbf{L}_2) & \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} - \widetilde{\boldsymbol{\beta}}_s Tr, \tag{11}$$

$$r\partial_r(r\boldsymbol{\tau}_r) - [\,\mathbf{D}_{21} \quad \mathbf{D}_{22}\,] \begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} + (\mathbf{K}_6^{\mathrm{T}} - \mathbf{L}_2^{\mathrm{T}})\widetilde{\boldsymbol{\beta}}_s Tr + \mathbf{F}r^2 = \mathbf{0}, \tag{12}$$

where

$$\mathbf{D}_{21} = (\mathbf{K}_6^{\mathrm{T}} - \mathbf{L}_2^{\mathrm{T}})\widetilde{\mathbf{C}}_{ss}(\mathbf{K}_6 + \mathbf{L}_2), \quad \widetilde{\mathbf{C}}_{ss} = \mathbf{C}_{ss} - \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1}\mathbf{C}_{rs},$$
$$\mathbf{D}_{22} = [(\mathbf{K}_3^{\mathrm{T}} - \mathbf{L}_1^{\mathrm{T}})\mathbf{C}_{rr} + (\mathbf{K}_6^{\mathrm{T}} - \mathbf{L}_2^{\mathrm{T}})\mathbf{C}_{rs}^{\mathrm{T}}]\mathbf{C}_{rr}^{-1}, \quad \widetilde{\boldsymbol{\beta}}_s = \boldsymbol{\beta}_s - \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1}\boldsymbol{\beta}_r.$$

Casting Eqs. (10) and (12) into a single matrix differential equation, we arrive at

$$r\frac{\partial}{\partial r}\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{rr}^{-1}\mathbf{D}_{11} & \mathbf{C}_{rr}^{-1} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{rr}^{-1}\boldsymbol{\beta}_r \\ (\mathbf{L}_2^{\mathrm{T}} - \mathbf{K}_6^{\mathrm{T}})\widetilde{\boldsymbol{\beta}}_s \end{bmatrix} Tr - \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix} r^2. \tag{13}$$

The state equation (13) and the output equation (11) embrace the three-dimensional equations of anisotropic elasticity in the cylindrical coordinates in full. The state equation plays a key role in the formalism, once it is solved, the displacement and stress follow. Determining the analytic solution of Eq. (13) in general is formidable, if not impossible. Yet its structure suggests that when the functional dependence on one of the coordinates is known or can be assumed a priori, the equation becomes two-dimensional and might be amenable to analytic treatment. Indeed, with the formalism, it is easy to derive the displacement as a function of z and obtain the general solution to the state equation for the generalized plane problem.

The foregoing formulation is based on the stiffness representation. The formulation could be based on the compliance representation as well. To this end, the constitutive equations are expressed as

$$\begin{bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\gamma}_s \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{rs} \\ \mathbf{S}_{rs}^{\mathrm{T}} & \mathbf{S}_{ss} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_r \\ \boldsymbol{\tau}_s \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}_r \\ \boldsymbol{\alpha}_s \end{bmatrix} T, \tag{14}$$

where the compliance matrix and the thermal expansion vector are

$$\begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{rs} \\ \mathbf{S}_{rs}^{\mathrm{T}} & \mathbf{S}_{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{rr} & \mathbf{C}_{rs} \\ \mathbf{C}_{rs}^{\mathrm{T}} & \mathbf{C}_{ss} \end{bmatrix}^{-1}, \quad \begin{bmatrix} \boldsymbol{\alpha}_r \\ \boldsymbol{\alpha}_s \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{rs} \\ \mathbf{S}_{rs}^{\mathrm{T}} & \mathbf{S}_{ss} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_r \\ \boldsymbol{\beta}_s \end{bmatrix}.$$

Substituting Eq. (7) in Eq. (14) and using the resulting equation along with Eq. (9) to represent $\boldsymbol{\tau}_s$ in terms of the state vector $[\mathbf{u}, r\boldsymbol{\tau}_r]$, one could derive the state equation and the output equation in terms of the elastic compliance. Following the same line as in Part I of the paper using the identities associated with the constitutive matrices, it can be shown that the equations are identical to Eqs. (11) and (13).

3. Generalized plane problems

3.1. State equation and output equation

The displacement field in the body in the state of generalized plane strain and generalized torsion can be obtained simply by transforming the expressions in the Cartesian coordinates (Eqs. (22)–(24) in Tarn, 2002) to the ones in the cylindrical coordinates using

$$u_r = u_1 \cos\theta + u_2 \sin\theta, \quad u_\theta = -u_1 \sin\theta + u_2 \cos\theta, \quad u_z = u_3 \tag{15}$$

to produce

$$u_r = u - \tfrac{1}{2}z^2(b_1 \cos\theta + b_2 \sin\theta) + z(\omega_2 \cos\theta - \omega_1 \sin\theta) + u_0 \cos\theta + v_0 \sin\theta, \tag{16}$$

$$u_\theta = v + \tfrac{1}{2}z^2(b_1 \sin\theta - b_2 \cos\theta) + \vartheta rz - z(\omega_2 \sin\theta + \omega_1 \cos\theta) + \omega_0 r - u_0 \sin\theta + v_0 \cos\theta, \tag{17}$$

$$u_z = w + z(b_1 r \cos\theta + b_2 r \sin\theta + \varepsilon) + r(\omega_1 \sin\theta - \omega_2 \cos\theta) + w_0, \tag{18}$$

where u, v, w are unknown functions of r and θ; u_0, v_0, w_0, ω_0, ω_1, ω_2 are constants characterizing the rigid body displacements; the constant ε is a uniform extension, ϑ is the twisting angle per unit length along z-axis, b_1 and b_2 are associated with bending. Eqs. (16)–(18) were obtained in Section 23 of Lekhnitskii's monograph (1981) by direct yet intricate manipulation of the displacement–stress relations.

On substituting Eqs. (16)–(18) in Eqs. (11) and (13), the state equation and the output equation read

$$\begin{aligned} r\frac{\partial}{\partial r}\begin{bmatrix} \tilde{\mathbf{u}} \\ r\boldsymbol{\tau}_r \end{bmatrix} = {} & \begin{bmatrix} \mathbf{L}_{11} & \mathbf{C}_{rr}^{-1} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ r\boldsymbol{\tau}_r \end{bmatrix} - \left(\varepsilon \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 \end{bmatrix} - T \begin{bmatrix} \mathbf{C}_{rr}^{-1}\boldsymbol{\beta}_r \\ \mathbf{K}_4 \widetilde{\boldsymbol{\beta}}_s \partial_\theta \end{bmatrix} \right) r - \left(\vartheta \begin{bmatrix} \mathbf{p}_2 \\ \tilde{c}_{24}\mathbf{k}_3 \end{bmatrix} \right. \\ & \left. + \operatorname{Re}\left\{ (b_1 - \mathrm{i}b_2) \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 - \mathrm{i}\mathbf{p}_3 \end{bmatrix} \mathrm{e}^{\mathrm{i}\theta} \right\} - \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \right) r^2, \end{aligned} \tag{19}$$

$$r\boldsymbol{\tau}_s = \begin{bmatrix} \mathbf{L}_3 & \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ r\boldsymbol{\tau}_r \end{bmatrix} + (\widetilde{\mathbf{C}}_{ss}\mathbf{k}_1\varepsilon - \widetilde{\boldsymbol{\beta}}_s T)r + \widetilde{\mathbf{C}}_{ss}[\mathbf{k}_2\vartheta + \mathbf{k}_1 \operatorname{Re}\{(b_1 - \mathrm{i}b_2)\mathrm{e}^{\mathrm{i}\theta}\}]r^2, \tag{20}$$

where

$$\tilde{\mathbf{u}} = [u \quad v \quad w]^{\mathrm{T}}, \qquad \mathbf{f} = [R \quad \Theta \quad 0]^{\mathrm{T}},$$

$$\begin{aligned}
\mathbf{L}_{11} &= -\mathbf{C}_{rr}^{-1}[\mathbf{C}_{rr}\mathbf{K}_3 + \mathbf{C}_{rs}\mathbf{K}_6 + (\mathbf{C}_{rr}\mathbf{K}_1 + \mathbf{C}_{rs}\mathbf{K}_4)\partial_\theta],\\
\mathbf{L}_{21} &= (\mathbf{K}_6^{\mathrm{T}} - \mathbf{K}_4^{\mathrm{T}}\partial_\theta)\widetilde{\mathbf{C}}_{ss}(\mathbf{K}_6 + \mathbf{K}_4\partial_\theta), \quad \mathbf{L}_3 = \widetilde{\mathbf{C}}_{ss}(\mathbf{K}_4\partial_\theta + \mathbf{K}_6),\\
\mathbf{L}_{22} &= [\mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{rr} + \mathbf{K}_6^{\mathrm{T}}\mathbf{C}_{rs}^{\mathrm{T}} - (\mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{rr} + \mathbf{K}_4^{\mathrm{T}}\mathbf{C}_{rs}^{\mathrm{T}})\partial_\theta]\mathbf{C}_{rr}^{-1},
\end{aligned}$$

$$\mathbf{k}_1 = [0 \quad 1 \quad 0]^{\mathrm{T}}, \quad \mathbf{k}_2 = [0 \quad 0 \quad 1]^{\mathrm{T}}, \quad \mathbf{k}_3 = [1 \quad 0 \quad 0]^{\mathrm{T}},$$

$$\mathbf{p}_1 = \mathbf{C}_{rr}^{-1}\mathbf{C}_{rs}\mathbf{k}_1, \quad \mathbf{p}_2 = \mathbf{C}_{rr}^{-1}\mathbf{C}_{rs}\mathbf{k}_2, \quad \mathbf{p}_3 = \tilde{c}_{23}\mathbf{k}_1 + \tilde{c}_{34}\mathbf{k}_2.$$

At this point before solving the state equation, it can be seen that extension and a uniform temperature change enter the stage through a linear function of r, producing axisymmetric deformation; torsion and bending enter through a quadratic function of r, producing axisymmetric deformation and θ-dependent deformation, respectively.

3.2. Homogeneous solution

We seek the homogeneous solution to Eq. (19) in the form of Fourier complex series as

$$\begin{bmatrix}\tilde{\mathbf{u}} & r\boldsymbol{\tau}_r\end{bmatrix}_h = \sum_{n=-\infty}^{\infty} [\mathbf{U}_n \quad \mathbf{X}_n] r^{\lambda} \mathrm{e}^{\mathrm{i}n\theta}, \tag{21}$$

where $\mathbf{U}_n$ and $\mathbf{X}_n$ are constant vectors, λ is a constant yet unknown.

Substituting Eq. (21) in Eq. (19) leads to an eigen relation

$$\begin{bmatrix}\mathbf{N}_{n1} & \mathbf{N}_{n2}\\ \mathbf{N}_{n3} & -\overline{\mathbf{N}}_{n1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{U}_n\\ \mathbf{X}_n\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{U}_n\\ \mathbf{X}_n\end{bmatrix}, \tag{22}$$

where

$$\mathbf{N}_{n1} = -\mathbf{C}_{rr}^{-1}(\mathbf{C}_1 + \mathrm{i}n\mathbf{C}_2), \quad \mathbf{N}_{n2} = \mathbf{C}_{rr}^{-1}, \quad \mathbf{N}_{n3} = \mathbf{C}_3 + (n^2-1)\mathbf{K}_4^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_4 + \mathrm{i}n\mathbf{C}_4,$$

$$\mathbf{C}_1 = \mathbf{C}_{rr}\mathbf{K}_3 + \mathbf{C}_{rs}\mathbf{K}_6, \quad \mathbf{C}_2 = \mathbf{C}_{rr}\mathbf{K}_1 + \mathbf{C}_{rs}\mathbf{K}_4, \quad \mathbf{C}_3 = \mathbf{K}_4^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_4 + \mathbf{K}_6^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_6, \quad \mathbf{C}_4 = \mathbf{K}_6^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_4 - \mathbf{K}_4^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_6,$$

Obviously, λ is the eigenvalue and $[\mathbf{U}_n, \mathbf{X}_n]^{\mathrm{T}}$ is the eigenvector of Eq. (22). It will be shown in the next section that if λ_i is an eigenvalue of Eq. (22) so is $-\overline{\lambda}_i$, where an over-bar denotes complex conjugate. Denoting the six eigenvalues for a specific value of n by λ_{nk} and $\lambda_{nk+3} = -\overline{\lambda}_{nk}$ $(k = 1, 2, 3)$, then the homogeneous solution is a linear combination of the six linearly independent eigenvectors associated with the eigenvalues. Since the displacement and the stress are real, there follows

$$\begin{bmatrix}\mathbf{U}\\ r\boldsymbol{\tau}_r\end{bmatrix}_h = \sum_{n=0}^{\infty}\sum_{k=1}^{3}\mathrm{Re}\left\{\left(c_{nk}r^{\lambda_{nk}}\begin{bmatrix}\mathbf{U}_{nk}\\ \mathbf{X}_{nk}\end{bmatrix} + d_{nk}r^{-\overline{\lambda}_{nk}}\begin{bmatrix}\mathbf{U}_{nk+3}\\ \mathbf{X}_{nk+3}\end{bmatrix}\right)\mathrm{e}^{\mathrm{i}n\theta}\right\}, \tag{23}$$

where $\mathrm{Re}\{f\}$ denotes the real part of the complex function f; λ_{nk} denotes the kth eigenvalue associated with the nth term of the Fourier series, $[\mathbf{U}_{nk}, \mathbf{X}_{nk}]^{\mathrm{T}}$ is the associated eigenvector, c_{nk} and d_{nk} are complex coefficients of linear combination.

3.3. Particular solution

The particular solution of Eq. (19) are determined in an elementary way. When the body is subjected to extension, torsion, bending, a uniform temperature change and body force, the nonhomogeneous terms in Eq. (19) consist of the r and r^2 terms. The particular solution is

$$\begin{bmatrix} \tilde{\mathbf{u}} \\ r\boldsymbol{\tau}_r \end{bmatrix}_p = \varepsilon r \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} + \vartheta r^2 \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + r^2 \mathrm{Re}\left\{ \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\theta} \right\}, \tag{24}$$

where the terms of ε and ϑ are associated with extension and torsion, respectively; the term of $\mathrm{e}^{\mathrm{i}\theta}$ is associated with bending; the coefficients are determined from

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{rr}^{-1}(\mathbf{C}_1 + \mathbf{C}_{rr}) & \mathbf{C}_{rr}^{-1} \\ \mathbf{K}_6^{\mathrm{T}} \widetilde{\mathbf{C}}_{ss} \mathbf{K}_6 & (\mathbf{C}_1^{\mathrm{T}} - \mathbf{C}_{rr})\mathbf{C}_{rr}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 \end{bmatrix}, \tag{25}$$

$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{rr}^{-1}(\mathbf{C}_1 + 2\mathbf{C}_{rr}) & \mathbf{C}_{rr}^{-1} \\ \mathbf{K}_6^{\mathrm{T}} \widetilde{\mathbf{C}}_{ss} \mathbf{K}_6 & (\mathbf{C}_1^{\mathrm{T}} - 2\mathbf{C}_{rr})\mathbf{C}_{rr}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}_2 \\ \tilde{c}_{24}\mathbf{k}_3 \end{bmatrix}, \tag{26}$$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{12} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{G} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}, \quad \mathbf{G} + \mathrm{i}\mathbf{H} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 + \mathrm{i}\mathbf{h}_1 \\ \mathbf{g}_2 + \mathrm{i}\mathbf{h}_2 \end{bmatrix}, \tag{27}$$

where

$$\mathbf{Q}_1 = b_1 \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 \end{bmatrix} - b_2 \begin{bmatrix} \mathbf{0} \\ \mathbf{p}_3 \end{bmatrix}, \quad \mathbf{Q}_2 = b_2 \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 \end{bmatrix} + b_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{p}_3 \end{bmatrix},$$

$$\mathbf{M}_{11} = \begin{bmatrix} -\mathbf{C}_{rr}^{-1}(\mathbf{C}_1 + 2\mathbf{C}_{rr}) & \mathbf{C}_{rr}^{-1} \\ \mathbf{C}_3 & (\mathbf{C}_1^{\mathrm{T}} - 2\mathbf{C}_{rr})\mathbf{C}_{rr}^{-1} \end{bmatrix}, \quad \mathbf{M}_{12} = \begin{bmatrix} \mathbf{C}_{rr}^{-1}\mathbf{C}_2 & \mathbf{0} \\ -\mathbf{C}_4 & \mathbf{C}_2^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix},$$

$$\mathbf{M}_{22} = \begin{bmatrix} \mathbf{C}_{rr}^{-1}(\mathbf{C}_1 + 2\mathbf{C}_{rr}) & \mathbf{C}_{rr}^{-1} \\ \mathbf{C}_3 & -(\mathbf{C}_1^{\mathrm{T}} - 2\mathbf{C}_{rr})\mathbf{C}_{rr}^{-1} \end{bmatrix}.$$

The $[\mathbf{b}_1, \mathbf{b}_2]^{\mathrm{T}}$ are obtained by replacing $[\mathbf{p}_1, \tilde{c}_{23}\mathbf{k}_3]^{\mathrm{T}}$ in Eq. (25) by $-[\mathbf{C}_{rr}^{-1}\boldsymbol{\beta}_r, \mathbf{K}_6\widetilde{\boldsymbol{\beta}}_s]^{\mathrm{T}}$. Eq. (24) holds provided that λ are not equal to 1 or 2. When $\lambda = 1$ or 2, the inverse matrices in Eqs. (25) and (26) are singular. This occurs because a member of the nonhomogeneous terms coincides with the homogeneous solution. In this situation the particular solution must be modified. The coincidental equality $\lambda = 2$ exists only mathematically, but $\lambda = 1$ occurs for cylindrically orthotropic, transverse isotropic and isotropic materials.

In case $\lambda = 1$ the particular solution corresponding to the nonhomogeneous terms of power r in Eq. (19) is modified to

$$\begin{bmatrix} \tilde{\mathbf{u}} \\ r\boldsymbol{\tau}_r \end{bmatrix}_p = \varepsilon\left(r\log r \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{bmatrix} + r \begin{bmatrix} \mathbf{a}_1'' \\ \mathbf{a}_2'' \end{bmatrix} \right) + \Delta T\left(r\log r \begin{bmatrix} \mathbf{b}_1' \\ \mathbf{b}_2' \end{bmatrix} + r \begin{bmatrix} \mathbf{b}_1'' \\ \mathbf{b}_2'' \end{bmatrix} \right). \tag{28}$$

By substitution and equating the terms associated with $\log r$ on both sides of the equation leads to

$$\begin{bmatrix} -\mathbf{C}_{rr}^{-1}\mathbf{C}_1 & \mathbf{C}_{rr}^{-1} \\ \mathbf{K}_6^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_6 & \mathbf{C}_1^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{bmatrix} \tag{29}$$

for the term of ε. This is exactly the eigen relation of Eq. (22) in case $\lambda = 1$ and $n = 0$. Hence $[\mathbf{a}_1', \mathbf{a}_2']^{\mathrm{T}}$, similarly $[\mathbf{b}_1', \mathbf{b}_2']^{\mathrm{T}}$, is the eigenvector of Eq. (22) for $\lambda = 1$ and $n = 0$.

Equating the other terms of ε gives

$$\begin{bmatrix} -\mathbf{C}_{rr}^{-1}(\mathbf{C}_1+\mathbf{C}_{rr}) & \mathbf{C}_{rr}^{-1} \\ \mathbf{K}_6^{\mathrm{T}}\widetilde{\mathbf{C}}_{ss}\mathbf{K}_6 & (\mathbf{C}_1^{\mathrm{T}}-\mathbf{C}_{rr})\mathbf{C}_{rr}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{a}_1'' \\ \mathbf{a}_2'' \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 \\ \tilde{c}_{23}\mathbf{k}_3 \end{bmatrix}. \tag{30}$$

The rank of the coefficient matrix in Eq. (30) is less than six because of Eq. (29). In order to have a solution the augmented matrix must be of the same rank (Hildebrand, 1965). This provides a necessary condition to determine a unique solution for the $[\mathbf{a}_1', \mathbf{a}_2']^{\mathrm{T}}$, $[\mathbf{a}_1'', \mathbf{a}_2'']^{\mathrm{T}}$ follows from Eq. (30). A unique $[\mathbf{b}_1', \mathbf{b}_2']^{\mathrm{T}}$ and $[\mathbf{b}_1'', \mathbf{b}_2'']^{\mathrm{T}}$ are determined in the same way.

4. Eigen relation

To facilitate examining the eigen relation, let us express Eq. (22) as

$$\mathbf{N}\boldsymbol{\phi}_i = \lambda_i\boldsymbol{\phi}_i, \tag{31}$$

where $\boldsymbol{\phi}_i$ is the eigenvector associated with the eigenvalue λ_i,

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{n1} & \mathbf{N}_{n2} \\ \mathbf{N}_{n3} & -\overline{\mathbf{N}}_{n1}^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{\phi}_i = \begin{bmatrix} \mathbf{U}_n \\ \mathbf{X}_n \end{bmatrix},$$

$\mathbf{N}_{n2}$ is a real and symmetric matrix; $\mathbf{N}_{n3}$ is a *Hermitian* matrix, $\mathbf{N}_{n3} = \overline{\mathbf{N}}_{n3}^{\mathrm{T}}$.

The eigenvalues of $\overline{\mathbf{N}}^{\mathrm{T}}$ and $\mathbf{N}$ are complex conjugate, hence

$$\overline{\mathbf{N}}^{\mathrm{T}}\boldsymbol{\varphi}_i = \bar{\lambda}_i\boldsymbol{\varphi}_i. \tag{32}$$

It is known that $\boldsymbol{\phi}_i$ is orthogonal with $\boldsymbol{\varphi}_i$ in the Hermitian sense (Hildebrand, 1965) such that

$$\overline{\boldsymbol{\varphi}}_i^{\mathrm{T}}\boldsymbol{\phi}_j = \delta_{ij}, \tag{33}$$

where δ_{ij} is the Kronecker delta.

The matrix $\mathbf{N}$ has the relation

$$\overline{\mathbf{N}}^{\mathrm{T}}\mathbf{J}^{\mathrm{T}} = \mathbf{J}\mathbf{N}, \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{34}$$

Post-multiplying Eq. (34) by $\boldsymbol{\phi}_i$, with $\mathbf{J} = -\mathbf{J}^{\mathrm{T}}$ and Eq. (31), one has

$$\overline{\mathbf{N}}^{\mathrm{T}}(\mathbf{J}^{\mathrm{T}}\boldsymbol{\phi}_i) = -\lambda_i(\mathbf{J}^{\mathrm{T}}\boldsymbol{\phi}_i). \tag{35}$$

This implies that $-\lambda_i$ is the eigenvalue of $\overline{\mathbf{N}}^{\mathrm{T}}$ and $\mathbf{J}^{\mathrm{T}}\boldsymbol{\phi}_i$ is the associated eigenvector. Since the eigenvalues of $\mathbf{N}$ and $\overline{\mathbf{N}}^{\mathrm{T}}$ are complex conjugate, it follows that if λ_i is an eigenvalue of $\mathbf{N}$, so is $-\bar{\lambda}_i$.

Denoting the six eigenvalues of $\mathbf{N}$ by λ_i and $\lambda_{i+3} = -\bar{\lambda}_i$ $(i = 1, 2, 3)$, then

$$\mathbf{N}\boldsymbol{\phi}_{i+3} = -\bar{\lambda}_i\boldsymbol{\phi}_{i+3}. \tag{36}$$

With Eqs. (35) and (36), replacing $\boldsymbol{\varphi}_i$ by $\mathbf{J}^{\mathrm{T}}\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_j$ by $\boldsymbol{\phi}_{j+3}$ in Eq. (33) gives the orthogonality property for the eigenvectors of $\mathbf{N}$:

$$\overline{\boldsymbol{\phi}}_i^{\mathrm{T}}\mathbf{J}\phi_{j+3} = \delta_{ij}. \tag{37}$$

Having examined the characteristics of the eigen relation, one could express the homogeneous solution as given by Eq. (23).

5. Thermoelastic analysis of a circular tube or bar

When a circular tube with inner and outer radii being a and b is subjected to end loads and surface tractions that do not vary in z axis, the boundary conditions are

$$r\boldsymbol{\tau}_r = [\,ap_a \quad a\tau_a \quad as_a\,]^{\mathrm{T}} \quad \text{on } r = a, \tag{38}$$

$$r\boldsymbol{\tau}_r = [\,bp_b \quad b\tau_b \quad bs_b\,]^{\mathrm{T}} \quad \text{on } r = b, \tag{39}$$

where p_a, p_b denote the internal and external radial forces; τ_a, τ_b the inplane shears, s_a, s_b the antiplane shears. In order to maintain static equilibrium the uniform shears acting on the lateral surfaces must satisfy the conditions $\tau_a a^2 = \tau_b b^2$ and $s_a a = s_b b$.

For a solid bar the state of generalized plane strain and generalized torsion does not allow for inplane and antiplane shears to be prescribed on the surfaces. Thus the boundary conditions are

$$r\boldsymbol{\tau}_r = [\,bp_b \quad 0 \quad 0\,]^{\mathrm{T}} \quad \text{on } r = b. \tag{40}$$

When the stress is independent of z, the stress resultants over the cross section Ω reduce to an axial force P_z, a torque M_t, and bi-axial bending moments M_1, M_2; the resultant shears vanish identically. The end conditions are

$$\int_\Omega \mathbf{H}\boldsymbol{\tau}_s r\,\mathrm{d}r\,\mathrm{d}\theta = \mathbf{P}, \tag{41}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & r\sin\theta & 0 \\ 0 & r\cos\theta & 0 \\ 0 & 0 & r \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} P_z \\ M_1 \\ M_2 \\ M_t \end{bmatrix}.$$

The general solution for the problem is given by adding the homogeneous solution and the particular solution along with Eqs. (16)–(18) as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = {} & \sum_{n=1}^{\infty}\sum_{k=1}^{3} \mathrm{Re}\left\{ \left(c_{nk} r^{\lambda_{nk}} \begin{bmatrix} \mathbf{U}_{nk} \\ \mathbf{X}_{nk} \end{bmatrix} + d_{nk} r^{-\bar{\lambda}_{nk}} \begin{bmatrix} \mathbf{U}_{nk+3} \\ \mathbf{X}_{nk+3} \end{bmatrix} \right) \mathrm{e}^{\mathrm{i}n\theta} \right\} + \varepsilon\left(r\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + z\begin{bmatrix} \mathbf{k}_2 \\ \mathbf{0} \end{bmatrix} \right) \\ & + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} + \vartheta r\left(r\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + z\begin{bmatrix} \mathbf{k}_1 \\ \mathbf{0} \end{bmatrix} \right) + \mathrm{Re}\left\{ \left(r^2\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} + z^2\begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \right) \mathrm{e}^{\mathrm{i}\theta} \right\}, \end{aligned} \tag{42}$$

where

$$\boldsymbol{\eta} = (b_1 - \mathrm{i}b_2)[\,-1/2 \quad -\mathrm{i}/2 \quad r/z\,]^{\mathrm{T}}.$$

In Eq. (42) the eigenvectors $[\mathbf{U}_{nk}, \mathbf{X}_{nk}]$ are easily determined from Eq. (22) with the aid of *Mathematica* or MATLAB. The eigensolution of a matrix is a standard routine in these software, in which the degenerate cases of repeated eigenvalues are considered as well. Explicit expressions for the eigenvalues and eigenvectors are obtainable for the cylindrically anisotropic material having elastic symmetry with respect to the cylindrical surfaces $r =$ constant. In this case, $c_{i5} = c_{i6} = 0$, $(i = 1, 2, 3, 4)$, Eq. (22) is simplified and the eigenvalues for $n = 0$ are found to be

$$\lambda_{01} = 0, \quad \lambda_{02} = 1, \quad \lambda_{03} = \kappa, \quad \lambda_{04} = 0, \quad \lambda_{05} = -1, \quad \lambda_{06} = -\kappa, \tag{43}$$

where $\kappa = (c_{22}/c_{11})^{1/2}$.

The associated linearly independent eigenvectors are

$$\lambda_{01} = 0, \quad \mathbf{U}_{01} = [0 \quad 0 \quad 1]^{\mathrm{T}}, \quad \mathbf{X}_{01} = [0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{44}$$

$$\lambda_{02} = 1, \quad \mathbf{U}_{02} = [0 \quad 1 \quad 0]^{\mathrm{T}}, \quad \mathbf{X}_{02} = [0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{45}$$

$$\lambda_{03} = \kappa, \quad \mathbf{U}_{03} = [h \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{X}_{03} = [\varkappa_1 h \quad 0 \quad 0]^{\mathrm{T}}, \tag{46}$$

$$\lambda_{04} = 0, \quad \mathbf{U}_{04} = \left[0 \quad -s_{56}s_{66}^{-1/2} \quad 0\right]^{\mathrm{T}}, \quad \mathbf{X}_{04} = \left[0 \quad 0 \quad s_{66}^{-1/2}\right]^{\mathrm{T}}, \tag{47}$$

$$\lambda_{05} = -1, \quad \mathbf{U}_{05} = [0 \quad -s_{55}/2 \quad -s_{56}]^{\mathrm{T}}, \quad \mathbf{X}_{05} = [0 \quad 1 \quad 0]^{\mathrm{T}}, \tag{48}$$

$$\lambda_{06} = -\kappa, \quad \mathbf{U}_{06} = [-h \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{X}_{06} = [\varkappa_2 h \quad 0 \quad 0]^{\mathrm{T}}, \tag{49}$$

where the generalized eigenvector associated with the repeated eigenvalue $\lambda_{04} = 0$ has been determined by means of the Jordan chain (Pease, 1965), $\mathbf{N}\boldsymbol{\phi}_{04} = \lambda_{04}\boldsymbol{\phi}_{04} + \boldsymbol{\phi}_{01}$; s_{ij} are the elastic compliances, and

$$\varkappa_1 = (c_{11}c_{22})^{1/2} + c_{12}, \quad \varkappa_2 = (c_{11}c_{22})^{1/2} - c_{12}, \quad h = (4c_{11}c_{22})^{-1/4}.$$

Apart from the situation $\lambda_{01} = \lambda_{04} = 0$, repeated eigenvalues occur when the material is cylindrically orthotropic, transverse isotropic or isotropic. In these cases, $c_{22} = c_{11}$, $\kappa = 1$, then $\lambda = \pm 1$ are also repeated eigenvalues. Linearly independent solutions corresponding to $\lambda_{03} = 1$ and $\lambda_{06} = -1$ are determined from the Jordan chain as

$$\left[\tilde{\mathbf{u}} \quad r\boldsymbol{\tau}_r\right] = r(\log r[\mathbf{U}_{02} \quad \mathbf{X}_{02}] + [\mathbf{U}_{03} \quad \mathbf{X}_{03}]), \tag{50}$$

$$\left[\tilde{\mathbf{u}} \quad r\boldsymbol{\tau}_r\right] = r^{-1}(\log r[\mathbf{U}_{05} \quad \mathbf{X}_{05}] + [\mathbf{U}_{06} \quad \mathbf{X}_{06}]), \tag{51}$$

where

$$(\mathbf{N} - \lambda_{0j}\mathbf{I})^2\boldsymbol{\phi}_j = \mathbf{0}, \quad (j = 3, 6), \quad \lambda_{03} = 1, \quad \lambda_{06} = -1, \tag{52}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{01} & \mathbf{N}_{02} \\ \mathbf{N}_{03} & -\overline{\mathbf{N}}_{01}^{\mathrm{T}} \end{bmatrix}, \qquad \boldsymbol{\phi}_j = \begin{bmatrix} \mathbf{U}_{0j} \\ \mathbf{X}_{0j} \end{bmatrix}. \tag{53}$$

The general solution contains the ε, ϑ, b_1 and b_2. These parameters can be determined through the end conditions for a prescribed set of P_z, M_t, M_1 and M_2. For plane deformation we may set $\varepsilon = \vartheta = b_1 = b_2 = 0$ in advance and determine subsequently the end loads that are required to maintain such a deformation. As there is a one to one correspondence between them and the applied loads, they may be regarded as known a priori. The remaining task is to determine the c_{nk} using the boundary conditions for a specific problem. This can be done for the general case of combined loading and specialize it to particular cases. Alternatively, one could solve various particular loading cases and obtain the solution for combined loading by superposition. In the following we consider various cases of particular loading conditions.

5.1. Axisymmetric deformations

Extension, torsion, internal and external pressures, uniform shearing, and a uniform temperature change give rise to axisymmetric deformations. The end loads include P_z and M_t, but not M_1 and M_2. The general solution is given by the terms that are independent of θ in Eq. (42):

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \sum_{k=1}^{3} \left(c_k r^{\lambda_k} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + d_k r^{-\bar{\lambda}_k} \begin{bmatrix} \mathbf{U}_{0k+3} \\ \mathbf{X}_{0k+3} \end{bmatrix} \right) + \varepsilon \left(r \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_2 \\ \mathbf{0} \end{bmatrix} \right) + \vartheta r \left(r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{0} \end{bmatrix} \right) + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \tag{54}$$

Imposing the boundary conditions Eqs. (38) and (39) on Eq. (54) yields the following equations to determine the six unknowns c_k and c_{k+3} in the solution:

$$\sum_{k=1}^{3} (c_k a^{\lambda_k - 1} \mathbf{X}_{0k} + d_k a^{-\bar{\lambda}_k - 1} \mathbf{X}_{0k+3}) + \varepsilon \mathbf{a}_2 + \mathbf{b}_2 \Delta T + \vartheta a \mathbf{c}_2 = [p_a \quad \tau_a \quad s_a]^{\mathrm{T}}, \tag{55}$$

$$\sum_{k=1}^{3} (c_k b^{\lambda_k - 1} \mathbf{X}_{0k} + d_k b^{-\bar{\lambda}_k - 1} \mathbf{X}_{0k+3}) + \varepsilon \mathbf{a}_2 + \mathbf{b}_2 \Delta T + \vartheta b \mathbf{c}_2 = [p_b \quad \tau_b \quad s_b]^{\mathrm{T}}. \tag{56}$$

The stress components $\boldsymbol{\tau}_s$are obtained from the output equation as

$$\boldsymbol{\tau}_s = \begin{bmatrix} \widetilde{\mathbf{C}}_{ss}\mathbf{K}_6 & \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} + \varepsilon \widetilde{\mathbf{C}}_{ss}\mathbf{k}_1 - \Delta T \widetilde{\boldsymbol{\beta}}_s + \vartheta \widetilde{\mathbf{C}}_{ss}\mathbf{k}_2 r, \tag{57}$$

where

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \sum_{k=1}^{3} \left(c_k r^{\lambda_k - 1} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + d_k r^{-\bar{\lambda}_k - 1} \begin{bmatrix} \mathbf{U}_{0k+3} \\ \mathbf{X}_{0k+3} \end{bmatrix} \right) + \varepsilon \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + \vartheta r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

The solution for a solid bar cannot be obtained from that of a tube by specifying $a \to 0$. This results in a cylinder with a pin hole, not a solid bar. For a solid bar the terms of negative power of r should be excluded from the solution in order that the displacement and stress remain finite at the center $r = 0$. As a result, the solution is

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \sum_{k=1}^{3} c_k r^{\lambda_k} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + \varepsilon \left(r \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_2 \\ \mathbf{0} \end{bmatrix} \right) + \vartheta r \left(r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{0} \end{bmatrix} \right) + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \tag{58}$$

in which the c_k are determined from the conditions obtained from imposing the boundary condition Eq. (40) on Eq. (58):

$$\sum_{k=1}^{3} c_k b^{\lambda_k - 1} \mathbf{X}_{0k} + \varepsilon \mathbf{a}_2 + \mathbf{b}_2 \Delta T + \vartheta b \mathbf{c}_2 = [p_b \quad 0 \quad 0]^{\mathrm{T}}. \tag{59}$$

The stress components $\boldsymbol{\tau}_s$ are

$$\boldsymbol{\tau}_s = \begin{bmatrix} \widetilde{\mathbf{C}}_{ss}\mathbf{K}_6 & \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} + \varepsilon \widetilde{\mathbf{C}}_{ss}\mathbf{k}_1 - \Delta T \widetilde{\boldsymbol{\beta}}_s + \vartheta \widetilde{\mathbf{C}}_{ss}\mathbf{k}_2 r, \tag{60}$$

where

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \sum_{k=1}^{3} c_k r^{\lambda_k - 1} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + \vartheta r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

5.2. Bending of a tube or bar

When a circular tube is subjected to pure bending at the ends, the inner and outer surfaces are traction-free. The only loading is M_1 and M_2 at the ends. The general solution to the problem is given by the $n = \pm 1$ terms of Eq. (42):

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \mathrm{Re}\left\{ \left[\sum_{k=1}^{3} \left(c_{1k} r^{\lambda_{1k}} \begin{bmatrix} \mathbf{U}_{1k} \\ \mathbf{X}_{1k} \end{bmatrix} + d_{1k} r^{-\bar{\lambda}_{1k}} \begin{bmatrix} \mathbf{U}_{1k+3} \\ \mathbf{X}_{1k+3} \end{bmatrix} \right) + r^2 \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} + z^2 \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \right] \mathrm{e}^{\mathrm{i}\theta} \right\}, \tag{61}$$

in which the c_{1k} and d_{1k} are complex number.

Imposing the boundary conditions

$$r\boldsymbol{\tau}_r = \mathbf{0} \quad \text{on } r = a \text{ and } b \tag{62}$$

on the solution and equating the real part and imaginary part of these equations yields

$$\sum_{k=1}^{3} \mathrm{Re}\{c_{1k}\mathbf{X}_{1k}a^{\lambda_{1k}-2} + d_{1k}\mathbf{X}_{1k+3}a^{-\bar{\lambda}_{1k}-2}\} = -\mathbf{g}_2, \tag{63}$$

$$\sum_{k=1}^{3} \mathrm{Im}\{c_{1k}\mathbf{X}_{1k}b^{\lambda_{1k}-2} + d_{1k}\mathbf{X}_{1k+3}b^{-\bar{\lambda}_{1k}-2}\} = -\mathbf{h}_2, \tag{64}$$

for the real part and imaginary part of the unknown coefficients c_{1k} and $c_{1(k+3)}$, ($k = 1, 2, 3$) in Eq. (61), where $\mathbf{g}_2$ and $\mathbf{h}_2$ are the real part and imaginary part of $\mathbf{f}_2$ defined in (27).

The stress components $\boldsymbol{\tau}_s$ are obtained from the output equation as

$$\boldsymbol{\tau}_s = \mathrm{Re}\left\{ \left[\widetilde{\mathbf{C}}_{ss}(\mathbf{K}_6 + \mathrm{i}\mathbf{K}_4) \quad \mathbf{C}_{rs}^{\mathrm{T}}\mathbf{C}_{rr}^{-1} \right] \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \right\} + \widetilde{\mathbf{C}}_{ss}\mathbf{k}_1 r \mathrm{Re}\{(b_1 - \mathrm{i}b_2)\mathrm{e}^{\mathrm{i}\theta}\}, \tag{65}$$

where

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \mathrm{Re}\left\{ \left[\sum_{k=1}^{3} \left(c_{1k} r^{\lambda_{1k}-1} \begin{bmatrix} \mathbf{U}_{1k} \\ \mathbf{X}_{1k} \end{bmatrix} + d_{1k} r^{-\bar{\lambda}_{1k}-1} \begin{bmatrix} \mathbf{U}_{1k+3} \\ \mathbf{X}_{1k+3} \end{bmatrix} \right) + r^2 \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \right] \mathrm{e}^{\mathrm{i}\theta} \right\}. \tag{66}$$

For pure bending of a solid bar the terms of negative power of r must be excluded from the solution. Imposing the boundary condition $r\boldsymbol{\tau}_r = \mathbf{0}$ at $r = b$ on the general solution, one obtains

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \mathrm{Re}\left\{ \left(\sum_{k=1}^{3} c_{1k} r^{\lambda_{1k}} \begin{bmatrix} \mathbf{U}_{1k} \\ \mathbf{X}_{1k} \end{bmatrix} + r^2 \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} + z^2 \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{0} \end{bmatrix} \right) \mathrm{e}^{\mathrm{i}\theta} \right\}, \tag{67}$$

in which the real part and imaginary part of the unknown c_{1k}, ($k = 1, 2, 3$) are determined from

$$\sum_{k=1}^{3} \mathrm{Re}\{c_{1k}b^{\lambda_{1k}-2}\mathbf{X}_{1k}\} = -\mathbf{g}_2, \tag{68}$$

$$\sum_{k=1}^{3} \mathrm{Im}\{c_{1k}b^{\lambda_{1k}-2}\mathbf{X}_{1k}\} = -\mathbf{h}_2. \tag{69}$$

The stress $\boldsymbol{\tau}_s$ in the bar is obtained from Eq. (65) with

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \mathrm{Re}\left\{ \left(\sum_{k=1}^{3} c_k r^{\lambda_{1k}-1} \begin{bmatrix} \mathbf{U}_{1k} \\ \mathbf{X}_{1k} \end{bmatrix} + r^2 \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \right) \mathrm{e}^{\mathrm{i}\theta} \right\}. \tag{70}$$

5.3. A tube or bar subjected to nonuniform thermo-mechanical loads

When the tube or bar is subjected to nonuniform surface tractions and the temperature field, the deformation and stress fields depend on θ. Let us consider the case when the radial traction is prescribed on the surfaces. The boundary conditions are

$$r\boldsymbol{\tau}_r = [\,ap_a(\theta) \quad 0 \quad 0\,]^{\mathrm{T}} \quad \text{on } r = a, \tag{71}$$

$$r\boldsymbol{\tau}_r = [\,bp_b(\theta) \quad 0 \quad 0\,]^{\mathrm{T}} \quad \text{on } r = b, \tag{72}$$

for a tube, and

$$r\boldsymbol{\tau}_r = [\,bp_b(\theta) \quad 0 \quad 0\,]^{\mathrm{T}} \quad \text{on } r = b, \tag{73}$$

for a solid bar.

As long as the distribution of the radial traction and the temperature field are piecewise continuous, they may be expressed by Fourier series as

$$[\,p_a(\theta) \quad p_b(\theta) \quad T(r,\theta)\,] = \sum_{n=-\infty}^{\infty} [\,A_n \quad B_n \quad q_n(r)\,]\mathrm{e}^{\mathrm{i}n\theta}, \tag{74}$$

where

$$[\,A_n \quad B_n \quad q_n(r)\,] = \int_0^{2\pi} [\,p_a(\theta) \quad p_b(\theta) \quad T(r,\theta)\,]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta. \tag{75}$$

The general solution for the problem is

$$\begin{aligned}\begin{bmatrix}\mathbf{u}\\ r\boldsymbol{\tau}_r\end{bmatrix} = {}& \sum_{n=1}^{\infty}\mathrm{Re}\left\{\left[\sum_{k=1}^{3}\left(c_{nk}r^{\lambda_{nk}}\begin{bmatrix}\mathbf{U}_{nk}\\ \mathbf{X}_{nk}\end{bmatrix} + d_{nk}r^{-\bar{\lambda}_{nk}}\begin{bmatrix}\mathbf{U}_{nk+3}\\ \mathbf{X}_{nk+3}\end{bmatrix}\right) + rq_n(r)\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\right]\mathrm{e}^{\mathrm{i}n\theta}\right\} + \varepsilon\left(z\begin{bmatrix}\mathbf{k}_2\\ \mathbf{0}\end{bmatrix} + r\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix}\right)\\ &+ \vartheta r\left(z\begin{bmatrix}\mathbf{k}_1\\ \mathbf{0}\end{bmatrix} + r\begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{bmatrix}\right) + \mathrm{Re}\left\{\left(r^2\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix} + z^2\begin{bmatrix}\boldsymbol{\eta}\\ \mathbf{0}\end{bmatrix}\right)\mathrm{e}^{\mathrm{i}\theta}\right\}.\end{aligned} \tag{76}$$

Imposing the boundary conditions on the solution gives the following equations to determine the unknown c_{nk} and d_{nk}:

$$\sum_{k=1}^{3}(c_{nk}a^{\lambda_{nk}-1}\mathbf{X}_{nk} + d_{nk}a^{-\bar{\lambda}_{nk}-1}\mathbf{X}_{nk+3} + q_n(a)\mathbf{b}_2) = [\,A_n \quad 0 \quad 0\,]^{\mathrm{T}}, \tag{77}$$

$$\sum_{k=1}^{3}(c_{nk}b^{\lambda_{nk}-1}\mathbf{X}_{nk} + d_{nk}b^{-\bar{\lambda}_{nk}-1}\mathbf{X}_{nk+3} + q_n(b)\mathbf{b}_2) = [\,B_n \quad 0 \quad 0\,]^{\mathrm{T}}. \tag{78}$$

For a solid bar the negative power of r in the general solution must be discarded. The solution that satisfies the boundary conditions on $r = b$ is

$$\begin{bmatrix}\mathbf{u}\\ r\boldsymbol{\tau}_r\end{bmatrix} = \sum_{n=0}^{\infty}\mathrm{Re}\left\{\left(\sum_{k=1}^{3}c_{nk}r^{\lambda_{nk}}\begin{bmatrix}\mathbf{U}_{nk}\\ \mathbf{X}_{nk}\end{bmatrix} + rq_n(r)\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}\right)\mathrm{e}^{\mathrm{i}n\theta}\right\}, \tag{79}$$

where the c_{nk} are determined from

$$\sum_{k=1}^{3}c_{nk}b^{\lambda_{nk}-1}\mathbf{X}_{nk} + q_n(b)\mathbf{b}_2 = [\,B_n \quad 0 \quad 0\,]^{\mathrm{T}}. \tag{80}$$

The stress components $\boldsymbol{\tau}_s$ are readily obtained using the output equation.

5.4. A tube or bar with displacement boundary conditions

The displacement boundary conditions occurs in certain problems of cylindrical anisotropy, for example, in modeling the fiber-reinforced composite the interface between a rigid fiber and a soft matrix may be assumed to be constrained in the displacement. When the displacement is prescribed on the lateral surfaces of a tube, one might suspect that generalized plane strain may not exist. Let us examine the issue by considering a tube with mixed boundary conditions such that the inner surface is fixed and the outer surface is subjected to a uniform radial traction.

The lateral boundary conditions are

$$\mathbf{u} = \mathbf{0} \quad \text{on } r = a, \qquad r\boldsymbol{\tau}_r = [\,bp_b \quad 0 \quad 0\,]^{\mathrm{T}} \quad \text{on } r = b. \tag{81}$$

The general solution for the problem is

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \sum_{k=1}^{3} \left(c_k r^{\lambda_k} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + d_k r^{-\bar{\lambda}_k} \begin{bmatrix} \mathbf{U}_{0k+3} \\ \mathbf{X}_{0k+3} \end{bmatrix} \right) + \varepsilon \left(r \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_2 \\ \mathbf{0} \end{bmatrix} \right) + \vartheta r \left(r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{0} \end{bmatrix} \right) + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \tag{82}$$

Imposing the boundary conditions on the solution yields

$$\sum_{k=1}^{3} (c_k a^{\lambda_k - 1} \mathbf{U}_{0k} + c_{k+3} a^{-\bar{\lambda}_k - 1} \mathbf{U}_{0k+3}) + \varepsilon(\mathbf{a}_1 + \mathbf{k}_2 z/a) + \vartheta(a\mathbf{c}_1 + \mathbf{k}_1 z) + \mathbf{b}_1 \Delta T = \mathbf{0}, \tag{83}$$

$$\sum_{k=1}^{3} (c_k b^{\lambda_k - 1} \mathbf{X}_{0k} + c_{k+3} b^{-\bar{\lambda}_k - 1} \mathbf{X}_{0k+3}) + \varepsilon \mathbf{a}_2 + \vartheta b \mathbf{c}_2 + \mathbf{b}_2 \Delta T = [\,p_b \quad 0 \quad 0\,]^{\mathrm{T}}. \tag{84}$$

Eq. (83) holds for any z provided that

$$\varepsilon \mathbf{k}_2 + \vartheta a \mathbf{k}_1 = \mathbf{0}, \tag{85}$$

giving $\varepsilon = \vartheta = 0$. This indicates that the uniform extension and twisting angle per unit length must vanish for the state of generalized plane strain to be admissible. It does not mean that extension and torsion are inadmissible when the displacement is prescribed, rather it suggests that an appropriate axial force and torque must be applied at the ends in order to maintain the deformation $\varepsilon = \vartheta = 0$, otherwise the tube will not be in the state of generalized plane strain. When the condition is meet, the c_k and c_{k+3} in the solution can be determined from Eqs. (83) and (84).

For a solid bar with the lateral surface fixed, the general solution is

$$\begin{bmatrix} \mathbf{u} \\ r\boldsymbol{\tau}_r \end{bmatrix} = \sum_{k=1}^{3} c_k r^{\lambda_k} \begin{bmatrix} \mathbf{U}_{0k} \\ \mathbf{X}_{0k} \end{bmatrix} + \varepsilon \left(r \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_2 \\ \mathbf{0} \end{bmatrix} \right) + \vartheta r \left(r \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + z \begin{bmatrix} \mathbf{k}_1 \\ \mathbf{0} \end{bmatrix} \right) + r\Delta T \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}. \tag{86}$$

Imposing the boundary conditions $\mathbf{u} = \mathbf{0}$ at $r = b$ on the solution gives

$$\sum_{k=1}^{3} c_k b^{\lambda_k - 1} \mathbf{U}_k + \varepsilon(b\mathbf{a}_1 + \mathbf{k}_2 z/b) + \vartheta(b\mathbf{c}_1 + \mathbf{k}_1 z) + \mathbf{b}_1 \Delta T = \mathbf{0}, \tag{87}$$

which holds for any z if

$$\varepsilon \mathbf{k}_2 + \vartheta b \mathbf{k}_1 = \mathbf{0}, \tag{88}$$

indicating that $\varepsilon = \vartheta = 0$. Again, an axial force and a torque must be applied at the ends in order to maintain the state of generalized plane strain. When it does, the c_k in the solution can be determined from Eq. (87), the stress components $\boldsymbol{\tau}_s$ are obtained using the output equation.

6. Closure

In the formulation $[\mathbf{u}, r\boldsymbol{\tau}_r]$ has been taken as the state vector. This is advantageous for multilayered systems in which the interfacial and boundary conditions on the cylindrical surfaces $r =$ constant are directly expressed by $[\mathbf{u}, r\boldsymbol{\tau}_r]$. These conditions are easily satisfied using the transfer matrix without resort to a layerwise approach. For analysis of a multilayered cylindrical tube using the method of transfer matrix, see Tarn and Wang (2001). The state space formalism in conjunction with the transfer matrix is an elegant and effective approach for problems of a multilayered system.

A subject area of long standing is the three-dimensional anisotropic elasticity. An explicit formulation based on the basic three-dimensional equations as they stand would be intractable, especially in the cylindrical coordinate system. The present formalism paves a new way to treat the three-dimensional problems. The state equation and the output equation embrace all the three-dimensional equations of anisotropic elasticity. Only two independent unknown quantities, $\mathbf{u}$ and $r\boldsymbol{\tau}_r$ in the cylindrical coordinates; $\mathbf{u}$ and $\boldsymbol{\tau}_2$ in the Cartesian coordinates, (Tarn, 2002) play the key roles in the formalism. The elastic properties of the anisotropic material are characterized by four sub-matrices of the material matrix. With the formalism, there is no need to deal with the individual components of the displacement and stress and the elastic constants. The simplicity of the formulation makes treatment of a three-dimensional problem less formidable. Much can be gained if extension to three dimensions is viewed in the state space framework.

In closing, we remark that the Lekhnitskii and Stroh formalisms were seldom implemented in computational mechanics due to the use of stress functions. Since the stress components are obtained by differentiating the stress functions twice, a computational model based on them demands C^2 continuity of the assumed functions. By contrast, in the state space formalism the stresses as well as the displacements are regarded as the state variables, a computational model based on it in conjunction with a variational approach requires only C^0 continuity. A preliminary study shows that numerical modeling in the state space framework is worthy pursuing.

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