

Analysis of A Composite Time Integration Scheme

Sandeep Kumar¹, Sachin S. Gautam²

¹ Graduate Student, Department of Mechanical Engineering, IIT Guwahati,
Guwahati, Assam, 781039, India

² Asst. Professor, Department of Mechanical Engineering, IIT Guwahati,
Guwahati, Assam, 781039, India

Presenting authors email: sandeepkumar9995@gmail.com

Other Author's Email(s): ssg@iitg.ernet.in

Abstract

This paper proposes a simple extension to a collocation based composite time integration proposed by Silva and Bezerra [14]. In this scheme, each time step is further divided into two substeps which may not be necessarily equal. In the first substep, the Newmark scheme is employed and the three point backward Euler scheme is used in the second substep. Stability analysis of the scheme is performed and spectral radius, period elongation, and amplitude decay are discussed. The influence of Newmark parameters and substep sizes on stability and accuracy is studied in detail.

Keyword: Implicit time integration scheme; composite time integration; numerical stability.

1 Introduction

The distinctive nature between static and dynamic problem is the presence of inertia forces which opposes the motion generated by the applied dynamic loading. The dynamic nature of a problem is dominant if the inertia forces are large compared to the total applied forces [13]. In order to investigate the characteristics of transient dynamic problems, the resulting motion of a structural dynamic problem is studied for a given load distribution in space and time. Transient response analysis is used to compute the dynamic response of structure subjected to time-varying excitation. In order to obtain the time history of transient response time integration schemes are widely used. Time integration schemes are broadly classified into two categories: explicit and implicit [1, 5, 13, 6]. Some of the examples of explicit schemes are central difference method, Runge-Kutta method etc. Some of the examples of implicit schemes are Newmark scheme [11], Bathe composite scheme [3, 2, 4] etc.

For conservation of energy and momentum, trapezoidal scheme is combined to three-point backward Euler scheme. Trapezoidal scheme ensures second order accuracy and the backward Euler scheme ensures high-frequency numerical dissipation. Numerical dissipation is considered to be advantageous as it ensures better numerical stability for time integration schemes. Silva and Bezerra [14] proposed a scheme which is based on the Bathe scheme [2] but with generalised substep sizes instead of equal substep size used in the Bathe scheme. It was shown that for too large time step, the scheme remains stable but numerical dissipations are also large. Klarmann and Wagner [8] have further analyzed the Bathe scheme for variable step sizes and have shown that at a particular value of the step size, the period elongation is minimum and the numerical dissipation is maximum.

In the present work, an extension to the implicit composite scheme of Silva and Bezerra [14] is proposed. In this composite scheme, Newmark scheme [11] has been coupled with three-point backward Euler scheme thus making it a three parameter based composite time integration scheme. The paper is organized as follows. In section 2 the proposed scheme is explained where the basic equations of the scheme are discussed. In section 3 and 4, the stability and accuracy aspects of the proposed scheme are presented. Section 5 discusses the results for different parameters of the proposed scheme. Section 6 concludes this paper.

2 Proposed Implicit Composite Scheme

The scheme proposed in the present work extends the composite scheme proposed by Silva and Bezerra [14] where the variable time substep sizes are used. The proposed implicit composite scheme is a parameter based time integration scheme in which the Newmark scheme [11] is applied in the first substep and three-point backward Euler method for the second substep. The composite scheme is shown schematically in Figure (1).

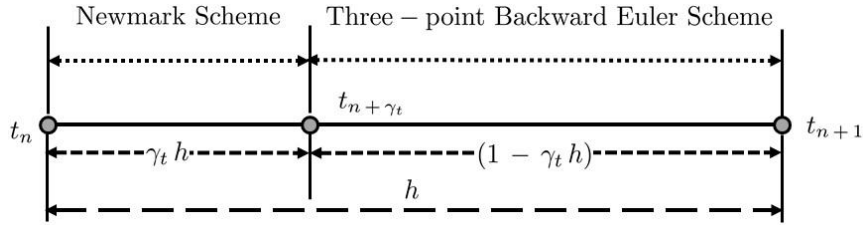


Figure 1: Proposed Composite Scheme. The time step is denoted by $t_{n+1} - t_n = h$.

The governing equations of equilibrium for nonlinear transient structural dynamic problems is expressed as follows:

$$M\ddot{u} + C\dot{u} + N(u, t) = F(t) \quad (1)$$

where M is the mass matrix, C is the damping matrix, $N(u, t)$ is the internal force vector which is, in general, a function of displacement vector u and time t and $F(t)$ is the external force vector. The vectors of velocity and acceleration are represented by \dot{u} , and \ddot{u} respectively. Note that for linear dynamic analysis, the internal force vector $N(u, t)$ can be written as $K u$ where K is the stiffness matrix. Next, the proposed scheme is explained in detail by applying it to Eq.(1).

Considering $t_{n+\gamma_t} = t_n + \gamma_t h$ (where h is the time step size) as an instance of time between t_n and t_{n+1} for $\gamma_t \in (0, 1)$, Newmark scheme is applied over the first substep, $\gamma_t h$ (see Fig. 1). The approximations for displacement and velocity at time $t_{n+\gamma_t}$ for Newmark scheme are given by

$$\begin{aligned} \dot{u}_{n+\gamma_t} &= \dot{u}_n + \gamma_t h [(1 - \gamma) \ddot{u}_n + \gamma \ddot{u}_{n+\gamma_t}] \\ u_{n+\gamma_t} &= u_n + (\gamma_t h) \dot{u}_n + \frac{(\gamma_t h)^2}{2} [(1 - 2\beta) \ddot{u}_n + (2\beta) \ddot{u}_{n+\gamma_t}] \end{aligned} \quad (2)$$

where β, γ are Newmark scheme parameters.

Then, the residual is defined as

$$R_{n+\gamma_t} = M \left[\frac{1}{\beta (\gamma_t h)^2} (u_{n+\gamma_t} - u_n) - \frac{1}{\beta (\gamma_t h)} \dot{u}_n - \left(\frac{1}{2\beta} - 1 \right) \ddot{u}_n \right] + N(u, t)_{n+\gamma_t} - F_{n+\gamma_t} \quad (3)$$

In the second substep the three point backward Euler scheme is applied over the second substep $(1 - \gamma_t) h$. The approximation for velocity and acceleration at time t_{n+1} for the three point backward Euler scheme is given by

$$\begin{aligned}\dot{u}_{n+1} &= c_1 u_n + c_2 u_{n+\gamma_t} + c_3 u_{n+1}, \\ \ddot{u}_{n+1} &= c_1 \dot{u}_n + c_2 \dot{u}_{n+\gamma_t} + c_3 \dot{u}_{n+1},\end{aligned}\quad (4)$$

where the constants are given expressed as

$$\begin{aligned}c_1 &= \frac{(1 - \gamma_t)}{\gamma_t h}, \\ c_2 &= \frac{-1}{(1 - \gamma_t) \gamma_t h}, \\ c_3 &= \frac{(2 - \gamma_t)}{(1 - \gamma_t) h}.\end{aligned}\quad (5)$$

Then, the residual is defined as

$$R_{n+1} = M \left[c_1 \dot{u}_n + c_2 \dot{u}_{n+\gamma_t} + c_3 c_1 u_n + c_3 c_2 u_{n+\gamma_t} + c_3^2 u_{n+1} \right] + N(u, t)_{n+1} - F_{n+1} \quad (6)$$

3 Stability of the Proposed Scheme

Stability can be loosely defined as the property of an integration method to keep the errors in the integration process of a given equation bounded at subsequent time steps. Spectral stability is concerned with the rate of growth, or decay of powers of the amplification matrix. In spectral stability, the dissipation can be measured by spectral radius $\rho(\mathbf{A})$. Hence, conditions for spectral stability can be summarized as [7] :

1. The spectral radius $\rho(\mathbf{A})$, which is the maximum of the eigen values of amplification matrix, \mathbf{A} , should be less than or equal to one i.e., $\rho(\mathbf{A}) \leq 1$.
2. Eigenvalues of \mathbf{A} of multiplicity greater than one, are strictly less than one in modulus.

Stability analysis uses the equation

$$\begin{pmatrix} \ddot{u}_{n+1} \\ \dot{u}_{n+1} \\ u_{n+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \ddot{u}_n \\ \dot{u}_n \\ u_n \end{pmatrix}$$

where \mathbf{A} is the amplification matrix and is expressed as following:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The characteristic equation of the amplification matrix is given by the following equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (7)$$

where λ and \mathbf{I} are the eigen values of amplification matrix \mathbf{A} and unit diagonal matrix, respectively.

Amplitude error can be measured by an equivalent viscous damping coefficient ξ , which is given by [13]

$$\xi = - \frac{\ln(\rho(\mathbf{A}))}{\varphi} \quad (8)$$

The period error [13], introduced by the time integration scheme, is defined as,

$$\frac{\delta T}{T} = \frac{\omega h}{\varphi} - 1. \quad (9)$$

4 Results and discussions

The stability and accuracy characteristics of the proposed scheme is studied for the following β , γ parameters of Newmark scheme as shown in Table 1. These values are chosen as per the relation given by Bathe [1].

Table 1: Newmark parameters.

S.No	β	γ
1	0.25	0.5
2	0.3025	0.6
3	0.36	0.7
4	0.4225	0.8
5	0.49	0.9

The values of γ_t considered for stability and accuracy analysis are chosen as 0.2, 0.4, 0.5, 0.6 and 0.8. The results for the proposed scheme are compared with those of the Bathe composite scheme [2].

Figs. 2(a), 3(a), 4(a), 5(a), and 6(a) show the plot of spectral radius with normalized time step size for different values of γ_t for the proposed scheme. In Fig. 2(a), proposed scheme with Newmark scheme paramters $\beta=0.25$, $\gamma=0.5$ and $\beta=0.3025$, $\gamma=0.6$ proves to be more efficient in preserving the low frequency modes compared to Bathe scheme [2]. Also, it is observed that as the value of Newmark parameters increases from $\beta=0.25$, $\gamma=0.5$ to $\beta=0.49$, $\gamma=0.9$, the stability reduces in preserving the low frequency modes and damping the high frequency modes. Proposed scheme with parameters $\gamma_t=0.8$, $\beta=0.49$ and $\gamma=0.9$ (See: Fig. 6(a)) is least efficient.

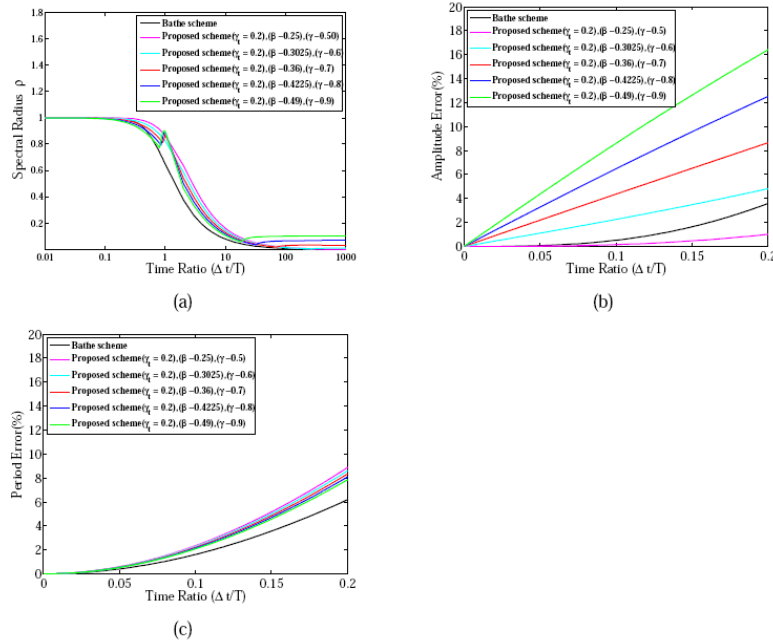


Figure 2: Variation of spectral radii, amplitude error and period error for $\gamma_t = 0.2$.

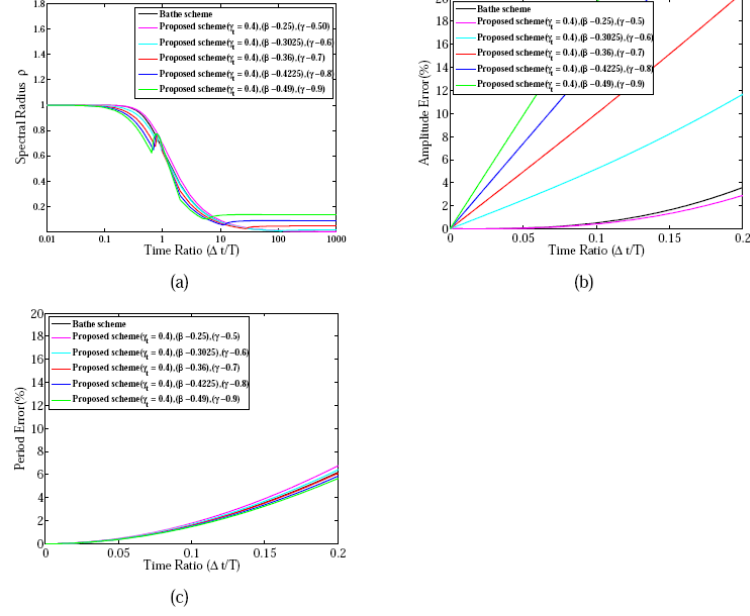


Figure 3: Variation of spectral radii, amplitude error and period error for $\gamma_t = 0.4$.

Figs. 2(b), 3(b), 4(b), 5(b), and 6(b) show the plot of amplitude error with normalized time step size for different values of γ_t for different Newmark parameters and time step ratio (γ_t) values. In Fig. 2(b), 3(b), and 6(b), for Newmark parameters $\beta=0.25, \gamma=0.5$ amplitude decay is quite less than that of the Bathe scheme [2]. For all other Newmark parameters of the proposed scheme (for all γ_t values) except for $\beta=0.25, \gamma=0.5$, amplitude decays are high. For $\gamma_t=0.6, \beta=0.25$ and $\gamma=0.5$ (Fig. 5(b)), the amplitude decay of the proposed scheme is slightly more than the Bathe scheme [2].

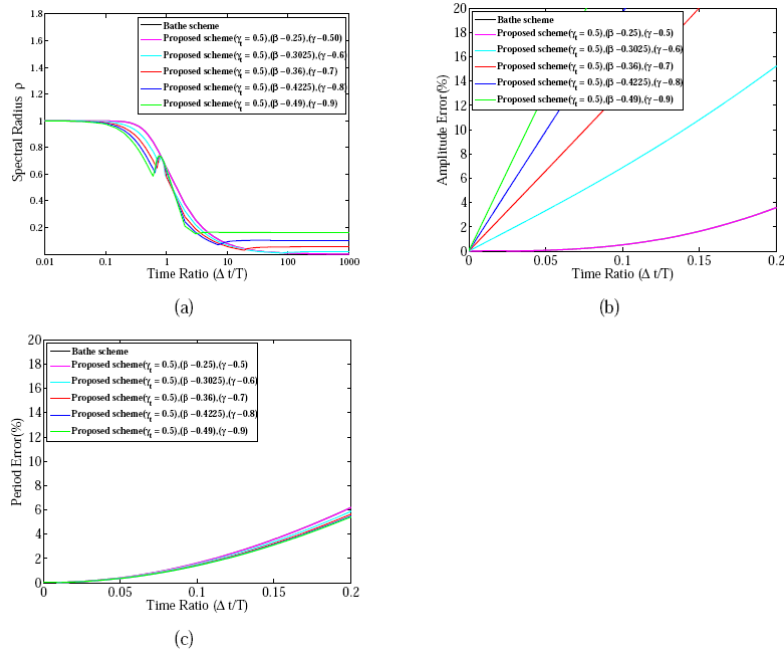


Figure 4: Variation of spectral radii, amplitude error and period error for $\gamma_t = 0.5$.

Figs. 2(c) to 6(c) show the period error with normalized time step size for different values of γ_t . In Figs. 2(c) and 3(c), the period elongation for the proposed scheme is maximum for Newmark parameters $\beta=0.25$ and $\gamma=0.5$. For all other values of Newmark parameters, period elongation is less than that of Bathe scheme. For $\gamma_t = 0.6$

(Fig. 5(c)), period elongation of the proposed scheme is less than that of Bathe scheme [2] for all values of Newmark parameters. For $\gamma_t = 0.4$ (Fig. 3(c)) and Newmark parameters- $\beta=0.4225$, $\gamma=0.8$ and $\beta=0.49$, $\gamma=0.9$, period elongation is less than that of Bathe scheme and for Newmark parameters- $\beta=0.36$, $\gamma=0.7$, period elongation almost coincides with the Bathe scheme.

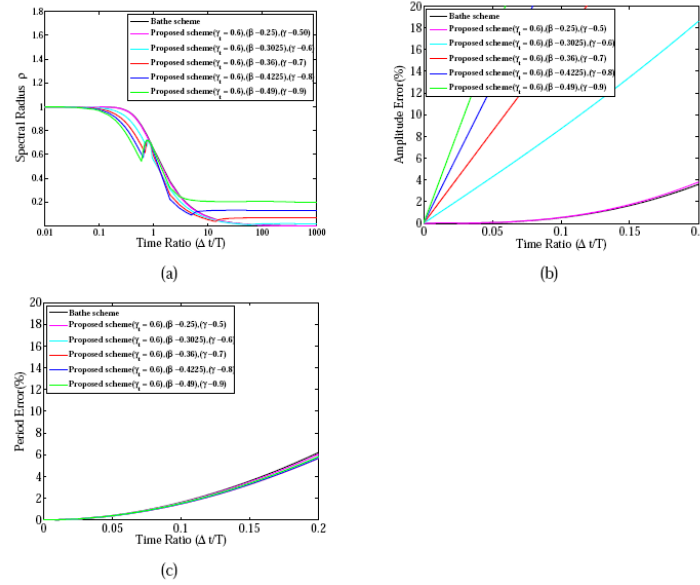


Figure 5: Variation of spectral radii, amplitude error and period error for $\gamma_t = 0.6$.

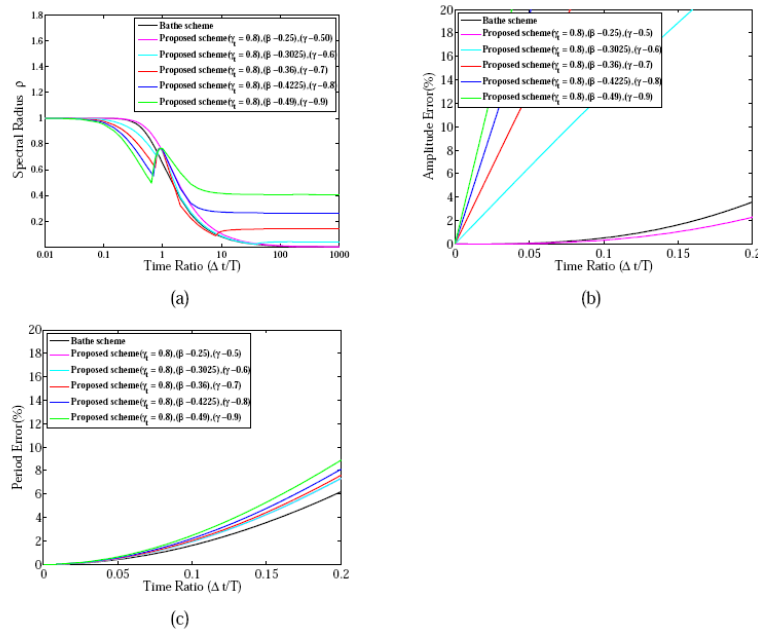


Figure 6: Variation of spectral radii, amplitude error and period error for $\gamma_t = 0.8$.

4 Conclusion

The present work is an extension of a composite scheme proposed by Silva and Bezerra [14]. This scheme gives freedom to choose any combinations of Newmark parameters and substep sizes to control the high frequency

dissipation. For some combinations of Newmark parameters and substep sizes, the proposed scheme gives better results in terms of stability and accuracy when compared with Bathe composite scheme.

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