Nonlinear Elastic Inclusions in Anisotropic Solids*

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Abstract

In this paper we study the stress and deformation fields generated by nonlinear inclusions with finite eigenstrains in anisotropic solids. In particular, we consider finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars made of both compressible and incompressible solids. We show that the stress field in a spherical inclusion with uniform pure dilatational eigenstrain in a spherical ball made of an incompressible transversely isotropic solid such that the material preferred direction is radial at any point is uniform and hydrostatic. Similarly, the stress in a cylindrical inclusion contained in an incompressible orthotropic cylindrical bar is uniform hydrostatic if the radial and circumferential eigenstrains are equal and the axial stretch is equal to a value determined by the axial eigenstrain. We also prove that for a compressible isotropic spherical ball and a cylindrical bar containing a spherical and a cylindrical inclusion, respectively, with uniform eigenstrains the stress in the inclusion is uniform (and hydrostatic for the spherical inclusion) if the radial and circumferential eigenstrains are equal. For compressible transversely isotropic and orthotropic solids, we show that the stress field in an inclusion with uniform eigenstrain is not uniform, in general. Nevertheless, in some special cases the material can be designed in order to maintain a uniform stress field in the inclusion. As particular examples to investigate such special cases, we consider compressible Mooney-Rivlin and Blatz-Ko reinforced models and find analytical expressions for the stress field in the inclusion.

Keywords: Transversely isotropic solids; orthotropic solids; finite eigenstrains; geometric mechanics; anisotropic inclusions; nonlinear elasticity.

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1 Introduction

Inclusions are regions of a body that have stress-free configurations different from that of the body and can be modeled using distributed eigenstrains. The anelastic part of any measure of strain that represents distortions, referential rearrangements, phase changes, etc., is called eigenstrain. Eigenstrains can model a host of anelastic effects in solids, such as swelling and cavitation [Pence and Tsai, 2005, 2006, 2007, Goriely et al., 2010, Moulton and Goriely, 2011], bulk and surface growth [Amar and Goriely, 2005, Yavari, 2010, Sozio and Yavari, 2017], thermal strains [Stojanovic et al., 1964, Ozakin and Yavari, 2010, Sadik and Yavari, 2015], and defects [Yavari, 2016, Sadik and Yavari, 2016].

In the setting of linear elasticity, Eshelby [1957] showed that the stress field in an ellipsoidal inclusion with uniform eigenstrains embedded in an infinite linear elastic medium is uniform. Since then the study of inclusions has been mainly restricted to linear elasticity. There are some recent 2D solutions for the inclusion problem in the case of harmonic solids [Ru and Schiavone, 1996, Ru et al., 2005, Kim and Schiavone, 2007, 2008, Kim et al., 2008]. In 3D, Yavari and Goriely [2013] investigated the nonlinear inclusion problem in isotropic solids. They

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showed that the stress field inside spherical and cylindrical inclusions with finite pure dilatational eigenstrains in spherical balls and cylindrical bars, receptively, is uniform for both incompressible isotropic solids and some special classes of compressible isotropic solids. Finite shear and torsional eigenstrains in nonlinear solids were studied by Yavari and Goriely [2015b]. As an example, they solved the problem of a cylindrical inhomogeneity with finite shear eigenstrains and examined the effect of torsional eigensrains on the stiffness of a circular cylindrical bar.

Willis [1964] formulated the two-dimensional linear inclusion problem for an infinite anisotropic medium. He obtained explicit solutions for an elliptic inclusion in a medium with cubic symmetry. He showed that the stress field inside such an inclusion is uniform. In the setting of 3D linear elasticity, Li and Dunn [1998] investigated the inclusion and inhomogeneity problem in an infinite anisotropic solid using Eshelby's approach. They found closed-form expressions for the Eshelby tensors in the case of transversely isotropic media containing cylindrical and thin-disk inclusions. Kinoshita and Mura [1971] obtained the displacement and stress fields induced by an inclusion with a uniform distribution of eigenstrains in an infinitely extended homogeneous linear anisotropic elastic medium. Their expressions are valid for the general case of material anisotropy and different shapes of inclusions. In a series of papers [Jiang and Pan, 2004, Pan, 2004b,a, Yue et al., 2015, Lee et al., 2015], two-dimensional Eshelby's problem for linear polygonal inclusions in anisotropic full and half-planes were studied. Giordano et al. [2009] investigated the elastic properties of composites consisting of isotropic spherical and cylindrical inhomogeneities embedded in a linear isotropic solid matrix. They obtained the elastic properties of the overall material in terms of the elastic constants of the constituents and their volume fractions under the simplifying assumptions of small strains for the body and small volume fractions of the embedded phase.

To our best knowledge, the problem of nonlinear inclusions in anisotropic solids has not been studied in the literature. In this paper, we consider finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars for both incompressible and compressible solids. We then determine conditions that guarantee that the stress field in spherical and cylindrical inclusions with uniform dilatational eigenstrains is uniform. In particular, we show that the results given in [Yavari and Goriely, 2013] for some special classes of compressible isotropic solids can be generalized to an arbitrary compressible isotropic solid. In the case of compressible transversely isotropic and orthotropic solids, we show that there are some nontrivial special cases for which uniform stress can be maintained in the inclusion when the radial and circumferential eigenstrains are not equal (or the axial stretch satisfies some conditions in the case of cylindrical bars). To investigate these cases, we employ the so called standard reinforcing model (see, e.g., [Merodio and Ogden, 2003]) and find the stress field in the inclusion in the case of compressible Mooney-Rivlin and Blatz-Ko materials for several reinforcement combinations.

This paper is organized as follows. In section 2 we tersely review some fundamental concepts of geometric nonlinear elasticity for anisotropic solids. In section 3.1 we consider finite eigenstrains in an incompressible transversely isotropic spherical ball. In section 3.2 the corresponding problem in the case of compressible transversely isotropic and compressible isotropic solids is discussed. Finite eigenstrains in an incompressible orthotropic cylindrical bar is studied in section 3.3. Section 3.4 is devoted to compressible orthotropic cylindrical bars with finite eigenstrains. We conclude the paper with some remarks in section 4.

2 Elements of Geometric Anelasticity for Anisotropic Bodies

In this section, we briefly review some fundamental concepts of the geometric theory of nonlinear elasticity for anisotropic solids (see [Marsden and Hughes, 1994, Yavari et al., 2006] for more detailed discussions).

Kinematics. A body $\mathcal{B}$ is identified with a three-dimensional Riemannian manifold $(\mathcal{B}, \mathbf{G})$, and a deformation of the body is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ denotes the ambient space. The deformation gradient $\mathbf{F}$ is the derivative map of φ defined as $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$. The adjoint of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathsf{T}}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}\,, \qquad \mathbf{g}\left(\mathbf{F}\mathbf{V}, \mathbf{v}\right) = \mathbf{G}\left(\mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v}\right), \quad \forall \mathbf{V} \in T_X\mathcal{B},\, \mathbf{v} \in T_{\varphi_t(X)}\mathcal{S}\,. \tag{2.1}$$

The right Cauchy-Green deformation tensor is defined as $\mathbf{C}(X,t) = \mathbf{F}^{\mathsf{T}}(X,t)\mathbf{F}(X,t) : T_X\mathcal{B} \to T_X\mathcal{B}$. The material and spatial Riemannian volume elements are related by the Jacobian of the motion as $dv(x, \mathbf{g}) =$

$JdV(X,\mathbf{G})$, where J is given by

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F}\,. \tag{2.2}$$

Equilibrium Equations. The localized balance of linear momentum of a body in static equilibrium and in the absence of body forces in terms of the Cauchy stress tensor reads

$$\operatorname{div}\boldsymbol{\sigma}=\mathbf{0}\,, \tag{2.3}$$

where div is the spatial divergence operator, which in components reads

$$(\operatorname{div}\boldsymbol{\sigma})^a={\sigma^{ab}}_{|b}=\frac{\partial\sigma^{ab}}{\partial x^b}+\sigma^{ac}{\gamma^b}_{cb}+\sigma^{cb}{\gamma^a}_{cb}\,, \tag{2.4}$$

and ${\gamma^a}_{bc}$ is the Christoffel symbol of the Levi-Civita connection $\nabla^{\mathbf{g}}$ asscoiated with the spatial metric $\mathbf{g}$ in the local chart $\{x^a\}$, defined as $\nabla^{\mathbf{g}}{}_{\partial_b}\partial_c={\gamma^a}_{bc}\partial_a$.

Constitutive Equations. In this paper we restrict our calculations to compressible and incompressible transversely isotropic and orthotropic materials. We use structural tensors to establish a materially covariant strain energy density function corresponding to the symmetry group of the material. See [Spencer, 1971, 1982, Liu et al., 1982, Zheng and Spencer, 1993, Lu and Papadopoulos, 2000] for detailed discussions of structural tensors and the determination of the integrity basis for the invariants of a collection of tensors.

Transverse isotropy. Let us consider a compressible transversely isotropic solid with the unit vector $\mathbf{N}(\mathrm{X})$ identifying the material preferred direction at a point X in the reference configuration. The strain energy density function (per unit volume) is given by (see, e.g., [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000])

$$W=W(\mathrm{X},\mathbf{G},\mathbf{C}^\flat,\mathbf{A})\,, \tag{2.5}$$

where $\mathbf{A}=\mathbf{N}\otimes\mathbf{N}$ is a structural tensor associated with the transverse isotropy material symmetry group. The second Piola-Kirchhoff stress tensor is written as

$$\mathbf{S}=2\frac{\partial W}{\partial\mathbf{C}^\flat}\,. \tag{2.6}$$

The energy function W depends on five independent invariants defined as follows

$$I_1=\operatorname{tr}\mathbf{C}\,,\quad I_2=\det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}\,,\quad I_3=\det\mathbf{C}\,,\quad I_4=\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,,\quad I_5=\mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,. \tag{2.7}$$

In components, $I_1={C^A}_A$, $I_2=\det({C^A}_B)({C^{-1}})^D{}_D$, $I_3=\det({C^A}_B)$, $I_4=N^AN^BC_{AB}$, and $I_5=N^AN^BC_{BQ}{C^Q}_A$. Using (2.6), one has[1]

$$\mathbf{S}=2W_{I_n}\frac{\partial I_n}{\partial\mathbf{C}^\flat}\,,\qquad W_{I_n}:=\frac{\partial W}{\partial I_n}\,,\quad n=1,\dots,5\,. \tag{2.8}$$

It then follows that

$$\frac{\partial I_1}{\partial\mathbf{C}^\flat}=\mathbf{G}^\sharp\,,\quad \frac{\partial I_2}{\partial\mathbf{C}^\flat}=I_2\mathbf{C}^{-1}-I_3\mathbf{C}^{-2}\,,\quad \frac{\partial I_3}{\partial\mathbf{C}^\flat}=I_3\mathbf{C}^{-1}\,,\quad \frac{\partial I_4}{\partial\mathbf{C}^\flat}=\mathbf{N}\otimes\mathbf{N}\,,\quad \frac{\partial I_5}{\partial\mathbf{C}^\flat}=\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N}+\mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\,. \tag{2.9}$$

Therefore, (2.8) and (2.9) give the following representation for $\mathbf{S}$.

$$\mathbf{S}=2\left\{W_{I_1}\mathbf{G}^\sharp+W_{I_2}\left(I_2\mathbf{C}^{-1}-I_3\mathbf{C}^{-2}\right)+W_{I_3}I_3\mathbf{C}^{-1}+W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right)+W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N}+\mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\}\,. \tag{2.10}$$

In the case of incompressible materials $I_3=1$, and hence, $W=W(\mathrm{X},I_1,I_2,I_4,I_5)$. Thus, from (2.10), one expresses $\mathbf{S}$ as

$$\mathbf{S}=2\left\{W_{I_1}\mathbf{G}^\sharp+W_{I_2}\left(I_2\mathbf{C}^{-1}-\mathbf{C}^{-2}\right)+W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right)+W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N}+\mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\}-p\mathbf{C}^{-1}\,, \tag{2.11}$$

[1] For the sake of simplicity of calculations, here we do not consider an explicit dependence of W on X, which is needed in the case of inhomogeneous bodies. Instead, we assume that the material is piece-wise homogeneous and model an inhomogeneity by using different energy functions in different regions of the body.

where p is the Lagrange multiplier associated with the incompressibility constraint $J = 1$. The Cauchy stress $\sigma^{ab} = \frac{1}{J} F^a{}_A F^b{}_B S^{AB}$ has the following representation in component form[2]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B \left[(W_{I_1} + I_1 W_{I_2}) G^{AB} - W_{I_2} C^{AB} + W_{I_4} N^A N^B + W_{I_5} \left(N^Q N^A C^B{}_Q + N^P N^B C_P{}^A \right) \right] - p g^{ab}. \tag{2.13}$$

Orthotropy. We next consider a compressible orthotropic material such that $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ are three $\mathbf{G}$-orthonormal vectors specifying the orthotropic axes in the reference configuration at a point X. A choice of structural tensors for this case is given by $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3 \otimes \mathbf{N}_3$, only two of which are independent.[3] Therefore, the energy function is written as [Doyle and Ericksen, 1956, Spencer, 1982, Lu and Papadopoulos, 2000]

$$W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)\,. \tag{2.14}$$

The energy function W depends on the following seven independent invariants.

$$\begin{gathered} I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\,\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}_1 \cdot \mathbf{C} \cdot \mathbf{N}_1\,, \\ I_5 = \mathbf{N}_1 \cdot \mathbf{C}^2 \cdot \mathbf{N}_1\,, \quad I_6 = \mathbf{N}_2 \cdot \mathbf{C} \cdot \mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2 \cdot \mathbf{C}^2 \cdot \mathbf{N}_2\,. \end{gathered} \tag{2.15}$$

From (2.6), one obtains

$$\mathbf{S} = 2W_{I_n} \frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \quad n = 1, \ldots, 7\,. \tag{2.16}$$

Substituting (2.9) into (2.16), the second Piola-Kirchhoff stress tensor is written as

$$\begin{aligned} \mathbf{S} = 2\Big\{ & W_{I_1} \mathbf{G}^\sharp + W_{I_2} \left(I_2 \mathbf{C}^{-1} - I_3 \mathbf{C}^{-2} \right) + W_{I_3} I_3 \mathbf{C}^{-1} + W_{I_4} (\mathbf{N}_1 \otimes \mathbf{N}_1) + W_{I_5} (\mathbf{N}_1 \otimes \mathbf{C} \cdot \mathbf{N}_1 + \mathbf{N}_1 \cdot \mathbf{C} \otimes \mathbf{N}_1) \\ & + W_{I_6} (\mathbf{N}_2 \otimes \mathbf{N}_2) + W_{I_7} (\mathbf{N}_2 \otimes \mathbf{C} \cdot \mathbf{N}_2 + \mathbf{N}_2 \cdot \mathbf{C} \otimes \mathbf{N}_2) \Big\}\,. \end{aligned} \tag{2.17}$$

If the material is assumed to be incompressible, then it follows that $I_3 = 1$ and $W = W(\mathrm{X}, I_1, I_2, I_4, I_5, I_6, I_7)$. Hence, from (2.17), one obtains the following representation for the second Piola-Kirchhoff stress tensor

$$\begin{aligned} \mathbf{S} = 2\Big\{ & W_{I_1} \mathbf{G}^\sharp + W_{I_2} \left(I_2 \mathbf{C}^{-1} - \mathbf{C}^{-2} \right) + W_{I_4} (\mathbf{N}_1 \otimes \mathbf{N}_1) + W_{I_5} (\mathbf{N}_1 \otimes \mathbf{C} \cdot \mathbf{N}_1 + \mathbf{N}_1 \cdot \mathbf{C} \otimes \mathbf{N}_1) \\ & + W_{I_6} (\mathbf{N}_2 \otimes \mathbf{N}_2) + W_{I_7} (\mathbf{N}_2 \otimes \mathbf{C} \cdot \mathbf{N}_2 + \mathbf{N}_2 \cdot \mathbf{C} \otimes \mathbf{N}_2) \Big\} - p\mathbf{C}^{-1}\,. \end{aligned} \tag{2.18}$$

The Cauchy stress tensor is given in components by

$$\begin{aligned} \sigma^{ab} = 2F^a{}_A F^b{}_B \Big[& (W_{I_1} + I_1 W_{I_2}) G^{AB} - W_{I_2} C^{AB} + W_{I_4} N_1{}^A N_1{}^B + W_{I_5} \left(N_1{}^Q N_1{}^A C^B{}_Q + N_1{}^P N_1{}^B C_P{}^A \right) \\ & + W_{I_6} N_2{}^A N_2{}^B + W_{I_7} \left(N_2{}^S N_2{}^A C^B{}_S + N_2{}^K N_2{}^B C_K{}^A \right) \Big] - p g^{ab}\,. \end{aligned} \tag{2.19}$$

3 Examples of Anisotropic Bodies with Finite Eigenstrains

In this section, we consider several examples of inclusions in transversely isotropic spherical balls and orthotropic cylindrical bars. We start with spherically and cylindrically symmetric distributions of finite dilatational eigenstrains in a spherical ball and a solid cylinder, respectively. We study the inclusion problem by considering uniform distribution of finite anisotropic eigenstrains in the inclusion region. We then investigate the conditions under which the stress inside the inclusion is uniform. We also identify those cases that exhibit stress singularities, depending on the values of the radial and circumferential eigenstrains, along with the axial eigenstrain in the case of cylindrical bars.

[2]Note that using the Cayley-Hamilton theorem, one can write

$$\frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2 (\mathbf{C}^{-1})^\sharp - I_3 (\mathbf{C}^{-2})^\sharp = I_1 \mathbf{G}^\sharp - \mathbf{C}^\sharp. \tag{2.12}$$

[3]Note that $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$.

3.1 Finite Eigenstrains in an Incompressible Transversely Isotropic Spherical Ball

Consider a ball of radius R_o made of a nonlinear incompressible transversely isotropic material with a given spherically symmetric distribution of radial and circumferential eigenstrains. We assume that the material preferred direction is radial, i.e., $\mathbf{N} = \hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ is a unit vector in the radial direction. The material metric for the eigenstrain-free configuration in the spherical coordinates (R, Θ, Φ) reads $\mathbf{G}_o = \operatorname{diag}\left(1, R^2, R^2 \sin^2\Theta\right)$. To preserve the spherical symmetry, we require that the azimuthal and circumferential eigenstrains be equal. Therefore, the material metric for the ball with dilatational eigenstrains is written as[4]

$$\mathbf{G} = \begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta(R)} & 0 \\ 0 & 0 & e^{2\omega_\Theta(R)} R^2 \sin^2\Theta \end{pmatrix}, \tag{3.1}$$

where ω_R and ω_Θ describe the radial and circumferential eigenstrains, respectively. We endow the ambient space with the flat Euclidean metric $\mathbf{g} = \operatorname{diag}\left(1, r^2, r^2 \sin^2\theta\right)$ in the spherical coordinates (r, θ, ϕ). We then assume an embedding of the material manifold into the ambient space with the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, and hence, $\mathbf{F} = \operatorname{diag}(r'(R), 1, 1)$. Assuming incompressibility, i.e., $J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = 1$, one obtains

$$\frac{r^2(R) r'(R)}{R^2 e^{\omega_R(R)+2\omega_\Theta(R)}} = 1\,. \tag{3.2}$$

Eliminating the rigid body translation by setting $r(0) = 0$ gives

$$r(R) = \left(\int_0^R 3\eta^2 e^{\omega_R(\eta)+2\omega_\Theta(\eta)} d\eta\right)^{\frac{1}{3}}. \tag{3.3}$$

Therefore, the right Cauchy-Green deformation tensor is written as[5]

$$\mathbf{C} = \begin{pmatrix} \frac{R^4 e^{4\omega_\Theta(R)}}{r^4(R)} & 0 & 0 \\ 0 & \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} & 0 \\ 0 & 0 & \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} \end{pmatrix}. \tag{3.4}$$

Using (2.7), the invariants of the strain energy function are simplified to read[6]

$$I_1 = \operatorname{tr}(\mathbf{C}) = \frac{2r^2(R)}{R^2} e^{-2\omega_\Theta(R)} + \frac{R^4}{r^4(R)} e^{4\omega_\Theta(R)}, \tag{3.5}$$

$$I_2 = \frac{1}{2}\left(\operatorname{tr}(\mathbf{C}^2) - \operatorname{tr}(\mathbf{C})^2\right) = \frac{2R^2}{r^2(R)} e^{2\omega_\Theta(R)} + \frac{r^4(R)}{R^4} e^{-4\omega_\Theta(R)}, \tag{3.6}$$

$$I_4 = \left(\frac{e^{\omega_\Theta(R)} R}{r(R)}\right)^4, \tag{3.7}$$

$$I_5 = \left(\frac{e^{\omega_\Theta(R)} R}{r(R)}\right)^8. \tag{3.8}$$

[4]Similar constructions using nontrivial material manifolds with the explicit dependence of the material metric on the type of anelasticity were discussed in [Yavari, 2010, Ozakin and Yavari, 2010, Yavari and Goriely, 2013, Sadik and Yavari, 2015, Golgoon et al., 2016, Golgoon and Yavari, 2017].

[5]All the symbolic computations in this paper were performed using Mathematica [Wolfram Research, 2016].

[6]Note that $\hat{\mathbf{N}} = e^{-\omega_R(R)}\mathbf{E}_R$ is the unit vector defining the material preferred direction, where $\mathbf{E}_R = \frac{\partial}{\partial R}$ is a radial basis vector for $T_X\mathcal{B}$ such that $\langle\!\langle \mathbf{E}_R, \mathbf{E}_R \rangle\!\rangle_{\mathbf{G}} = G_{RR}$.

Following (2.13), the non-zero components of the Cauchy stress tensor read

$$\sigma^{rr} = -p + 2\left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^4 (W_{I_1} + W_{I_4}) + \left(\frac{2e^{\omega_\Theta(R)}R}{r(R)}\right)^2 W_{I_2} + 4W_{I_5}\left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^8 , \tag{3.9}$$

$$\sigma^{\theta\theta} = \frac{2e^{-2\omega_\Theta(R)}W_{I_1}}{R^2} - \frac{p}{r^2(R)} + 2W_{I_2}\left(\frac{R^2e^{2\omega_\Theta(R)}}{r^4(R)} + \frac{r^2(R)}{R^4}e^{-4\omega_\Theta(R)}\right) , \tag{3.10}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta} . \tag{3.11}$$

Note that when the body is eigenstrain-free, $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$. Assuming that the stress vanishes for this case, we obtain (similar conditions were derived in [Merodio and Ogden, 2003, Vergori et al., 2013])

$$(2W_{I_5} + W_{I_4})\,|_{I_1=I_2=3, I_4=I_5=1} = 0 . \tag{3.12}$$

The physical components of the Cauchy stress tensor, i.e., $\hat{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}}$ (no summation) [Truesdell, 1953] are written as

$$\hat{\sigma}^{rr} = \sigma^{rr} , \quad \hat{\sigma}^{\theta\theta} = r^2(R)\sigma^{\theta\theta} , \quad \hat{\sigma}^{\phi\phi} = r^2(R)\sin^2\theta\,\sigma^{\phi\phi} . \tag{3.13}$$

In the absence of body forces and inertial effects, the only non-trivial equilibrium equation is ${\sigma^{rb}}_{|b} = 0$ following (2.4). Note that $p = p(R)$ is implied from the other two equilibrium equations. Therefore

$$\sigma^{rr}{}_{,r} + \frac{2}{r}\sigma^{rr} - r\sigma^{\theta\theta} - r\sin^2\theta\,\sigma^{\phi\phi} = 0 . \tag{3.14}$$

Using (3.11), equation (3.14) is rewritten as

$$\frac{1}{r'(R)}\sigma^{rr}{}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0 . \tag{3.15}$$

Therefore, substituting (3.9) and (3.10) into (3.15), one obtains $p'(R) = h(R)$, where

$$\begin{aligned} h(R) = -\frac{4e^{-2\omega_\Theta}}{R^3r^{19}}\Big(& -8R^{18}W_{I_5I_5}r^3e^{18\omega_\Theta}(R\omega'_\Theta+1) + 8R^{17}(W_{I_4I_5}+W_{I_1I_5})r^4e^{16\omega_\Theta+\omega_R} \\ & + 12R^{15}W_{I_2I_5}r^6e^{14\omega_\Theta+\omega_R} + 2R^{13}(3W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1})r^8e^{12\omega_\Theta+\omega_R} \\ & - 8R^{14}(W_{I_4I_5}+W_{I_1I_5})r^7e^{14\omega_\Theta}(R\omega'_\Theta+1) - 12R^{12}W_{I_2I_5}r^9e^{12\omega_\Theta}(R\omega'_\Theta+1) \\ & + 2R^{11}(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2})r^{10}e^{10\omega_\Theta+\omega_R} - 2R^5(W_{I_2I_4}+3W_{I_1I_2})r^{16}e^{4\omega_\Theta+\omega_R} \\ & - 2R^8(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2})r^{13}e^{8\omega_\Theta}(R\omega'_\Theta+1) + 8R^{21}W_{I_5I_5}e^{20\omega_\Theta+\omega_R} \\ & + 4W_{I_2I_2}r^{21}(R\omega'_\Theta+1) - 2R^6(W_{I_4}-2W_{I_2I_5}+2W_{I_2I_2}+W_{I_1})r^{15}e^{6\omega_\Theta}(R\omega'_\Theta+1) \\ & - 2R^4(W_{I_2}-W_{I_1I_4}-W_{I_1I_1})r^{17}e^{4\omega_\Theta}(R\omega'_\Theta+1) - R^3(4W_{I_2I_2}-W_{I_1})r^{18}e^{2\omega_\Theta+\omega_R} \\ & - 2R^{10}(4W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1})r^{11}e^{10\omega_\Theta}(R\omega'_\Theta+1) + RW_{I_2}r^{20}e^{\omega_R} \\ & + 2R^2(W_{I_2I_4}+3W_{I_1I_2})r^{19}e^{2\omega_\Theta(R)}(R\omega'_\Theta+1) + R^7(W_{I_2}-2(W_{I_1I_4}+W_{I_1I_1}))r^{14}e^{6\omega_\Theta+\omega_R} \\ & + R^9(W_{I_4}-4W_{I_2I_5}+4W_{I_2I_2}+W_{I_1})r^{12}e^{8\omega_\Theta+\omega_R}\Big) . \end{aligned} \tag{3.16}$$

If one assumes that the ball is subject to a uniform pressure p_∞ at its outer boundary, i.e, $\sigma^{rr}(R_o) = -p_\infty$, one obtains

$$\begin{aligned} p(R) = p_\infty + \int_{R_o}^R h(\zeta)d\zeta + 2\left(\frac{e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^4 (W_{I_1}|_{R=R_o} + W_{I_4}|_{R=R_o}) + \left(\frac{2e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^2 W_{I_2}|_{R=R_o} \\ + 4W_{I_5}|_{R=R_o}\left(\frac{e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^8 . \end{aligned} \tag{3.17}$$

Spherical inclusion in a transversely isotropic ball. Let us consider the following distributions of eigenstrains

$$\omega_R(R) = \begin{cases} \omega_1\,, & 0 \le R \le R_i \\ 0\,, & R_i \le R \le R_o \end{cases}\,, \quad \omega_\Theta(R) = \begin{cases} \omega_2\,, & 0 \le R \le R_i \\ 0\,, & R_i \le R \le R_o \end{cases}\,. \tag{3.18}$$

This corresponds to having an inclusion with radius R_i at the center of the ball. It follows from (3.2) that

$$r(R) = \begin{cases} e^{\frac{\omega_1}{3}+\frac{2\omega_2}{3}} R\,, & 0 \le R \le R_i \\ \left(R^3 + \left(e^{\omega_1+2\omega_2}-1\right) R_i^3\right)^{\frac{1}{3}}\,, & R_i \le R \le R_o \end{cases}\,. \tag{3.19}$$

Using (3.16) and (3.19), one has $p'(R) = h_0/R$ in the inclusion ($0 \le R \le R_i$), where

$$\begin{aligned} h_0 = 4e^{-\frac{8\omega_1}{3}-\frac{4\omega_2}{3}} \Big(& 2e^{4\omega_2} W_{I_5} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}} W_{I_4} - e^{4\omega_1} W_{I_2} + e^{2\omega_1+2\omega_2} W_{I_2} \\ & - e^{\frac{10\omega_1}{3}+\frac{2\omega_2}{3}} W_{I_1} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}} W_{I_1} \Big) \Big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}}\,, \end{aligned} \tag{3.20}$$

and $I_a = e^{\frac{2}{3}(\omega_1-\omega_2)}$. Moreover, in the matrix, $p'(R) = \hat{h}(R)$, where for $R_i \le R \le R_o$

$$\begin{aligned} \hat{h}(R) = -\frac{4}{R^3 r(R)^{19}} \Big(& -8R^{18} W_{I_5I_5} r^3 + 8R^{17}\left(W_{I_4I_5} + W_{I_1I_5}\right) r^4 - 8R^{14}\left(W_{I_4I_5} + W_{I_1I_5}\right) r^7 \\ & + 2R^{13}\left(3W_{I_5} + W_{I_4I_4} + 2W_{I_1I_4} + W_{I_1I_1}\right) r^8 - 12R^{12} W_{I_2I_5} r^9 + 4W_{I_2I_2} r^{21} + 8R^{21} W_{I_5I_5} \\ & + 2R^{11}\left(3W_{I_2I_4} - 2W_{I_1I_5} + 3W_{I_1I_2}\right) r^{10} - 2R^{10}\left(4W_{I_5} + W_{I_4I_4} + 2W_{I_1I_4} + W_{I_1I_1}\right) r^{11} \\ & + R^9\left(W_{I_4} - 4W_{I_2I_5} + 4W_{I_2I_2} + W_{I_1}\right) r^{12} - 2R^8\left(3W_{I_2I_4} - 2W_{I_1I_5} + 3W_{I_1I_2}\right) r^{13} + RW_{I_2} r^{20} \\ & + R^7\left(W_{I_2} - 2\left(W_{I_1I_4} + W_{I_1I_1}\right)\right) r^{14} - 2R^6\left(W_{I_4} - 2W_{I_2I_5} + 2W_{I_2I_2} + W_{I_1}\right) r^{15} \\ & + 2R^4\left(W_{I_1I_4} + W_{I_1I_1} - W_{I_2}\right) r^{17} + R^3\left(W_{I_1} - 4W_{I_2I_2}\right) r^{18} + 2R^2\left(W_{I_2I_4} + 3W_{I_1I_2}\right) r^{19} \\ & + 12R^{15} W_{I_2I_5} r^6 - 2R^5\left(W_{I_2I_4} + 3W_{I_1I_1}\right) r^{16} \Big)\,. \end{aligned} \tag{3.21}$$

Therefore, the pressure field distribution is given by

$$p(R) = \begin{cases} h_0 \ln\left(\frac{R}{R_i}\right) - c_i\,, & 0 \le R \le R_i\,, \\ \int_{R_o}^{R} \hat{h}(\zeta) d\zeta - c_o\,, & R_i \le R \le R_o\,, \end{cases} \tag{3.22}$$

where c_i and c_o are constants of integration to be determined after imposing the boundary conditions. The physical components of the Cauchy stress have the following distributions

$$\hat{\sigma}^{rr}(R) = \begin{cases} h_0 \ln\left(\frac{R_i}{R}\right) + 2e^{\frac{2}{3}(\omega_2-4\omega_1)}\Big(2e^{2\omega_2} W_{I_5} + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_4} + 2e^{2\omega_1} W_{I_2} \\ \qquad + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}} + c_i\,, & 0 \le R \le R_i\,, \\ c_o + \Big(\int_R^{R_o} \hat{h}(\zeta) d\zeta + 2\frac{R^4}{r^4}\left(W_{I_4} + W_{I_1}\right) + 4\frac{R^2}{r^2} W_{I_2} \\ \qquad + 4\frac{R^8}{r^8} W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R)+I^4(R),I_2=2I^2(R)+I^{-4}(R),I_4^2=I_5=I^8(R)}\,, & R_i \le R \le R_o\,, \end{cases} \tag{3.23}$$

$$\hat{\sigma}^{\theta\theta}(R) = \begin{cases} h_0 \ln\left(\frac{R_i}{R}\right) + c_i + 2e^{-\frac{2}{3}(\omega_1+2\omega_2)}\Big[\left(e^{2\omega_1} + e^{2\omega_2}\right) W_{I_2} \\ \qquad + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_1}\Big]\Big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}}\,, & 0 \le R \le R_i\,, \\ c_o + \Big[\int_R^{R_o} \hat{h}(\zeta) d\zeta + 2W_{I_2}\left(\frac{r^4}{R^4} + \frac{R^2}{r^2}\right) \\ \qquad + \frac{2W_{I_1} r^2}{R^2}\Big]\Big|_{I_1=2I^{-2}(R)+I^4(R),I_2=2I^2(R)+I^{-4}(R),I_4^2=I_5=I^8(R)}\,, & R_i \le R \le R_o\,, \end{cases} \tag{3.24}$$

where $I(R) = R/r(R)$, and note that $\hat{\sigma}^{\theta\theta}(R) = \hat{\sigma}^{\phi\phi}(R)$. The boundary condition $\sigma^{rr}(R_o) = -p_\infty$ gives us

$$\begin{aligned} c_o = -p_\infty - \Big(& \frac{2R_o^4}{r^4(R_o)}\left(W_{I_4} + W_{I_1}\right) + \frac{4R_o^2}{r^2(R_o)} W_{I_2} \\ & + \frac{4R_o^8}{r^8(R_o)} W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R_o)+I^4(R_o),I_2=2I^2(R_o)+I^{-4}(R_o),I_4^2=I_5=I^8(R_o)}\,. \end{aligned} \tag{3.25}$$

The continuity of the traction vector at the inclusion-matrix interface implies that σ^{rr} must be continuous at $R = R_i$. Using the expression for c_o in (3.25), this condition gives c_i as

$$
\begin{aligned}
c_i = &\int_{R_i}^{R_o} \hat{h}(\zeta)d\zeta + 2e^{-\frac{8}{3}(\omega_1+2\omega_2)}\Big(e^{\frac{4}{3}(\omega_1+2\omega_2)}\left(2e^{\frac{2}{3}(\omega_1+2\omega_2)}W_{I_2} + W_{I_4} + W_{I_1}\right)\\
&+ 2W_{I_5}\Big)\Big|_{I_1=2I_b^{-2}+I_b^4, I_2=2I_b^2+I_b^{-4}, I_4^2=I_5=I_b^8} - 2e^{\frac{2}{3}(\omega_2-4\omega_1)}\Big(2e^{2\omega_2}W_{I_5} + e^{\frac{2}{3}(2\omega_1+\omega_2)}W_{I_4}\\
&+ 2e^{2\omega_1}W_{I_2} + e^{\frac{2}{3}(2\omega_1+\omega_2)}W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} - p_\infty - \Big(\frac{2R_o^4}{r^4(R_o)}(W_{I_4}+W_{I_1})\\
&+ \frac{4R_o^2}{r^2(R_o)}W_{I_2} + \frac{4R_o^8}{r^8(R_o)}W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R_o)+I^4(R_o), I_2=2I^2(R_o)+I^{-4}(R_o), I_4^2=I_5=I^8(R_o)}\,,
\end{aligned}
\tag{3.26}
$$

where $I_b = I(R_i) = e^{-\frac{1}{3}(\omega_1+2\omega_2)}$.

Remark 3.1. Evidently, if $h_0 = 0$, from (3.23) and (3.24), the stress field in the inclusion will be uniform and hydrostatic. Note that when $\omega_1 = \omega_2$, one has $I_a = 1$, and hence, from (3.20) and (3.12)

$$
h_0 = 4\Big(2W_{I_5} + W_{I_4}\Big)\Big|_{I_1=I_2=3, I_4=I_5=1} = 0\,. \tag{3.27}
$$

Therefore, if $\omega_1 = \omega_2$, then $h_0 = 0$ for any nonlinear incompressible transversely isotropic solid. If $\omega_1 \neq \omega_2$, however, for h_0 to be zero the strain energy function must satisfy the following condition, which in turn puts a restriction on the energy function (cf. (3.20))

$$
\begin{aligned}
\Big(2e^{4\omega_2}W_{I_5} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_4} - e^{4\omega_1}W_{I_2} + e^{2\omega_1+2\omega_2}W_{I_2}&\\
- e^{\frac{10\omega_1}{3}+\frac{2\omega_2}{3}}W_{I_1} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} &= 0\,.
\end{aligned}
\tag{3.28}
$$

Therefore, we have proved the following proposition.

Proposition 3.2. *Consider a nonlinear incompressible transversely isotropic spherical ball such that the material preferred direction is radial. Suppose that the ball is subject to a uniform pressure on its boundary. Assume that the ball contains a spherical inclusion at its center with uniform radial and circumferential eigenstrains. The stress field inside the inclusion exhibits a logarithmic singularity at the center of the ball unless the radial and circumferential eigenstrains are equal or the energy function satisfies* (3.28)*. Moreover, the stress inside the inclusion is uniform and hydrostatic if the eigenstrains are pure dilatational.*

Remark 3.3. Given a nonlinear incompressible transversely isotropic spherical ball with the radial material preferred direction and a radially-symmetric distribution of radial and circumferential eigenstrains $e^{\omega_R(R)}$ and $e^{\omega_\Theta(R)}$, respectively, the stress exhibits a logarithmic singularity at the center of the ball unless $\omega_R(0) = \omega_\Theta(0)$. To see this, let $\omega_R(0) = \omega_1$ and $\omega_\Theta(0) = \omega_2$. Note that as $R \to 0$ (see also [Yavari and Goriely, 2015a])

$$
\omega_R(R) = \omega_1 + \mathcal{O}(R)\,, \qquad \omega_\Theta(R) = \omega_2 + \mathcal{O}(R)\,, \qquad r(R) = e^{\frac{\omega_1}{3}+\frac{2\omega_2}{3}}R + \mathcal{O}(R^2)\,. \tag{3.29}
$$

Moreover

$$
I_1(R) = 2I_a + I_a^{-2} + \mathcal{O}(R)\,,\ I_2(R) = 2I_a^{-1} + I_a^2 + \mathcal{O}(R)\,,\ I_4(R) = I_a^{-2} + \mathcal{O}(R)\,,\ I_5(R) = I_a^{-4} + \mathcal{O}(R)\,. \tag{3.30}
$$

Thus

$$
W_{I_i}(R) = W_{I_i}\big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} + \mathcal{O}(R)\,, \quad i = 1,2,4,5\,. \tag{3.31}
$$

Similarly

$$
W_{I_iI_j}(R) = W_{I_iI_j}\big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} + \mathcal{O}(R)\,, \quad i,j = 1,2,4,5\,. \tag{3.32}
$$

Therefore, using the above asymptotic expansions, from (3.16), one obtains

$$
h(R) = \frac{h_0}{R} + \mathcal{O}(1)\,. \tag{3.33}
$$

Hence, $p(R) = h_0 \ln R + \mathcal{O}(R)$, i.e., the stress field has a logarithmic singularity at the origin only if $\omega_R(0) \neq \omega_\Theta(0)$.

3.2 Finite Eigenstrains in a Compressible Transversely Isotropic Spherical Ball

Next, we consider a compressible transversely isotropic material with a radial material preferred direction. Given an embedding of the form $(r,\theta,\phi)=(r\left(R\right),\Theta,\Phi)$, the right Cauchy-Green deformation tensor reads

$$\mathbf{C}=\begin{pmatrix} r'(R)^2e^{-2\omega_R(R)} & 0 & 0 \\ 0 & \frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2} & 0 \\ 0 & 0 & \frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2} \end{pmatrix}. \tag{3.34}$$

The Jacobean is written as

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F}=\frac{r^2(R)r'(R)}{R^2e^{\omega_R(R)+2\omega_\Theta(R)}}\,. \tag{3.35}$$

The invariants are found using (2.7) and read

$$I_1=\mathrm{tr}(\mathbf{C})=r'(R)^2e^{-2\omega_R(R)}+\frac{2r^2(R)e^{-2\omega_\Theta(R)}}{R^2}\,, \tag{3.36}$$

$$I_2=\frac{1}{2}(\mathrm{tr}(\mathbf{C}^2)-\mathrm{tr}(\mathbf{C})^2)=\frac{r^4(R)}{R^4}r'(R)^2e^{-2(2\omega_\Theta(R)+\omega_R(R))}\left(\frac{e^{2\omega_R(R)}}{r'(R)^2}+\frac{2R^2e^{2\omega_\Theta(R)}}{r(R)^2}\right)\,, \tag{3.37}$$

$$I_3=\det(\mathbf{C})=\frac{r^4(R)}{R^4}r'(R)^2e^{-2(2\omega_\Theta(R)+\omega_R(R))}\,, \tag{3.38}$$

$$I_4=e^{-2\omega_R(R)}r'(R)^2\,, \tag{3.39}$$

$$I_5=e^{-4\omega_R(R)}r'(R)^4\,. \tag{3.40}$$

The non-zero components of the Cauchy stress tensor are written as

$$\sigma^{rr}=\frac{2r'(R)e^{-2\omega_\Theta(R)-3\omega_R(R)}}{R^2r^2(R)}\Big[R^4e^{4\omega_\Theta(R)}\left(e^{2\omega_R(R)}\left(W_{I_1}+W_{I_4}\right)+2W_{I_5}r'(R)^2\right) + 2R^2W_{I_2}r^2(R)e^{2(\omega_\Theta(R)+\omega_R(R))}+W_{I_3}r^4(R)e^{2\omega_R(R)}\Big]\,, \tag{3.41}$$

$$\sigma^{\theta\theta}=\frac{2e^{-\omega_R(R)}}{r'(R)r^2(R)}\left[W_{I_1}e^{2\omega_R(R)}+W_{I_2}r'(R)^2+\frac{e^{-2\omega_\Theta(R)}r^2(R)}{R^2}\left(W_{I_2}e^{2\omega_R(R)}+W_{I_3}r'(R)^2\right)\right]\,, \tag{3.42}$$

$$\sigma^{\phi\phi}=\frac{1}{\sin^2\Theta}\sigma^{\theta\theta}\,. \tag{3.43}$$

When the body is eigenstrain-free, we assume that the stress vanishes. Therefore

$$(W_{I_4}+2W_{I_5})\,|_{I_1=I_2=3,I_3=I_4=I_5=1}=0\,,\quad\text{and}\quad (W_{I_1}+2W_{I_2}+W_{I_3})\,|_{I_1=I_2=3,I_3=I_4=I_5=1}=0\,. \tag{3.44}$$

Substituting the stress components into (3.14), the simplified radial equilibrium equation is given in Appendix A.

Next, we consider the eigenstrain distribution (3.18) and solve the problem of a spherical inclusion with uniform anisotropic eigenstrains in a compressible transversely isotropic spherical ball. We then explore conditions under which the induced stress field in the inclusion is uniform. These conditions would impose some restrictions on the energy function, in general. Let us assume that the stress field in the inclusion is uniform, i.e., $\hat{\sigma}^{rr}=C_1$ and $\hat{\sigma}^{\theta\theta}=C_2$, where C_1 and C_2 are constants. It then follows from (3.41) and (3.42) for $0\leq R\leq R_i$ that:

$$C_1=\frac{2e^{-2\omega_2-3\omega_1}r'(R)}{R^2r(R)^2}\Big[R^4e^{4\omega_2}\left(e^{2\omega_1}\left(W_{I_1}+W_{I_4}\right)+2W_{I_5}r'(R)^2\right) + 2R^2e^{2(\omega_1+\omega_2)}W_{I_2}r(R)^2+e^{2\omega_1}W_{I_3}r(R)^4\Big]\,, \tag{3.45}$$

and

$$C_2=\frac{2e^{-\omega_1-2\omega_2}}{R^2r'(R)}\Big[R^2e^{2(\omega_1+\omega_2)}W_{I_1}+r(R)^2\left(e^{2\omega_1}W_{I_2}+W_{I_3}r'(R)^2\right)+R^2e^{2\omega_2}W_{I_2}r'(R)^2\Big]\,. \tag{3.46}$$

The first-order[7] nonlinear ODEs (3.45) and (3.46) are subject to the boundary condition $r(0) = 0$. We note that for $r(R) = \beta R$ in the inclusion, with β a constant, all the invariants of deformation are constant in the inclusion, and so are the partial derivatives of the energy function with respect to the invariants. Therefore, one can immediately see that $r(R) = \beta R$ is a solution of both initial-value problems (IVPs).[8] That is, the stress field in the inclusion is uniform if $r(R) = \beta R$ for $0 \leq R \leq R_i$. Note that when the stress in the inclusion is uniform, it then immediately follows from the equilibrium equation (3.15) that the stress is hydrostatic as well, i.e., $C_1 = C_2$. Now, we examine the conditions that guarantee that $r(R) = \beta R$ satisfies the radial equilibrium equation ($C_1 = C_2$). Using (3.45) and (3.46), one obtains the following condition in the inclusion.

$$\begin{aligned}\Big[e^{2\omega_1}\left(e^{2\omega_2}-e^{2\omega_1}\right)\left(\beta^2 W_{I_2}+e^{2\omega_2}W_{I_1}\right)+e^{4\omega_2}\left(2\beta^2 W_{I_5}\right.\\ \left.+e^{2\omega_1}W_{I_4}\right)\Big]\Big|_{I_1=\beta^2(e^{-2\omega_1}+2e^{-2\omega_2}),I_2=\beta^4 e^{-4\omega_2}(2e^{2\omega_2-2\omega_1}+1),I_3=\beta^6 e^{-2(\omega_1+2\omega_2)},I_4^2=I_5=e^{-4\omega_1}\beta^4}=0\,. \qquad (3.47)\end{aligned}$$

Note that when the radial and circumferential eigenstrains are equal ($\omega_1 = \omega_2$), the above condition is satisfied without imposing any restrictions on the energy function or β if the material is compressible and isotropic, i.e., $W = W(I_1, I_2, I_3)$, and hence, $W_{I_4} = W_{I_5} = 0$. This observation suggests that if the material is compressible and isotropic, and the inclusion has a uniform distribution of pure dilatational eigenstrains, then the stress inside the inclusion is uniform and hydrostatic. This generalizes the result of Yavari and Goriely [2013] that was proved for harmonic solids and class II and III materials according to Carroll [1988]. For compressible isotropic solids, (A.1) gives us the following second-order nonlinear ODE in the matrix (for $R_i \leq R \leq R_o$)

$$\begin{aligned}&R^5\left(R^4W_{I_1}r''+2R^2r^2W_{I_2}r''+r^4W_{I_3}r''-2R^2rW_{I_1}-2r^3W_{I_2}\right)\\&+4rr'^3\left(Rr'-r\right)\left[R^4r^2\left(2W_{I_2I_2}+W_{I_1I_3}\right)+3R^2r^4W_{I_2I_3}+r^6W_{I_3I_3}+R^6W_{I_1I_2}\right]\\&-r'\left[4R^6r^2W_{I_1I_1}+2R^4r^4\left(6W_{I_1I_2}+W_{I_3}\right)+4R^2r^6\left(2W_{I_2I_2}+W_{I_1I_3}\right)+4r^8W_{I_2I_3}-2R^8W_{I_1}\right]\\&+2Rr'^2\Big\{R^8W_{I_1I_1}r''+4R^6r^2W_{I_1I_2}r''+2R^4r^4\left(2W_{I_2I_2}+W_{I_1I_3}\right)r''+4R^2r^6W_{I_2I_3}r''+r^8W_{I_3I_3}r''\\&+2r^7W_{I_2I_3}+R^6r\left(2W_{I_1I_1}+W_{I_2}\right)+R^4r^3\left(6W_{I_1I_2}+W_{I_3}\right)+2R^2r^5\left(2W_{I_2I_2}+W_{I_1I_3}\right)\Big\}=0\,, \qquad (3.48)\end{aligned}$$

for which we need two boundary conditions, and given that β is also an unknown, we need three boundary conditions in total. These are given by continuity of $r(R)$ and the traction vector at $R = R_i$, and the boundary condition $\hat{\sigma}^{rr}(R_o) = -p_\infty$. Therefore, we have proved the following proposition.

Proposition 3.4. *Consider a spherical ball made of a compressible isotropic solid subject to a uniform pressure on its boundary sphere. Assume that the ball contains a spherical inclusion at its center with uniform radial and circumferential eigenstrains. The stress field in the inclusion is uniform and hydrostatic if the eigenstrains are pure dilatational.*

Remark 3.5. Consider the conditions in Proposition 3.4 for compressible isotropic solids and assume that the stress field inside the inclusion is uniform. We observed that $r(R) = \beta R$, where $0 \leq R \leq R_i$ is a solution for (3.45) and (3.46) subject to the boundary condition $r(0) = 0$. Therefore, the simplified equilibrium equation (3.47) implies that the radial and circumferential eigenstrains must be equal. Otherwise, from (3.47) the energy function and β must satisfy the following relation

$$\left[\beta^2 W_{I_2}+e^{2\omega_2}W_{I_1}\right]\Big|_{I_1=\beta^2(e^{-2\omega_1}+2e^{-2\omega_2}),I_2=\beta^4 e^{-4\omega_2}(2e^{2\omega_2-2\omega_1}+1),I_3=\beta^6 e^{-2(\omega_1+2\omega_2)}}=0\,. \qquad (3.49)$$

The boundary conditions and the above relation in turn put a restriction on the energy function.

For a compressible transversely isotropic material if the radial and circumferential eigenstrains are equal ($\omega_1 = \omega_2 = \omega$), from (3.47), we obtain

$$\left(W_{I_4}+2a^2W_{I_5}\right)\big|_{I_1=3a^2,I_2=3a^4,I_3=a^6,I_4^2=I_5=a^4}=0\,, \qquad (3.50)$$

[7]Note that the invariants of deformation, and thus, the energy function and its partial derivatives with respect to the invariants depend on the first and not higher order derivatives of r.

[8]Note that it is straightforward to show that there are no other solutions of the form $r(R) = \beta R^\alpha$, $\alpha > 1$ to these IVPs.

where $a = \beta e^{-\omega}$. Clearly, from the first equation in (3.44), $a = 1$ is a trivial solution of the above equation, which is stress-free and volume preserving ($I_3 = 1$). If we assume that the traction in the fiber direction is tensile for extension ($a > 1$) and compressive for contraction ($a < 1$), e.g., see [Merodio and Ogden, 2002], then $a = 1$ is the only solution of (3.50). This result simply suggests that for compressible transversely isotropic materials the induced stress field inside the inclusion with uniform pure dilatational eigenstrains is uniform in the trivial case $R_i = R_o$, i.e., when the entire ball has a uniform distribution of pure dilatational eigenstrains, which is stress-free.

Nonetheless, there are some nontrivial cases that can only occur if the radial and circumferential eigenstrains are different ($\omega_1 \neq \omega_2$). Such cases are special in the sense that a specific pressure must be applied on the boundary to maintain a uniform hydrostatic stress field inside the inclusion, or for a given pressure applied on the outer boundary, the ratio R_i/R_o is determined. This is because β is determined from (3.47) when ($\omega_1 \neq \omega_2$), and as the equilibrium equation in the matrix is a nonlinear second-order ODE, we only need two boundary conditions to find its solution. These are given by the continuity of $r(R)$ and the traction vector at $R = R_i$. To see this, we note that when β is determined from (3.47), the stress and deformation fields in the inclusion will be fully known. Therefore, the two boundary conditions of the equilibrium equation in the matrix are written as

$$r(R_i^+) = \beta R_i \,, \qquad \hat{\sigma}^{rr}(R_i^+) = \hat{\sigma}^{rr}(R_i^-) \,. \tag{3.51}$$

Hence, one may fix R_i/R_o and find the pressure that must be applied on the outer boundary using the relation $\hat{\sigma}^{rr}(R_o) = -p_\infty$. Alternatively, using this relation, one can find R_i/R_o by prescribing the pressure p_∞.

Next, we consider some specific strain energy functions to explore (3.47), where a choice of energy function determines β when $\omega_1 \neq \omega_2$. In doing so, we employ the so called *standard reinforcing model* for compressible materials, defined as [Merodio and Ogden, 2003, 2005]

$$W = W\left(I_1, I_2, I_3, I_4, I_5\right) = W_{\text{iso}}\left(I_1, I_2, I_3\right) + W_{\text{fib}}\left(I_4, I_5\right) \,, \tag{3.52}$$

where the first term denotes the isotropic base material, whereas the second term represents the anisotropic effects due to the fiber reinforcement. Let us consider the following strain energy functions (see, e.g., [Merodio and Ogden, 2005]):

i) Compressible Mooney-Rivlin reinforced model (I_4 reinforcement) for which

$$W(I_1, I_2, I_3, I_4) = C_1\left(I_1 - 3\right) + C_2\left(I_2 - 3\right) - \left(C_1 + 2C_2\right)\left(I_3 - 1\right) + \frac{\mu}{2}\left(I_4 - 1\right)^2 \,, \tag{3.53}$$

where C_1, C_2, and μ are constants, and $\mu > 0$ is an anisotropy parameter describing the reinforcement property. Therefore, from (3.47), we have

$$\beta = e^{\omega_1 + \omega_2}\left[\frac{C_1\delta + \mu e^{2\omega_2}}{\mu e^{4\omega_2} - C_2 e^{2\omega_1}\delta}\right]^{\frac{1}{2}} \,, \tag{3.54}$$

where $\delta = e^{2\omega_1} - e^{2\omega_2}$. We need to have the following constraint on μ for β to be a real positive number.

$$\begin{aligned} &\mu > C_2 e^{2(\omega_1 - \omega_2)}\left(e^{2(\omega_1 - \omega_2)} - 1\right) \,, && \text{for} \quad \omega_1 > \omega_2 \,, \\ &\mu > C_1\left(1 - e^{2(\omega_1 - \omega_2)}\right) \,, && \text{for} \quad \omega_1 < \omega_2 \,. \end{aligned} \tag{3.55}$$

As expected, the stress field in the inclusion is uniform and hydrostatic, i.e., $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = \sigma_o$, where

$$\begin{aligned} \sigma_o = -\frac{2\delta e^{\omega_2}}{\left(\mu e^{4\omega_2} - C_2\delta e^{2\omega_1}\right)^2}\sqrt{\frac{\mu e^{4\omega_2} - C_2\delta e^{2\omega_1}}{C_1\delta + \mu e^{2\omega_2}}}\Big[&C_1\left(C_2\mu e^{2\omega_2}\left(\delta + 4e^{2\omega_1}\right) + C_2^2\delta e^{2\omega_1} + \mu^2 e^{4\omega_2}\right) \\ &+ 2C_1^2 e^{2\omega_1}\left(C_2\delta + \mu e^{2\omega_2}\right) + C_2\mu\left(C_2 e^{2\omega_1}\left(\delta + 2e^{2\omega_2}\right) + \mu e^{4\omega_2}\right) + C_1^3\delta e^{2\omega_1}\Big] \,. \end{aligned} \tag{3.56}$$

ii) Compressible Mooney-Rivlin reinforced model (I_5 reinforcement) that has the following energy function

$$W(I_1, I_2, I_3, I_5) = C_1\left(I_1 - 3\right) + C_2\left(I_2 - 3\right) - \left(C_1 + 2C_2\right)\left(I_3 - 1\right) + \frac{\mu}{2}\left(I_5 - 1\right)^2 \,. \tag{3.57}$$

Substituting (3.57) into (3.47) gives us

$$2\beta^2\mu e^{4\omega_2}\left(\beta^4 e^{-4\omega_1}-1\right)-e^{2\omega_1}\left(e^{2\omega_1}-e^{2\omega_2}\right)\left(C_1e^{2\omega_2}+\beta^2C_2\right)=0\,. \tag{3.58}$$

Therefore

$$\beta=\frac{e^{-2\omega_2}}{6^{\frac{1}{3}}\mu^{\frac{1}{2}}}\left(\frac{6^{\frac{1}{3}}\mu e^{4\omega_2}}{\Delta}\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)+e^{4\omega_1}\Delta\right)^{\frac{1}{2}}\,, \tag{3.59}$$

where Δ is defined as[9]

$$\Delta=e^{2(\omega_2-\omega_1)}\left[\sqrt{3}\mu^{3/2}\sqrt{27C_1^2\delta^2\mu e^{8\omega_2}-2\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)^3}+9C_1\delta\mu^2e^{4\omega_2}\right]^{\frac{1}{3}}\,. \tag{3.60}$$

The value of the hydrostatic stress in the inclusion is

$$\sigma_o=\frac{e^{-7\omega_1-2\omega_2}}{\beta}\left[2C_1e^{6\omega_1}\left(e^{4\omega_2}-\beta^4\right)+4\beta^2\left\{C_2e^{6\omega_1}\left(e^{2\omega_2}-\beta^2\right)-\mu e^{4\omega_2}\left(e^{4\omega_1}-\beta^4\right)\right\}\right]\,. \tag{3.61}$$

iii) Blatz-Ko reinforced model (I_4 reinforcement) for which the energy function is written as

$$W(I_2,I_3,I_4)=\frac{\mu_o}{2}\left(\frac{I_2}{I_3}+2I_3^{\frac{1}{2}}-5\right)+\frac{\mu}{2}\left(I_4-1\right)^2\,, \tag{3.62}$$

where $\mu_1,\mu_2>0$. From (3.47), we have

$$\mu_oe^{4\omega_1}\delta+2\beta^4\mu\left(e^{2\omega_1}-\beta^2\right)=0\,. \tag{3.63}$$

Hence

$$\beta=\frac{1}{\sqrt{3}}\left(\frac{2^{2/3}\mu e^{4\omega_1}}{\eta}+\frac{\eta}{2^{2/3}\mu}+e^{2\omega_1}\right)^{\frac{1}{2}}\,, \tag{3.64}$$

where η is given by

$$\eta=e^{\frac{4\omega_1}{3}}\mu^{\frac{2}{3}}\left[3\sqrt{3\mu_o}\sqrt{27\delta^2\mu_o+8\mu\delta e^{2\omega_1}}+27\delta\mu_o+4\mu e^{2\omega_1}\right]^{\frac{1}{3}}\,. \tag{3.65}$$

For β to be physical, i.e., $\beta\in\mathbb{R}^+$, it can be shown that one must have $\omega_1>\omega_2$. In that case, the hydrostatic stress in the inclusion reads

$$\sigma_o=\frac{e^{2\omega_2}}{\beta^5}\left[\mu_o\left(\beta^5e^{-2\omega_2}-e^{3\omega_1}\right)-2e^{-3\omega_1}\beta^4\mu\left(e^{2\omega_1}-\beta^2\right)\right]\,. \tag{3.66}$$

3.3 Finite Eigenstrains in a Finite Incompressible Orthotropic Cylindrical Bar

Let us consider a finite circular cylindrical bar of radius R_o made of a nonlinear incompressible orthotropic solid with a cylindrically-symmetric distribution of radial and circumferential eigenstrains in the reference configuration. Assume that the material orthotropic axes are in the R, Θ, and Z directions in the cylindrical coordinates (R,Θ,Z). Given the eigenstrain-free material metric, i.e., $\mathbf{G}_o=\operatorname{diag}\left(1,R^2,1\right)$, the material metric for the bar with eigenstrains is written as

$$\mathbf{G}=\begin{pmatrix} e^{2\omega_R(R)} & 0 & 0\\ 0 & R^2e^{2\omega_\Theta(R)} & 0\\ 0 & 0 & e^{2\omega_Z(R)}\end{pmatrix}\,, \tag{3.67}$$

where ω_R, ω_Θ, and ω_Z are some functions describing the radial, circumferential, and axial eigenstrains, respectively. The ambient space is endowed with the Euclidean metric $\mathbf{g}=\left(1,r^2,1\right)$. We embed the material

[9]Note that $\frac{6^{\frac{1}{3}}\mu e^{4\omega_2}}{\Delta}\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)+e^{4\omega_1}\Delta>0$ puts a constraint on the elastic constants.

manifold into the ambient space by looking for mappings of the form $(r,\theta,z)=(r(R),\Theta,\alpha Z)$, where α is a constant representing the axial stretch of the bar that depends on the axial boundary conditions.[10] Therefore, the deformation gradient reads $\mathbf{F}=\operatorname{diag}(r'(R),1,\alpha)$. Incompressibility constraint is written as

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F}=\frac{\alpha r(R)r'(R)}{Re^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}=1\,. \tag{3.68}$$

Requiring $r(0)=0$, one obtains

$$r(R)=\left(\int_0^R\frac{2\eta}{\alpha}e^{\omega_R(\eta)+\omega_\Theta(\eta)+\omega_Z(\eta)}d\eta\right)^{\frac{1}{2}}. \tag{3.69}$$

The right Cauchy-Green deformation tensor reads

$$\mathbf{C}=\begin{pmatrix}\frac{e^{2\omega_Z(R)+2\omega_\Theta(R)}R^2}{\alpha^2r^2(R)} & 0 & 0\\ 0 & \frac{e^{-2\omega_\Theta(R)}r^2(R)}{R^2} & 0\\ 0 & 0 & e^{-2\omega_Z(R)}\alpha^2\end{pmatrix}. \tag{3.70}$$

Let us denote the orthotropic axes by $\mathbf{N}_1=\hat{\mathbf{R}}$, $\mathbf{N}_2=\hat{\mathbf{Z}}$, and $\mathbf{N}_3=\hat{\boldsymbol{\Theta}}$, where $\hat{\mathbf{R}}$, $\hat{\mathbf{Z}}$, and $\hat{\boldsymbol{\Theta}}$ denote the unit vectors in the radial, longitudinal, and circumferential directions, respectively. Thus, $\mathbf{N}_1=e^{-\omega_R(R)}\mathbf{E}_R$, $\mathbf{N}_2=e^{-\omega_Z(R)}\mathbf{E}_Z$, and $\mathbf{N}_3=e^{-\omega_\Theta(R)}\mathbf{E}_\Theta/R$, where $\mathbf{E}_R=\partial/\partial R$, $\mathbf{E}_Z=\partial/\partial Z$, and $\mathbf{E}_\Theta=\partial/\partial\Theta$ form a basis for $T_X\mathcal{B}$. In light of (2.15), the invariants are written as

$$I_1=\frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2}+\frac{R^2e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\alpha^2r^2(R)}+\alpha^2e^{-2\omega_Z(R)}\,, \tag{3.71}$$

$$I_2=\frac{R^2e^{2\omega_\Theta(R)}}{r^2(R)}+\frac{\alpha^2r^2(R)e^{-2\omega_\Theta(R)-2\omega_Z(R)}}{R^2}+\frac{e^{2\omega_Z(R)}}{\alpha^2}\,, \tag{3.72}$$

$$I_4=\left(\frac{Re^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^2, \tag{3.73}$$

$$I_5=\left(\frac{Re^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^4, \tag{3.74}$$

$$I_6=e^{-2\omega_Z(R)}\alpha^2\,, \tag{3.75}$$

$$I_7=e^{-4\omega_Z(R)}\alpha^4\,. \tag{3.76}$$

The Cauchy stress components given by (2.19) read

$$\sigma^{rr}=\frac{2R^2e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\alpha^2r^2(R)}\left(W_{I_1}+W_{I_4}\right)+2W_{I_2}\left(\frac{R^2e^{2\omega_\Theta(R)}}{r^2(R)}+\frac{e^{2\omega_Z(R)}}{\alpha^2}\right)+4W_{I_5}\left(\frac{Re^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^4-p\,, \tag{3.77}$$

$$\sigma^{\theta\theta}=\frac{2e^{-2(\omega_\Theta(R)+\omega_Z(R))}}{R^2}\left(W_{I_1}e^{2\omega_Z(R)}+\alpha^2W_{I_2}\right)+\frac{2W_{I_2}e^{2\omega_Z(R)}}{\alpha^2r^2(R)}-\frac{p}{r^2(R)}\,, \tag{3.78}$$

$$\sigma^{zz}=2e^{-4\omega_Z(R)}\alpha^2\left(\left(W_{I_1}+W_{I_6}\right)e^{2\omega_Z(R)}+2\alpha^2W_{I_7}\right)+\frac{2R^2W_{I_2}e^{2\omega_\Theta(R)}}{r^2(R)}+2W_{I_2}\left(\frac{\alpha r(R)}{e^{\omega_\Theta(R)+\omega_Z(R)}R}\right)^2-p\,. \tag{3.79}$$

Assuming that the eigenstrain-free body is stress-free gives the following conditions (see also [Merodio and Ogden, 2003, Vergori et al., 2013] for more details)

$$\left(W_{I_4}+2W_{I_5}\right)|_{I_1=I_2=3,I_4=I_5=I_6=I_7=1}=0\,,\quad\text{and}\quad\left(W_{I_6}+2W_{I_7}\right)|_{I_1=I_2=3,I_4=I_5=I_6=I_7=1}=0\,. \tag{3.80}$$

[10]Note that mappings of this form correspond to the bar being subject to a displacement control loading with the axial stretch α.

From (2.4), the only nontrivial equilibrium equation is written as

$$\sigma^{rr}{}_{,r} + \frac{\sigma^{rr}}{r} - r\sigma^{\theta\theta} = 0\,. \tag{3.81}$$

Therefore, after some simplifications, $p'(R) = k(R)$, where the expression for $k(R)$ is given in Appendix B. Assuming that the bar is subject to a uniform pressure on its boundary cylinder, i.e., $\sigma^{rr}(R_o) = -p_\infty$, gives

$$p(R) = p_\infty + \int_{R_o}^{R} k(\zeta)d\zeta + \frac{2R_o^2 e^{2\omega_\Theta(R_o)+2\omega_Z(R_o)}}{\alpha^2 r^2(R_o)}\left(W_{I_1}+W_{I_4}\right)|_{R=R_o} + 2\left(\frac{R_o^2 e^{2\omega_\Theta(R_o)}}{r^2(R_o)} + \frac{e^{2\omega_Z(R_o)}}{\alpha^2}\right)W_{I_2}|_{R=R_o} + 4\left(\frac{R_o e^{\omega_\Theta(R_o)+\omega_Z(R_o)}}{\alpha r(R_o)}\right)^4 W_{I_5}|_{R=R_o}\,. \tag{3.82}$$

A cylindrical inclusion in a finite orthotropic cylindrical bar. We next consider the following distribution of eigenstrains in a cylindrical bar, corresponding to a cylindrical inclusion with radius R_i along the axis of the bar.

$$\omega_R(R) = \begin{cases}\omega_1\,, & 0\le R\le R_i\\ 0\,, & R_i\le R\le R_o\end{cases},\quad \omega_\Theta(R) = \begin{cases}\omega_2\,, & 0\le R\le R_i\\ 0\,, & R_i\le R\le R_o\end{cases},\quad \omega_Z(R) = \begin{cases}\omega_3\,, & 0\le R\le R_i\\ 0\,, & R_i\le R\le R_o\end{cases}. \tag{3.83}$$

Using (3.68), one finds

$$r(R) = \frac{1}{\alpha^{\frac{1}{2}}}\begin{cases} e^{\frac{1}{2}(\omega_1+\omega_2+\omega_3)}R\,, & 0\le R\le R_i\\ \left(R^2 + \left(e^{\omega_1+\omega_2+\omega_3}-1\right)R_i^2\right)^{\frac{1}{2}}\,, & R_i\le R\le R_o\end{cases}. \tag{3.84}$$

Simplifying (B.1), it follows that in the inclusion $p'(R) = k_0/R$, where

$$k_0 = \frac{2e^{-2\omega_1-\omega_2-\omega_3}}{\alpha^2}\Big[\alpha e^{\omega_1}\left(e^{2(\omega_2+\omega_3)}W_{I_4} - \left(e^{2\omega_1}-e^{2\omega_2}\right)\left(\alpha^2 W_{I_2} + e^{2\omega_3}W_{I_1}\right)\right) + 2e^{3(\omega_2+\omega_3)}W_{I_5}\Big]\Big|_{I_1=a^{-1}+b^{-1}+ab,\,I_2=a+b+(ab)^{-1},\,I_4^2=I_5=a^{-2},\,I_6^2=I_7=a^2b^2}\,, \tag{3.85}$$

and $a = e^{\omega_1-\omega_2-\omega_3}\alpha$ and $b = e^{\omega_2-\omega_1-\omega_3}\alpha$. Also, in the matrix, $p'(R) = \hat{k}(R)$, where for $R_i \le R \le R_o$

$$\begin{aligned}\hat{k}(R) = \frac{2}{\alpha^9 r^{10} R^3}\Big(& -8R^{10}W_{I_5I_5}\left(R^2-\alpha r^2\right) + 8\alpha^2 r^2 R^8 W_{I_4I_5}\left(\alpha r^2 - R^2\right) - \alpha^8 r^6 W_{I_2}\left(R^3-\alpha r^2 R\right)^2\\ & -\alpha^6 r^6 W_{I_1}\left(R^3-\alpha r^2R\right)^2 + \alpha^6 r^6 R^4 W_{I_4}\left(2\alpha r^2 - R^2\right) + 2\alpha^4 r^4 R^6 W_{I_4I_4}\left(\alpha r^2-R^2\right) + 2\alpha^4 r^4 R^6 W_{I_5}\left(4\alpha r^2 - 3R^2\right)\\ & -2\alpha^6 r^4 W_{I_2I_2}\left(R^2-\alpha r^2\right)^2\left[\alpha r^4 + \left(\alpha^3+1\right)r^2R^2 + \alpha^2R^4\right] - 2\alpha^4 r^4 W_{I_1I_2}\left(R^2-\alpha r^2\right)^2\left(\alpha r^4 + \left(2\alpha^3+1\right)r^2R^2 + 2\alpha^2 R^4\right)\\ & -2\alpha^4 r^4 W_{I_1I_1}\left(\alpha r^2 + R^2\right)\left(R^3-\alpha r^2 R\right)^2 - 4\alpha^2 r^2 R^4 W_{I_1I_5}\left(\alpha^3 r^6 - \alpha^2 r^4 R^2 - 2\alpha r^2 R^4 + 2R^6\right)\\ & -4\alpha^2 r^2 R^4 W_{I_2I_5}\left(\alpha^5 r^6 - \alpha\left(\alpha^3+1\right)r^4R^2 + \left(1-2\alpha^3\right)r^2R^4 + 2\alpha^2R^6\right) - 2\alpha^4 r^4 R^2 W_{I_1I_4}\Big\{\alpha^3 r^6 - \alpha^2 r^4 R^2 - 2\alpha r^2 R^4\\ & + 2R^6\Big\} - 2\alpha^4 r^4 R^2 W_{I_2I_4}\left(\alpha^5 r^6 - \alpha\left(\alpha^3+1\right)r^4R^2 + \left(1-2\alpha^3\right)r^2R^4 + 2\alpha^2 R^6\right)\Big)\,.\end{aligned} \tag{3.86}$$

Therefore, the pressure field is given by

$$p(R) = \begin{cases} k_0\ln\left(\frac{R}{R_i}\right) - p_i\,, & 0\le R\le R_i\,,\\ \int_{R_o}^{R}\hat{k}(\zeta)d\zeta - p_o\,, & R_i\le R\le R_o\,,\end{cases} \tag{3.87}$$

where p_i and p_o are integration constants to be determined. The physical components of the Cauchy stress read

$$\hat{\sigma}^{rr} = \begin{cases} k_0\ln\left(\frac{R_i}{R}\right) + p_i + \frac{2e^{-2\omega_1-\omega_3}}{\alpha^2}\Big(\alpha e^{\omega_1+\omega_2+2\omega_3}\left(W_{I_1}+W_{I_4}\right) + \alpha^3 e^{\omega_1+\omega_2}W_{I_2} + e^{2\omega_1+3\omega_3}W_{I_2}\\ \quad + 2e^{2\omega_2+3\omega_3}W_{I_5}\Big)\Big|_{I_1=a^{-1}+b^{-1}+ab,\,I_2=a+b+(ab)^{-1},\,I_4^2=I_5=a^{-2},\,I_6^2=I_7=a^2b^2}\,, & 0\le R\le R_i\,,\\ p_o + \Big[\int_R^{R_o}\hat{k}(\zeta)d\zeta + \frac{2}{\alpha^4 r^4}\Big\{\alpha^2R^2r^2\left(\alpha^2W_{I_2} + W_{I_1} + W_{I_4}\right) + 2R^4W_{I_5}\\ \quad + \alpha^2 W_{I_2} r^4\Big\}\Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2,\,I_2=\alpha^{-2}+I_R^2+\alpha^2I_R^{-2},\,I_4^2=I_5=\alpha^{-4}I_R^4,\,I_6^2=I_7=\alpha^4}\,, & R_i\le R\le R_o\,,\end{cases} \tag{3.88}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} k_0 \ln\left(\frac{R_i}{R}\right) + p_i + \Big[\left(\frac{2e^{2\omega_3}}{\alpha^2} + 2\alpha e^{\omega_1-\omega_2-\omega_3}\right) W_{I_2} \\ \qquad + \frac{2e^{\omega_1-\omega_2+\omega_3}}{\alpha} W_{I_1} \Big]\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2}, & 0 \le R \le R_i\,, \\ p_o + \Big[\int_R^{R_o} \hat{k}(\zeta)d\zeta + \frac{2r^2}{R^2}\left(W_{I_1} + \alpha^2 W_{I_2}\right) \\ \qquad + \frac{2}{\alpha^2} W_{I_2} \Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2, I_2=\alpha^{-2}+I_R^2+\alpha^2 I_R^{-2}, I_4^2=I_5=\alpha^{-4}I_R^4, I_6^2=I_7=\alpha^4}, & R_i \le R \le R_o\,, \end{cases} \tag{3.89}$$

$$\hat{\sigma}^{zz} = \begin{cases} k_0 \ln\left(\frac{R_i}{R}\right) + p_i + 2\alpha e^{-4\omega_3} \Big\{ \alpha e^{2\omega_3}\left(W_{I_1} + W_{I_6}\right) + e^{\omega_1-\omega_2+3\omega_3} W_{I_2} + e^{-\omega_1+\omega_2+3\omega_3} W_{I_2} \\ \qquad + 2\alpha^3 W_{I_7} \Big\}\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2}, & 0 \le R \le R_i\,, \\ p_o + \Big[\int_R^{R_o} \hat{k}(\zeta)d\zeta + 2\alpha^2 \Big(W_{I_1} + \frac{R^2}{\alpha^2 r^2} W_{I_2} + \frac{r^2}{R^2} W_{I_2} + W_{I_6} \\ \qquad + 2\alpha^2 W_{I_7} \Big) \Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2, I_2=\alpha^{-2}+I_R^2+\alpha^2 I_R^{-2}, I_4^2=I_5=\alpha^{-4}I_R^4, I_6^2=I_7=\alpha^4}, & R_i \le R \le R_o\,, \end{cases} \tag{3.90}$$

where $I_R = R/r(R)$. Imposing uniform pressure on the boundary cylinder, $\hat{\sigma}^{rr}(R_o) = -p_\infty$, gives

$$p_o = -p_\infty - \Big[\frac{2}{\alpha^4 r^4(R_o)} \Big\{ \alpha^2 R_o^2 r^2(R_o)\left(\alpha^2 W_{I_2} + W_{I_1} + W_{I_4}\right) + 2R_o^4 W_{I_5} \\ + \alpha^2 W_{I_2} r^4(R_o) \Big\} \Big]\Big|_{I_1=\alpha^2+I_{R_o}^{-2}+\alpha^{-2}I_{R_o}^2, I_2=\alpha^{-2}+I_{R_o}^2+\alpha^2 I_{R_o}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_o}^4, I_6^2=I_7=\alpha^4}. \tag{3.91}$$

The continuity of the traction vector at the inclusion-matrix interface requires that σ^{rr} be continuous at $R = R_i$. Therefore, p_i is calculated as

$$p_i = -p_\infty + \int_{R_i}^{R_o} \hat{k}(\zeta)d\zeta + \frac{2e^{-(\omega_1+\omega_2+\omega_3)}}{\alpha^2} \Big[\alpha\left(W_{I_1} + W_{I_4}\right) + \left(\alpha^3 + e^{\omega_1+\omega_2+\omega_3}\right) W_{I_2} \\ + 2e^{-(\omega_1+\omega_2+\omega_3)} W_{I_5} \Big]\Big|_{I_1=\alpha^2+I_{R_i}^{-2}+\alpha^{-2}I_{R_i}^2, I_2=\alpha^{-2}+I_{R_i}^2+\alpha^2 I_{R_i}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_i}^4, I_6^2=I_7=\alpha^4} \\ - \frac{2e^{-2\omega_1-\omega_3}}{\alpha^2} \Big(\alpha e^{\omega_1+\omega_2+2\omega_3}\left(W_{I_1} + W_{I_4}\right) + \alpha^3 e^{\omega_1+\omega_2} W_{I_2} + e^{2\omega_1+3\omega_3} W_{I_2} \\ + 2e^{2\omega_2+3\omega_3} W_{I_5} \Big)\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2} - \Big[\frac{2}{\alpha^4 r^4(R_o)} \Big\{ 2R_o^4 W_{I_5} + \alpha^2 W_{I_2} r^4(R_o) \\ + \alpha^2 R_o^2 r^2(R_o)\left(\alpha^2 W_{I_2} + W_{I_1} + W_{I_4}\right) \Big\} \Big]\Big|_{I_1=\alpha^2+I_{R_o}^{-2}+\alpha^{-2}I_{R_o}^2, I_2=\alpha^{-2}+I_{R_o}^2+\alpha^2 I_{R_o}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_o}^4, I_6^2=I_7=\alpha^4}. \tag{3.92}$$

Remark 3.6. For the stress to be uniform in the inclusion, k_0 must be zero (cf. (3.88), (3.89), and (3.90)). If $\omega_1 \neq \omega_2$, k_0 is zero only if the energy function satisfies the following condition

$$\Big[\alpha e^{\omega_1} \Big(e^{2(\omega_2+\omega_3)} W_{I_4} - \left(e^{2\omega_1} - e^{2\omega_2}\right)\left(\alpha^2 W_{I_2} + e^{2\omega_3} W_{I_1}\right) \Big) \\ + 2e^{3(\omega_2+\omega_3)} W_{I_5} \Big]\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2} = 0\,. \tag{3.93}$$

However, if $\omega_1 = \omega_2$, then $a = b = e^{-\omega_3}\alpha$, and (3.85) implies that k_0 is zero if

$$k_0 = 2a^{-1}\left(W_{I_4} + 2a^{-1} W_{I_5}\right)\big|_{I_1=2a^{-1}+a^2, I_2=2a+a^{-2}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^4} = 0\,. \tag{3.94}$$

In addition, if we assume that the traction in the radial fiber direction is tensile for extension $a < 1$ and compressive for contraction $a > 1$, then (3.94) implies that $a = b = 1$, or $\alpha = e^{\omega_3}$, and hence, for any nonlinear incompressible orthotropic material, $k_0 = 2\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0$, from (3.80). Therefore, we have proved the following proposition.

Proposition 3.7. *Consider a finite incompressible orthotropic elastic solid cylinder such that the material orthotropic axes are in the radial, circumferential, and longitudinal directions of the cylinder. Assume that the* *bar is subject to a uniform pressure on its boundary cylinder and contains an inclusion along its axis with uniform radial, circumferential, and longitudinal eigenstrains. The Cauchy stress exhibits a logarithmic singularity at the centerline of the cylinder unless the radial and circumferential eigenstrains are equal and the axial stretch α is equal to e^{ω_3}, or the energy function satisfies (3.93). If the radial and circumferential eigenstrains are equal and $\alpha = e^{\omega_3}$, then the stress inside the inclusion is uniform and hydrostatic.*

Note that Proposition 3.7 holds for a cylindrical bar made of any incompressible transversely isotropic solid with material preferred directions along the radial and circumferential directions as well. If the material preferred direction is longitudinal, then we do not need the condition $\alpha = e^{\omega_3}$ for the results of the proposition to hold.

3.4 Finite Eigenstrains in a Finite Compressible Orthotropic Cylindrical Bar

In this section, we release the incompressibility constraint of the problem of a bar with a finite cylindrically-symmetric eigenstrain distribution and consider a compressible orthotropic solid. Assuming that the material manifold is embedded into the ambient space using the mappings of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, the right Cauchy-Green deformation tensor is written as

$$\mathbf{C} = \begin{pmatrix} r'(R)^2 e^{-2\omega_R(R)} & 0 & 0 \\ 0 & \frac{e^{-2\omega_\Theta(R)} r^2(R)}{R^2} & 0 \\ 0 & 0 & e^{-2\omega_Z(R)}\alpha^2 \end{pmatrix}. \tag{3.95}$$

The Jacobean reads

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{\alpha r(R) r'(R)}{R e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}. \tag{3.96}$$

Using (2.15), one obtains the invariants of deformation as follows

$$I_1 = \mathrm{tr}(\mathbf{C}) = \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} + r'(R)^2 e^{-2\omega_R(R)} + \alpha^2 e^{-2\omega_Z(R)}, \tag{3.97}$$

$$I_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{\alpha^2 r^2(R) r'(R)^2}{e^{2(\omega_R(R)+\omega_\Theta(R)+\omega_Z(R))} R^2} \left[\frac{R^2 e^{2\omega_\Theta(R)}}{r^2(R)} + \frac{e^{2\omega_R(R)}}{r'(R)^2} + \frac{e^{2\omega_Z(R)}}{\alpha^2}\right], \tag{3.98}$$

$$I_3 = \frac{\alpha^2 r^2(R) r'(R)^2}{R^2} e^{-2(\omega_R(R)+\omega_\Theta(R)+\omega_Z(R))}, \tag{3.99}$$

$$I_4^2 = I_5 = e^{-4\omega_R(R)} r'(R)^4, \tag{3.100}$$

$$I_6^2 = I_7 = e^{-4\omega_Z(R)} \alpha^4. \tag{3.101}$$

Noting that the Jacobean is given by (3.96), the components of the Cauchy stress are written as

$$\sigma^{rr} = \frac{2r'(R) e^{-(\omega_\Theta(R)+3\omega_R(R)+\omega_Z(R))}}{\alpha R r(R)} \Big[R^2 e^{2\omega_\Theta(R)} \Big\{e^{2\omega_Z(R)} \left(e^{2\omega_R(R)} (W_{I_1} + W_{I_4}) + 2W_{I_5} r'(R)^2\right) + \alpha^2 W_{I_2} e^{2\omega_R(R)}\Big\} + r(R)^2 e^{2\omega_R(R)} \left(W_{I_2} e^{2\omega_Z(R)} + \alpha^2 W_{I_3}\right)\Big], \tag{3.102}$$

$$\sigma^{\theta\theta} = \frac{e^{-(\omega_\Theta(R)+\omega_R(R)+\omega_Z(R))}}{\alpha R r(R) r'(R)} \left[2e^{2\omega_R(R)} \left(W_{I_1} e^{2\omega_Z(R)} + \alpha^2 W_{I_2}\right) + 2r'(R)^2 \left(W_{I_2} e^{2\omega_Z(R)} + \alpha^2 W_{I_3}\right)\right], \tag{3.103}$$

$$\sigma^{zz} = \frac{2\alpha e^{-(\omega_\Theta(R)+\omega_R(R)+3\omega_Z(R))}}{R r(R) r'(R)} \Big[R^2 e^{2\omega_\Theta(R)} \left\{e^{2\omega_Z(R)} \left(e^{2\omega_R(R)} (W_{I_1} + W_{I_6}) + W_{I_2} r'(R)^2\right) + 2\alpha^2 W_{I_7} e^{2\omega_R(R)}\right\} + r(R)^2 e^{2\omega_Z(R)} \left(W_{I_2} e^{2\omega_R(R)} + W_{I_3} r'(R)^2\right)\Big]. \tag{3.104}$$

We need to have the following conditions in order for the eigenstrain-free body to be stress-free.

$$
\begin{gathered}
(W_{I_1}+2W_{I_2}+W_{I_3})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,, \\
(W_{I_4}+2W_{I_5})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,, \quad (W_{I_6}+2W_{I_7})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,.
\end{gathered}
\tag{3.105}
$$

Substituting for the stress components into (3.81) using (3.102) and (3.103), the radial equilibrium equation is simplified and is given in Appendix C.

We next consider the eigenstrain distribution (3.83) and solve the problem of a cylindrical inclusion with uniform anisotropic eigenstrains in a finite compressible orthotropic cylindrical bar. Following the same procedure that was explained in section 3.2, we first assume that the stress field inside the inclusion is uniform, i.e., $\hat{\sigma}^{rr}=C_1$, $\hat{\sigma}^{\theta\theta}=C_2$, $\hat{\sigma}^{zz}=C_3$, where C_i, $i=1,2,3$ are constants. Using (3.102), (3.103), and (3.104), we have the following three first-order ODEs for $0\leq R\leq R_i$ subject to the boundary condition $r(0)=0$:

$$
\begin{aligned}
C_1=\frac{2e^{-(3\omega_1+\omega_2+\omega_3)}r'(R)}{\alpha R r(R)}\Big[&R^2e^{2\omega_2}\left\{e^{2\omega_3}\left(e^{2\omega_1}\left(W_{I_1}+W_{I_4}\right)+2W_{I_5}r'(R)^2\right)+\alpha^2W_{I_2}e^{2\omega_1}\right\} \\
&+r(R)^2e^{2\omega_1}\left(W_{I_2}e^{2\omega_3}+\alpha^2W_{I_3}\right)\Big]\,,
\end{aligned}
\tag{3.106}
$$

$$
C_2=\frac{e^{-(\omega_1+\omega_2+\omega_3)}r(R)}{\alpha R r'(R)}\Big[2e^{2\omega_1}\left(W_{I_1}e^{2\omega_3}+\alpha^2W_{I_2}\right)+2r'(R)^2\left(W_{I_2}e^{2\omega_3}+\alpha^2W_{I_3}\right)\Big]\,,
\tag{3.107}
$$

$$
\begin{aligned}
C_3=\frac{2e^{-(\omega_1+\omega_2+3\omega_3)}\alpha}{R r(R) r'(R)}\Big[&R^2e^{2\omega_2}\left\{e^{2\omega_3}\left(e^{2\omega_1}\left(W_{I_1}+W_{I_6}\right)+W_{I_2}r'(R)^2\right)+2\alpha^2W_{I_7}e^{2\omega_1}\right\} \\
&+r(R)^2e^{2\omega_3}\left(W_{I_2}e^{2\omega_1}+W_{I_3}r'(R)^2\right)\Big]\,.
\end{aligned}
\tag{3.108}
$$

Note that $r(R)=\beta R$, with β a constant, is a solution of all the above IVPs, i.e., the stress inside the inclusion is uniform if $r(R)=\beta R$ for $0\leq R\leq R_i$. From the radial equilibrium equation (3.81), it follows that $C_1=C_2$ when the stress field in the inclusion is assumed to be uniform. Therefore, from (3.106) and (3.107), the equilibrium equation in the inclusion ($C_1=C_2$) for $r(R)=\beta R$ implies that

$$
\begin{aligned}
&\Big[e^{2(\omega_2+\omega_3)}\left(2\beta^2W_{I_5}+e^{2\omega_1}W_{I_4}\right)-e^{2\omega_1}\left(e^{2\omega_1}-e^{2\omega_2}\right)\left(\alpha^2W_{I_2}\right. \\
&\left.+e^{2\omega_3}W_{I_1}\right)\Big]\Big|_{I_1=\alpha^2e^{-2\omega_3}+\beta^2[e^{-2\omega_1}+e^{-2\omega_2}],I_2=\kappa\beta^2[\alpha^2(e^{2\omega_1}+e^{2\omega_2})+e^{2\omega_3}\beta^2],I_3=\kappa\alpha^2\beta^4,I_4^2=I_5=e^{-4\omega_1}\beta^4,I_6^2=I_7=e^{-4\omega_3}\alpha^4}=0\,,
\end{aligned}
\tag{3.109}
$$

where $\kappa=e^{-2(\omega_1+\omega_2+\omega_3)}$. If the material is compressible and isotropic, i.e., $W=W(I_1,I_2,I_3)$, then $W_{I_4}=W_{I_5}=0$, and (3.109) is clearly satisfied without restricting the longitudinal stretch α or the strain energy function if $\omega_1=\omega_2$. In this case, in the matrix, we have the following second-order nonlinear ODE from (C.1) for $R_i\leq R\leq R_o$:

$$
\begin{aligned}
&rR^3\Big[-r'^2\Big(2\Big\{\alpha^2\Big(r'^2\left(\alpha^2W_{I_2I_3}+W_{I_1I_3}\right)+W_{I_2I_2}(\alpha^2+r'^2)\Big)+W_{I_1I_2}(2\alpha^2+r'^2)+W_{I_1I_1}\Big\}+\alpha^2W_{I_3}\Big) \\
&+W_{I_2}(\alpha^2-r'^2)+W_{I_1}\Big]+2\alpha^4r^4r'^3W_{I_3I_3}+2\alpha^2r^4r'W_{I_1I_3}+2r^4r'W_{I_1I_2}-2r^3Rr'^2\Big[\alpha^4r'^2W_{I_3I_3}+\alpha^2W_{I_2I_3}(\alpha^2+2r'^2) \\
&+W_{I_2I_2}(\alpha^2+r'^2)+\alpha^2W_{I_1I_3}+W_{I_1I_2}\Big]-R^4\Big(\alpha^2W_{I_2}\left(Rr''+r'\right)+2Rr'^2r''\left(\alpha^4W_{I_2I_2}+2\alpha^2W_{I_1I_2}+W_{I_1I_1}\right)+W_{I_1}(Rr'' \\
&+r')\Big)+r^2R^2\Big[\alpha^2W_{I_3}\left(r'-Rr''\right)+2r'\Big\{\alpha^2\Big(r'\left(\alpha^2W_{I_2I_3}+W_{I_1I_3}\right)\left(r'-2Rr''\right)+W_{I_2I_2}\Big(\alpha^2+r'^2-2Rr'r''\Big)\Big) \\
&+W_{I_1I_2}\Big(2\alpha^2+r'^2-2Rr'r''\Big)+W_{I_1I_1}\Big\}+W_{I_2}\left(r'-Rr''\right)\Big]-2\alpha^4r^4Rr'^2r''W_{I_3I_3}+2\alpha^2r^4r'W_{I_2I_3}\Big(\alpha^2+2r'^2-2Rr'r''\Big) \\
&+2r^4r'W_{I_2I_2}\Big(\alpha^2+r'^2-Rr'r''\Big)=0\,.
\end{aligned}
\tag{3.110}
$$

The boundary conditions for the ODE (3.110) and determining the unknown β are given by continuity of $r(R)$ and the traction vector at the inclusion-matrix interface, i.e., $r(R)|_{R=R_i^+}=\beta R_i$ and $\sigma^{rr}|_{R=R_i^+}=\sigma^{rr}|_{R=R_i^-}$, respectively, along with the boundary condition $\hat{\sigma}^{rr}(R_o)=-p_\infty$. Thus, we have proved the following proposition.

Proposition 3.8. *Consider a cylindrical bar made of a compressible isotropic solid subject to a uniform pressure on its boundary cylinder. Assume that the bar contains a cylindrical inclusion along its axis with uniform radial, circumferential, and axial eigenstrains. The stress field in the inclusion is uniform if the radial and circumferential eigenstrains are equal.*

Returning to the compressible orthotropic solid case, if the radial and circumferential eigenstrains are equal $\omega_1 = \omega_2 = \omega$, (3.109) implies that

$$\left(2b^2 W_{I_5} + W_{I_4}\right)|_{I_1=a^2+2b^2, I_2=b^2(2a^2+b^2), I_3=a^2b^4, I_4^2=I_5=b^4, I_6^2=I_7=a^4} = 0\,, \tag{3.111}$$

where $a = \alpha e^{-\omega_3}$ and $b = \beta e^{-\omega}$. Note that if $\alpha = e^{\omega_3}$, i.e., $a = 1$, then $b = 1$ is trivially a solution of (3.111) from (3.105). If we further assume that the traction in the radial fiber direction is tensile for extension $b > 1$ and compressive for contraction $b < 1$, then $b = 1$ is the only solution of (3.111). This corresponds to the trivial stress-free case, where the entire bar has a uniform distribution of dilatational eigenstrains such that the radial and circumferential eigenstrains are equal, and the axial stretch is equal to e^{ω_3}. However, there are some nontrivial cases for which uniform stress can be maintained in the inclusion with uniform dilatational eigenstrains such that $\omega_1 \neq \omega_2$ or $\alpha \neq e^{\omega_3}$. As was already mentioned in section 3.2, these cases are special because a choice of energy function, in general, fully determines β from (3.109), which in turn specifies the kinematics and the stress field in the inclusion.

We next assume some specific energy functions analogous to (3.52) of the following form

$$W = W\left(I_1, I_2, I_3, I_4, I_5, I_6, I_7\right) = W_{\text{iso}}\left(I_1, I_2, I_3\right) + W_{\text{fib}}^R\left(I_4, I_5\right) + W_{\text{fib}}^Z\left(I_6, I_7\right)\,, \tag{3.112}$$

where the isotropic base material with the strain energy function W_{iso} is augmented by a fiber reinforcing model such that W_{fib}^R and W_{fib}^Z represent the reinforcing effects in the radial and longitudinal directions, respectively.

i) Compressible Mooney-Rivlin reinforced model (I_4, I_6, I_7 reinforcement) with the energy function

$$\begin{aligned} W(I_1, I_2, I_3, I_4, I_6, I_7) = {} & C_1\left(I_1 - 3\right) + C_2\left(I_2 - 3\right) - \left(C_1 + 2C_2\right)\left(I_3 - 1\right) \\ & + \frac{\mu}{2}\left(I_4 - 1\right)^2 + \frac{\mu_1}{2}\left(I_6 - 1\right)^2 + \frac{\mu_2}{2}\left(I_7 - 1\right)^2\,, \end{aligned} \tag{3.113}$$

where C_1 and C_2 are the constants of the Mooney-Rivlin base material and $\mu > 0$ is a material constant describing the strength of the reinforcement in the radial direction, while μ_1 and μ_2 are positive material constants pertaining to the reinforcement strength in the axial direction. Substituting (3.113) into (3.109) one gets

$$\beta = e^{\omega_1}\left[\frac{\delta}{\mu}\left(\alpha^2 e^{-2\omega_3} C_2 + C_1\right) e^{-2\omega_2} + 1\right]^{\frac{1}{2}}\,, \tag{3.114}$$

where $\delta = e^{2\omega_1} - e^{2\omega_2}$. For $\omega_1 \geq \omega_2$, there is no condition imposed on the material parameters for β to be positive, whereas for $\omega_1 < \omega_2$, one needs to have the following condition

$$\mu > \left(C_1 + \alpha^2 e^{-2\omega_3} C_2\right)\left[1 - e^{2(\omega_1 - \omega_2)}\right], \qquad \text{for} \quad \omega_1 < \omega_2\,. \tag{3.115}$$

The stress field in the inclusion is uniform and has the following non-zero components (cf. (3.102), (3.103), and (3.104))

$$\begin{aligned} \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \frac{2e^{-3\omega_1-\omega_2-\omega_3}}{\alpha}\Big[& C_1 e^{2\omega_1}\left(e^{2(\omega_2+\omega_3)} - \alpha^2\beta^2\right) + C_2 e^{2\omega_1}\left(\alpha^2 e^{2\omega_2} + \beta^2 e^{2\omega_3} - 2\alpha^2\beta^2\right) \\ & - \mu e^{2(\omega_2+\omega_3)}\left(e^{2\omega_1} - \beta^2\right)\Big]\,, \end{aligned} \tag{3.116a}$$

$$\begin{aligned} \hat{\sigma}^{zz} = \frac{2\alpha e^{-\omega_1-\omega_2-7\omega_3}}{\beta^2}\Big[& \beta^2 C_2 e^{6\omega_3}\left(e^{2\omega_1} + e^{2\omega_2} - 2\beta^2\right) + C_1 e^{6\omega_3}\left(e^{2(\omega_1+\omega_2)} - \beta^4\right) \\ & - e^{2(\omega_1+\omega_2)}\left(e^{2\omega_3} - \alpha^2\right)\left\{2\alpha^2\mu_2\left(\alpha^2 + e^{2\omega_3}\right) + \mu_1 e^{4\omega_3}\right\}\Big]\,. \end{aligned} \tag{3.116b}$$

ii) Compressible Mooney-Rivlin reinforced model (I_5, I_6, I_7 reinforcement) for which the energy function is written as

$$\begin{aligned} W(I_1, I_2, I_3, I_5, I_6, I_7) = C_1 (I_1 - 3) + C_2 (I_2 - 3) - (C_1 + 2C_2) (I_3 - 1) \\ + \frac{\mu}{2} (I_5 - 1)^2 + \frac{\mu_1}{2} (I_6 - 1)^2 + \frac{\mu_2}{2} (I_7 - 1)^2 . \end{aligned} \tag{3.117}$$

Using (3.117) and (3.109), one obtains

$$\beta = \frac{e^{\omega_1 - \omega_2 - \omega_3} \left(2\mu^2 6^{\frac{1}{3}} e^{4(\omega_2 + \omega_3)} + \left[\Delta^{\frac{1}{2}} + 9\delta\mu^2 e^{4(\omega_2+\omega_3)} \left(e^{2\omega_3} C_1 + \alpha^2 C_2 \right) \right]^{\frac{2}{3}} \right)^{\frac{1}{2}}}{6^{\frac{1}{3}} \mu^{\frac{1}{2}} \left[\Delta^{\frac{1}{2}} + 9\delta\mu^2 e^{4(\omega_2+\omega_3)} \left(e^{2\omega_3} C_1 + \alpha^2 C_2 \right) \right]^{\frac{1}{6}}} , \tag{3.118}$$

where[11]

$$\Delta = 3\mu^4 e^{8(\omega_2+\omega_3)} \left[27\delta^2 \left(\alpha^4 C_2^2 + C_1^2 e^{4\omega_3} \right) + 54\delta^2 \alpha^2 C_1 C_2 e^{2\omega_3} - 16\mu^2 e^{4(\omega_2+\omega_3)} \right] . \tag{3.119}$$

The non-zero components of the Cauchy stress in the inclusion read

$$\begin{aligned} \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \frac{2e^{-7\omega_1 - \omega_2 - \omega_3}}{\alpha} \Big[2\beta^2 \mu e^{2(\omega_2+\omega_3)} \left(\beta^4 - e^{4\omega_1} \right) + C_1 e^{6\omega_1} \left(e^{2(\omega_2+\omega_3)} - \alpha^2 \beta^2 \right) \\ + C_2 e^{6\omega_1} \left(\alpha^2 e^{2\omega_2} + \beta^2 e^{2\omega_3} - 2\alpha^2\beta^2 \right) \Big] , \end{aligned} \tag{3.120a}$$

$$\begin{aligned} \hat{\sigma}^{zz} = \frac{2\alpha e^{-\omega_1 - \omega_2 - 7\omega_3}}{\beta^2} \Big[\beta^2 C_2 e^{6\omega_3} \left(e^{2\omega_1} + e^{2\omega_2} - 2\beta^2 \right) + C_1 e^{6\omega_3} \left(e^{2(\omega_1+\omega_2)} - \beta^4 \right) \\ - e^{2(\omega_1+\omega_2)} \left(e^{2\omega_3} - \alpha^2 \right) \left\{ 2\alpha^2 \mu_2 \left(\alpha^2 + e^{2\omega_3} \right) + \mu_1 e^{4\omega_3} \right\} \Big] . \end{aligned} \tag{3.120b}$$

iii) Blatz-Ko reinforced model (I_4, I_6, I_7 reinforcement) with the following energy function

$$W(I_2, I_3, I_4, I_6, I_7) = \frac{\mu_o}{2} \left(\frac{I_2}{I_3} + 2I_3^{\frac{1}{2}} - 5 \right) + \frac{\mu}{2} (I_4 - 1)^2 + \frac{\mu_1}{2} (I_6 - 1)^2 + \frac{\mu_2}{2} (I_7 - 1)^2 , \tag{3.121}$$

where μ_o is a positive constant of the Blatz-Ko base material. From (3.121) and (3.109), one has

$$\mu_o e^{4\omega_1} \delta + 2\beta^4 \mu \left(e^{2\omega_1} - \beta^2 \right) = 0 , \tag{3.122}$$

which is the same as (3.63), obtained in the case of a spherical ball made of the same material. Therefore, β is given by (3.64), and one can show that β is physical (real and positive) only if $\omega_1 > \omega_2$. Note that in this case β is determined independently of the longitudinal stretch α and ω_3. The stress field in the inclusion is uniform with the non-zero stress components given as

$$\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \mu_o \left(1 - \frac{e^{3\omega_1 + \omega_2 + \omega_3}}{\alpha\beta^4} \right) - \frac{2\mu}{\alpha} \left(e^{2\omega_1} - \beta^2 \right) e^{-3\omega_1 + \omega_2 + \omega_3} , \tag{3.123a}$$

$$\hat{\sigma}^{zz} = -\frac{e^{\omega_1 + \omega_2 - 7\omega_3}}{\alpha^3 \beta^2} \left[4\alpha^6 \mu_2 \left(e^{4\omega_3} - \alpha^4 \right) + 2\alpha^4 \mu_1 e^{4\omega_3} \left(e^{2\omega_3} - \alpha^2 \right) + e^{7\omega_3} \mu_o \left(e^{3\omega_3} - \alpha^3 \beta^2 e^{-\omega_1 - \omega_2} \right) \right] . \tag{3.123b}$$

[11] Note that $\beta \in \mathbb{R}^+$ puts a constraint on the elastic constants.

4 Concluding Remarks

To this date the study of anisotropic inclusion problems in the literature has been restricted to linear elasticity. In this paper, we considered finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars made of both compressible and incompressible solids. We identified the conditions under which the stress field in the spherical and cylindrical inclusions with a uniform distribution of dilatational eigenstrains is uniform. We showed that the stress in a spherical inclusion with uniform eigenstrains contained in an incompressible transversely isotropic spherical ball with the material preferred direction being radial is uniform and hydrostatic if the radial and circumferential eigenstrains are equal. A similar result holds for cylindrical inclusions in incompressible orthotropic cylindrical bars when orthotropic axes are in the radial, circumferential, and axial directions, provided that the axial stretch is equal to a value determined by the longitudinal eigenstrain. Except for some special cases for which the energy function is constrained depending on the eigenstrains, in the case of incompressible solids a stress singularity emerges as a result of a mismatch between radial and circumferential eigenstrains at the center of a ball or on the axis of a cylindrical bar.

We generalized the results of Yavari and Goriely [2013] to any compressible isotropic material. Specifically, we showed that for compressible isotropic spherical balls and cylindrical bars with spherical and cylindrical inclusions with uniform eigenstrains, respectively, if the radial and circumferential eigenstrains are equal the stress in the inclusion is uniform (and hydrostatic for the spherical inclusion).

We observed that for compressible transversely isotropic and orthotropic solids the stress field in the inclusion with uniform dilatational eigenstrains is not necessarily uniform. We showed, however, that there are some energy functions for which for a given applied pressure on the outer boundary, the ratio R_i/R_o is determined if a uniform stress field is to be maintained in the inclusion. Similarly, for such special energy functions, fixing R_i/R_o uniquely determines the pressure that must be applied on the outer boundary to maintain a uniform stress field inside the inclusion. Moreover, material parameters must satisfy certain conditions depending on the eigenstrains (and the axial stretch in the case of cylindrical bars). To explore these special cases, we assumed some specific energy functions, namely compressible Mooney-Rivlin and Blatz-Ko reinforced models and found analytical expressions for the stress field in the inclusion. Investigating the nonlinear anisotropic inclusion problem for other types of material anisotropy and other geometries such as non-simply connected bodies will be the subjects of future communications.

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References

M. B. Amar and A. Goriely. Growth and instability in elastic tissues. *Journal of the Mechanics and Physics of Solids*, 53(10):2284–2319, 2005.

M. Carroll. Finite strain solutions in compressible isotropic elasticity. *Journal of Elasticity*, 20(1):65–92, 1988.

T. Doyle and J. Ericksen. Nonlinear elasticity. *Advances in Applied Mechanics*, 4:53–115, 1956.

J. D. Eshelby. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society of London A*, 241:376–396, 1957.

S. Giordano, P. Palla, and L. Colombo. Nonlinear elasticity of composite materials. *The European Physical Journal B*, 68(1):89–101, 2009.

A. Golgoon and A. Yavari. On the stress field of a nonlinear elastic solid torus with a toroidal inclusion. *Journal of Elasticity*, pages 1–31, 2017.

A. Golgoon, S. Sadik, and A. Yavari. Circumferentially-symmetric finite eigenstrains in incompressible isotropic nonlinear elastic wedges. *International Journal of Non-Linear Mechanics*, 84:116–129, 2016.

A. Goriely, D. E. Moulton, and R. Vandiver. Elastic cavitation, tube hollowing, and differential growth in plants and biological tissues. *Europhysics Letters*, 91(1):18001, 2010.

X. Jiang and E. Pan. Exact solution for 2D polygonal inclusion problem in anisotropic magnetoelectroelastic full-, half-, and bimaterial-planes. *International Journal of Solids and Structures*, 41(16):4361–4382, 2004.

C. Kim and P. Schiavone. A circular inhomogeneity subjected to non-uniform remote loading in finite plane elastostatics. *International Journal of Non-Linear Mechanics*, 42(8):989–999, 2007.

C. Kim and P. Schiavone. Designing an inhomogeneity with uniform interior stress in finite plane elastostatics. *Acta Mechanica*, 197(3-4):285–299, 2008.

C. Kim, M. Vasudevan, and P. Schiavone. Eshelby's conjecture in finite plane elastostatics. *The Quarterly Journal of Mechanics and Applied Mathematics*, 61(1):63–73, 2008.

N. Kinoshita and T. Mura. Elastic fields of inclusions in anisotropic media. *Physica Status Solidi (a)*, 5(3): 759–768, 1971.

Y.-G. Lee, W.-N. Zou, and E. Pan. Eshelby's problem of polygonal inclusions with polynomial eigenstrains in an anisotropic magneto-electro-elastic full plane. *Proceedings of the Royal Society of London A*, 471(2179): 20140827, 2015.

J. Y. Li and M. L. Dunn. Anisotropic coupled-field inclusion and inhomogeneity problems. *Philosophical Magazine A*, 77(5):1341–1350, 1998.

I. Liu et al. On representations of anisotropic invariants. *International Journal of Engineering Science*, 20(10): 1099–1109, 1982.

J. Lu and P. Papadopoulos. A covariant constitutive description of anisotropic non-linear elasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 51(2):204–217, 2000.

J. E. Marsden and T. Hughes. *Mathematical Foundations of Elasticity.* Dover Publications, New York, 1994.

J. Merodio and R. Ogden. Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation. *Archives of Mechanics*, 54(5-6):525–552, 2002.

J. Merodio and R. Ogden. Instabilities and loss of ellipticity in fiber-reinforced compressible non-linearly elastic solids under plane deformation. *International Journal of Solids and Structures*, 40(18):4707–4727, 2003.

J. Merodio and R. Ogden. Tensile instabilities and ellipticity in fiber-reinforced compressible non-linearly elastic solids. *International Journal of Engineering Science*, 43(8):697–706, 2005.

D. E. Moulton and A. Goriely. Anticavitation and differential growth in elastic shells. *Journal of Elasticity*, 102 (2):117–132, 2011.

A. Ozakin and A. Yavari. A geometric theory of thermal stresses. *Journal of Mathematical Physics*, 51(3): 032902, 2010.

E. Pan. Eshelby problem of polygonal inclusions in anisotropic piezoelectric bimaterials. *Proceedings of the Royal Society of London A*, 460(2042):537–559, 2004a.

E. Pan. Eshelby problem of polygonal inclusions in anisotropic piezoelectric full-and half-planes. *Journal of the Mechanics and Physics of Solids*, 52(3):567–589, 2004b.

T. J. Pence and H. Tsai. Swelling-induced microchannel formation in nonlinear elasticity. *IMA Journal of Applied Mathematics*, 70(1):173–189, 2005.

T. J. Pence and H. Tsai. Swelling-induced cavitation of elastic spheres. *Mathematics and Mechanics of Solids*, 11(5):527–551, 2006.

T. J. Pence and H. Tsai. Bulk cavitation and the possibility of localized interface deformation due to surface layer swelling. *Journal of Elasticity*, 87(2-3):161–185, 2007.

C. Ru, P. Schiavone, L. Sudak, and A. Mioduchowski. Uniformity of stresses inside an elliptic inclusion in finite plane elastostatics. *International Journal of Non-Linear Mechanics*, 40(2):281–287, 2005.

C.-Q. Ru and P. Schiavone. On the elliptic inclusion in anti-plane shear. *Mathematics and Mechanics of Solids*, 1(3):327–333, 1996.

S. Sadik and A. Yavari. Geometric nonlinear thermoelasticity and the time evolution of thermal stresses. *Mathematics and Mechanics of Solids*, 2015. doi: 10.1177/1081286515599458.

S. Sadik and A. Yavari. Small-on-large geometric anelasticity. *Proceedings of the Royal Society of London A*, 472(2195), 2016.

F. Sozio and A. Yavari. Nonlinear mechanics of surface growth for cylindrical and spherical elastic bodies. *Journal of the Mechanics and Physics of Solids*, 98:12 – 48, 2017.

A. Spencer. Part iii. Theory of invariants. *Continuum Physics*, 1:239–353, 1971.

A. Spencer. The formulation of constitutive equation for anisotropic solids. In *Mechanical Behavior of Anisotropic Solids/Comportment Méchanique des Solides Anisotropes*, pages 3–26. Springer, 1982.

R. Stojanovic, S. Djuric, and L. Vujosevic. On finite thermal deformations. *Archiwum Mechaniki Stosowanej*, 16:103–108, 1964.

C. Truesdell. The physical components of vectors and tensors. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, 33(10-11):345–356, 1953.

L. Vergori, M. Destrade, P. McGarry, and R. W. Ogden. On anisotropic elasticity and questions concerning its finite element implementation. *Computational Mechanics*, 52(5):1185–1197, 2013.

J. Willis. Anisotropic elastic inclusion problems. *The Quarterly Journal of Mechanics and Applied Mathematics*, 17(2):157–174, 1964.

I. Wolfram Research. *Mathematica*. Version 11.0. Wolfram Research, Inc., Champaign, Illinois, 2016.

A. Yavari. A geometric theory of growth mechanics. *Journal of Nonlinear Science*, 20:781–830, 2010.

A. Yavari. On the wedge dispiration in an inhomogeneous isotropic nonlinear elastic solid. *Mechanics Research Communications*, 78:55–59, 2016.

A. Yavari and A. Goriely. Nonlinear elastic inclusions in isotropic solids. *Proceedings of the Royal Society A*, 469(2160):20130415, 2013.

A. Yavari and A. Goriely. On the stress singularities generated by anisotropic eigenstrains and the hydrostatic stress due to annular inhomogeneities. *Journal of the Mechanics and Physics of Solids*, 76:325–337, 2015a.

A. Yavari and A. Goriely. The twist-fit problem: finite torsional and shear eigenstrains in nonlinear elastic solids. *Proceedings of the Royal Society A*, 471(2183), 2015b.

A. Yavari, J. E. Marsden, and M. Ortiz. On the spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47:85–112, 2006.

Y. Yue, K. Xu, Q. Chen, and E. Pan. Eshelby problem of an arbitrary polygonal inclusion in anisotropic piezoelectric media with quadratic eigenstrains. *Acta Mechanica*, 226(7):2365–2378, 2015.

Q.-S. Zheng and A. Spencer. Tensors which characterize anisotropies. *International Journal of Engineering Science*, 31(5):679–693, 1993.

Appendix A Radial equilibrium equation for the compressible transversely isotropic case

$$
\begin{aligned}
&8e^{8\omega_\Theta}W_{I_5I_5}r'^7\omega_R'R^9-e^{6\omega_R+4\omega_\Theta}\Big(e^{4\omega_\Theta}\left(W_{I_1}+W_{I_4}\right)r''R^4-2e^{2(\omega_R+\omega_\Theta)}rW_{I_1}R^2+2e^{2\omega_\Theta}r^2W_{I_2}r''R^2-2e^{2\omega_R}r^3W_{I_2}\\
&+r^4W_{I_3}r''\Big)R^5-8e^{4\omega_\Theta}r'^6\left(e^{4\omega_\Theta}W_{I_5I_5}r''R^4+e^{2\omega_R}r\left(e^{2\omega_\Theta}W_{I_2I_5}R^2+r^2W_{I_3I_5}\right)\right)R^5\\
&+8e^{2\omega_R+4\omega_\Theta}r'^5\left(e^{4\omega_\Theta}\left(W_{I_1I_5}+W_{I_4I_5}\right)\omega_R'R^5+e^{2\omega_\Theta}r^2W_{I_2I_5}\left(2R\omega_R'+R\omega_\Theta'+1\right)R^2+r^4W_{I_3I_5}\left(R\omega_R'+R\omega_\Theta'+1\right)\right)R^4\\
&-2e^{4\omega_R}r'^2\Big[e^{8\omega_\Theta}\left(W_{I_1I_1}+2W_{I_1I_4}+W_{I_4I_4}+3W_{I_5}\right)r''R^8+e^{2\omega_R+6\omega_\Theta}r\left(W_{I_2}+2\left(W_{I_1I_1}+W_{I_1I_4}\right)\right)R^6\\
&+4e^{6\omega_\Theta}r^2\left(W_{I_1I_2}+W_{I_2I_4}\right)r''R^6+e^{2\omega_R+4\omega_\Theta}r^3\left(6W_{I_1I_2}+W_{I_3}+2W_{I_2I_4}\right)R^4+2e^{4\omega_\Theta}r^4\left(2W_{I_2I_2}+W_{I_1I_3}+W_{I_3I_4}\right)r''R^4\\
&+2e^{2(\omega_R+\omega_\Theta)}r^5\left(2W_{I_2I_2}+W_{I_1I_3}\right)R^2+4e^{2\omega_\Theta}r^6W_{I_2I_3}r''R^2+2e^{2\omega_R}r^7W_{I_2I_3}+r^8W_{I_3I_3}r''\Big]R\\
&-4e^{2\omega_R}r'^4\Big\{2e^{4\omega_\Theta}\left(e^{4\omega_\Theta}W_{I_1I_5}R^4+e^{4\omega_\Theta}W_{I_4I_5}R^4+2e^{2\omega_\Theta}r^2W_{I_2I_5}R^2+r^4W_{I_3I_5}\right)r''R^4+e^{2\omega_R}r\Big(e^{6\omega_\Theta}(W_{I_1I_2}+W_{I_2I_4}\\
&+2W_{I_1I_5})R^6+e^{4\omega_\Theta}r^2\left(2W_{I_2I_2}+W_{I_1I_3}+W_{I_3I_4}+2W_{I_2I_5}\right)R^4+3e^{2\omega_\Theta}r^4W_{I_2I_3}R^2+r^6W_{I_3I_3}\Big)\Big\}R+2e^{4\omega_R}r'^3\Big[e^{8\omega_\Theta}W_{I_5}(3R\omega_R'\\
&-2R\omega_\Theta'-2)R^8+\Big(e^{8\omega_\Theta}\left(W_{I_1I_1}+2W_{I_1I_4}\right)R^8+e^{8\omega_\Theta}W_{I_4I_4}R^8+4e^{6\omega_\Theta}r^2\left(W_{I_1I_2}+W_{I_2I_4}\right)R^6+2e^{4\omega_\Theta}r^4(2W_{I_2I_2}+W_{I_1I_3}\\
&+W_{I_3I_4})R^4+4e^{2\omega_\Theta}r^6W_{I_2I_3}R^2+r^8W_{I_3I_3}\Big)\omega_R'R+2r^2\Big(e^{6\omega_\Theta}\left(W_{I_1I_2}+W_{I_2I_4}+2W_{I_1I_5}\right)R^6+e^{4\omega_\Theta}r^2(2W_{I_2I_2}+W_{I_1I_3}+W_{I_3I_4}\\
&+2W_{I_2I_5})R^4+3e^{2\omega_\Theta}r^4W_{I_2I_3}R^2+r^6W_{I_3I_3}\Big)\left(R\omega_\Theta'+1\right)\Big]+e^{6\omega_R}r'\Big\{e^{8\omega_\Theta}W_{I_1}\left(R\omega_R'-2R\omega_\Theta'-2\right)R^8+e^{8\omega_\Theta}W_{I_4}(R\omega_R'-2R\omega_\Theta'\\
&-2)R^8+2e^{6\omega_\Theta}r^2\left(RW_{I_2}\omega_R'+2\left(W_{I_1I_1}+W_{I_1I_4}\right)\left(R\omega_\Theta'+1\right)\right)R^6+e^{4\omega_\Theta}r^4\Big[4\left(3W_{I_1I_2}+W_{I_2I_4}\right)\left(R\omega_\Theta'+1\right)\\
&+W_{I_3}(R\omega_R'+2R\omega_\Theta'+2)\Big]R^4+4e^{2\omega_\Theta}r^6\left(2W_{I_2I_2}+W_{I_1I_3}\right)\left(R\omega_\Theta'+1\right)R^2+4r^8W_{I_2I_3}\left(R\omega_\Theta'+1\right)\Big\}=0\,. \qquad \text{(A.1)}
\end{aligned}
$$

Appendix B Analytical expression for $k(R)$

$$
\begin{aligned}
k(R) = &-\frac{2e^{-2(2\omega_Z+\omega_\Theta)}}{r^{10}R^3\alpha^9}\Big(2e^{4\omega_Z}\alpha^7\left(W_{I_2I_2}\left(R\omega'_\Theta+R\omega'_Z+1\right)\alpha^2+e^{2\omega_Z}W_{I_1I_2}\left(R\omega'_\Theta+1\right)\right)r^{12}\\
&+e^{2\omega_Z+\omega_\Theta}R^2\alpha^5\Big[2e^{\omega_\Theta}W_{I_2I_2}\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_Z\alpha^6+4e^{\omega_\Theta}RW_{I_2I_7}\omega'_Z\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^6+4e^{2\omega_Z+\omega_\Theta}W_{I_1I_2}\alpha^4\\
&+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_2I_6}\omega'_Z\alpha^4+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta\alpha^4+2e^{2\omega_Z+\omega_\Theta}W_{I_2I_4}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^4\\
&+e^{\omega_R+3\omega_Z}W_{I_1}\alpha^3-2e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^3+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_1}\alpha^2+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_4}\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_\Theta\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_\Theta\alpha^2\\
&-2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_Z+W_{I_2}\left(e^{\omega_R+\omega_Z}\alpha^5-2e^{4\omega_Z+\omega_\Theta}R\alpha^2\omega'_Z\right)\Big]r^{10}-2e^{3\omega_\Theta}R^4\alpha^5\Big\{-2e^{\omega_\Theta}RW_{I_2I_7}\omega'_Z\alpha^8\\
&-e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^6-e^{2\omega_Z+\omega_\Theta}RW_{I_2I_6}\omega'_Z\alpha^6-2e^{2\omega_Z+\omega_\Theta}RW_{I_1I_7}\omega'_Z\alpha^6-2e^{2\omega_Z+\omega_\Theta}RW_{I_4I_7}\omega'_Z\alpha^6+e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^5\\
&-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_Z\alpha^4-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_Z\alpha^4-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_6}\omega'_Z\alpha^4-e^{4\omega_Z+\omega_\Theta}RW_{I_4I_6}\omega'_Z\alpha^4+e^{4\omega_Z+\omega_\Theta}W_{I_2}\left(R\omega'_\Theta+1\right)\alpha^4\\
&+2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha^3+e^{\omega_R+5\omega_Z}W_{I_2I_4}\alpha^3+e^{6\omega_Z+\omega_\Theta}W_{I_1}\alpha^2+e^{6\omega_Z+\omega_\Theta}W_{I_2I_2}\alpha^2-2e^{6\omega_Z+\omega_\Theta}W_{I_2I_5}\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_1}\omega'_Z\alpha^2\\
&+e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_Z\alpha^2-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_1}\omega'_\Theta\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^2-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\alpha^2\\
&+e^{6\omega_Z+\omega_\Theta}W_{I_4}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2+e^{\omega_R+7\omega_Z}W_{I_1I_1}\alpha+e^{\omega_R+7\omega_Z}W_{I_1I_4}\alpha+e^{8\omega_Z+\omega_\Theta}W_{I_1I_2}+e^{8\omega_Z+\omega_\Theta}W_{I_2I_4}-2e^{8\omega_Z+\omega_\Theta}W_{I_1I_5}\\
&+2e^{8\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z+2e^{8\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_Z+e^{8\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta+e^{8\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_\Theta-2e^{8\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_\Theta\Big\}r^8\\
&-e^{4\omega_Z+5\omega_\Theta}R^6\alpha^3\Big[2e^{\omega_\Theta}W_{I_2I_2}\alpha^6-8e^{\omega_\Theta}RW_{I_5I_7}\omega'_Z\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^6-e^{\omega_R+\omega_Z}W_{I_2}\alpha^5+4e^{2\omega_Z+\omega_\Theta}W_{I_1I_2}\alpha^4\\
&+4e^{2\omega_Z+\omega_\Theta}W_{I_2I_4}\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_Z\alpha^4-4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_Z\alpha^4-4e^{2\omega_Z+\omega_\Theta}RW_{I_5I_6}\omega'_Z\alpha^4\\
&+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta\alpha^4+4e^{2\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_\Theta\alpha^4-e^{\omega_R+3\omega_Z}W_{I_1}\alpha^3-2e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^3-e^{\omega_R+3\omega_Z}W_{I_4}\alpha^3+4e^{\omega_R+3\omega_Z}W_{I_2I_5}\alpha^3\\
&+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_1}\alpha^2+4e^{4\omega_Z+\omega_\Theta}W_{I_1I_4}\alpha^2+2e^{4\omega_Z+\omega_\Theta}W_{I_4I_4}\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_Z\alpha^2+4e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_Z\alpha^2\\
&+2e^{4\omega_Z+\omega_\Theta}RW_{I_4I_4}\omega'_Z\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_\Theta\alpha^2+4e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_\Theta\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_4I_4}\omega'_\Theta\alpha^2\\
&+8e^{4\omega_Z+\omega_\Theta}W_{I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2-2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha-2e^{\omega_R+5\omega_Z}W_{I_2I_4}\alpha+4e^{\omega_R+5\omega_Z}W_{I_1I_5}\alpha+4e^{6\omega_Z+\omega_\Theta}W_{I_2I_5}\\
&+8e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z+4e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\Big]r^6+2e^{5\omega_Z+7\omega_\Theta}R^8\alpha^2\Big\{e^{\omega_R}W_{I_2I_2}\alpha^6+2e^{\omega_R+2\omega_Z}W_{I_1I_2}\alpha^4+2e^{\omega_R+2\omega_Z}W_{I_2I_4}\alpha^4\\
&-4e^{3\omega_Z+\omega_\Theta}W_{I_2I_5}\alpha^3-2e^{3\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z\alpha^3-4e^{3\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\alpha^3+e^{\omega_R+4\omega_Z}W_{I_1I_1}\alpha^2+2e^{\omega_R+4\omega_Z}W_{I_1I_4}\alpha^2+e^{\omega_R+4\omega_Z}W_{I_4I_4}\alpha^2\\
&+3e^{\omega_R+4\omega_Z}W_{I_5}\alpha^2-4e^{5\omega_Z+\omega_\Theta}W_{I_1I_5}\alpha-4e^{5\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_Z\alpha-4e^{5\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_\Theta\alpha-4e^{5\omega_Z+\omega_\Theta}W_{I_4I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha\\
&+2e^{\omega_R+6\omega_Z}W_{I_2I_5}\Big\}r^4-8e^{9(\omega_Z+\omega_\Theta)}R^{10}\alpha\left(e^{3\omega_Z+\omega_\Theta}W_{I_5I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)-e^{\omega_R}\alpha\left(W_{I_2I_5}\alpha^2+e^{2\omega_Z}W_{I_1I_5}+e^{2\omega_Z}W_{I_4I_5}\right)\right)r^2\\
&+8e^{\omega_R+13\omega_Z+11\omega_\Theta}R^{12}W_{I_5I_5}\Big). \qquad \text{(B.1)}
\end{aligned}
$$

Appendix C Radial equilibrium equation for the compressible orthotropic case

$$
\begin{aligned}
&4e^{6\omega_R+4\omega_\Theta}R^5W_{I_2I_7}r'\omega'_Z\alpha^6+4e^{6\omega_R+2\omega_\Theta}r^2R^3W_{I_3I_7}r'\omega'_Z\alpha^6+8e^{4\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_5I_7}{r'}^3\omega'_Z\alpha^4\\
&+2e^{2(3\omega_R+\omega_Z+\omega_\Theta)}r^2R^3W_{I_3I_6}r'\omega'_Z\alpha^4+4e^{6\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_1I_7}r'\omega'_Z\alpha^4+4e^{2(3\omega_R+\omega_Z+\omega_\Theta)}r^2R^3W_{I_2I_7}r'\omega'_Z\alpha^4\\
&+4e^{6\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_4I_7}r'\omega'_Z\alpha^4+2e^{6\omega_R+2\omega_Z}r^4W_{I_2I_3}r'\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^4-2e^{4\omega_R+4\omega_Z+2\omega_\Theta}rR^3W_{I_1I_3}{r'}^4\alpha^2\\
&-2e^{4\omega_R+2\omega_Z}rR\left(2e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}{r'}^4\alpha^2+2e^{4\omega_R+2\omega_Z}r^2R\left(e^{2\omega_Z}r^2+2e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}{r'}^3\omega'_Z\alpha^2\\
&+4e^{4(\omega_R+\omega_Z+\omega_\Theta)}R^5W_{I_5I_6}{r'}^3\omega'_Z\alpha^2+2e^{6\omega_R+4(\omega_Z+\omega_\Theta)}R^5\left(W_{I_1I_6}+W_{I_4I_6}\right)r'\omega'_Z\alpha^2\\
&+2e^{4\omega_R+2\omega_Z}r^2\left(2e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}{r'}^3\left(R\omega'_\Theta+1\right)\alpha^2+4e^{4\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_2I_5}{r'}^3\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2\\
&+2e^{6\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_2I_4}r'\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2+2e^{4\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_1I_3}{r'}^3\left(2R\omega'_R+R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2\\
&+e^{6\omega_R+4(\omega_Z+\omega_\Theta)}R^4W_{I_2}\left(r'\left(R\omega'_R+R\omega'_Z-R\omega'_\Theta-1\right)-Rr''\right)\alpha^2+2e^{2(\omega_R+2\omega_Z+\omega_\Theta)}rR^2{r'}^2\left(2W_{I_3I_5}{r'}^2+e^{2\omega_R}W_{I_3I_4}\right)\Big(-R{r'}^2\\
&+r\left(2R\omega'_R+R\omega'_Z+R\omega'_\Theta+1\right)r'-2rRr''\Big)\alpha^2-4e^{2(\omega_R+3\omega_Z+\omega_\Theta)}rR^3W_{I_2I_5}{r'}^6-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}rR^3W_{I_1I_2}{r'}^4\\
&-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}rR^3W_{I_2I_4}{r'}^4-e^{6\omega_R+4\omega_Z+2\omega_\Theta}rR^3W_{I_2}\left(e^{2\omega_Z}{r'}^2-e^{2\omega_R}\alpha^2\right)+8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_5}{r'}^5\omega'_R\\
&+8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_5}{r'}^5\omega'_R+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_1}{r'}^3\omega'_R+4e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_4}{r'}^3\omega'_R\\
&+2e^{4\omega_R+2\omega_Z}r'\Big[e^{2(\omega_R+\omega_Z+\omega_\Theta)}W_{I_1I_1}R^2+e^{2(\omega_Z+\omega_\Theta)}\left(2W_{I_1I_5}{r'}^2+e^{2\omega_R}W_{I_1I_4}\right)R^2+e^{2\omega_R}r^2\alpha^2W_{I_1I_3}\Big]\Big\{e^{2\omega_\Theta}\alpha^2\omega'_ZR^3\\
&+e^{2\omega_Z}r^2\left(R\omega'_\Theta+1\right)\Big\}+2e^{2(\omega_R+2\omega_Z+\omega_\Theta)}R^2{r'}^3\Big[2W_{I_2I_5}{r'}^2+e^{2\omega_R}W_{I_1I_2}+e^{2\omega_R}W_{I_2I_4}\Big]\Big(e^{2\omega_\Theta}\alpha^2\omega'_ZR^3\\
&+2\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)\omega'_RR+e^{2\omega_Z}r^2\left(R\omega'_\Theta+1\right)\Big)+2e^{6\omega_R+2\omega_Z}W_{I_1I_2}r'\Big[e^{2\omega_Z}\left(e^{2\omega_Z}r^2+2e^{2\omega_\Theta}R^2\alpha^2\right)\left(R\omega'_\Theta+1\right)r^2\\
&+\Big(e^{4\omega_\Theta}\alpha^4R^5+2e^{2(\omega_Z+\omega_\Theta)}r^2\alpha^2R^3\Big)\omega'_Z\Big]+2e^{4\omega_R+2\omega_Z}\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)r'\Big[R\Big(e^{2(\omega_R+\omega_\Theta)}W_{I_2I_6}\omega'_ZR^2+2r^2W_{I_2I_3}{r'}^2\omega'_R\Big)\alpha^2\\
&+W_{I_2I_2}\Big(-e^{2\omega_Z}rR{r'}^3+\left(e^{2\omega_\Theta}\alpha^2\omega'_ZR^3+e^{2\omega_Z}r^2\left(R\omega'_\Theta+1\right)\right){r'}^2-e^{2\omega_R}rR\alpha^2r'+e^{2\omega_R}r^2\alpha^2\left(R\omega'_Z+R\omega'_\Theta+1\right)\Big)\Big]\\
&+8e^{6\omega_Z+4\omega_\Theta}R^5W_{I_5I_5}{r'}^6\left(r'\omega'_R-r''\right)+2e^{4\omega_R+2\omega_Z}R\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)^2W_{I_2I_2}{r'}^2\left(r'\omega'_R-r''\right)\\
&+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_4}{r'}^2\left(r'\omega'_R-r''\right)-8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_5}{r'}^4r''+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^4W_{I_5}{r'}^2[r'(3R\omega'_R-R\omega'_Z\\
&-R\omega'_\Theta-1)-3Rr'']+e^{6\omega_R+6\omega_Z+4\omega_\Theta}R^4W_{I_4}\left(r'\left(R\omega'_R-R\omega'_Z-R\omega'_\Theta-1\right)-Rr''\right)+e^{6\omega_R+6\omega_Z+2\omega_\Theta}r^2R^2W_{I_2}[r'(R\omega'_R-R\omega'_Z\\
&+R\omega'_\Theta+1)-Rr'']+\Big\{2e^{4\omega_R+2\omega_Z}r^3W_{I_3I_3}{r'}^2\alpha^4+e^{6\omega_R+4\omega_Z+2\omega_\Theta}rR^2W_{I_3}\alpha^2\Big\}\Big(-R{r'}^2+r\left(R\omega'_R+R\omega'_Z+R\omega'_\Theta+1\right)r'-rRr''\Big)\\
&+e^{6\omega_R+6\omega_Z+2\omega_\Theta}R^3W_{I_1}\left(-e^{2\omega_\Theta}r''R^2+e^{2\omega_\Theta}r'\left(R\omega'_R-R\omega'_Z-R\omega'_\Theta-1\right)R+e^{2\omega_R}r\right)-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}R^3W_{I_1I_1}{r'}^2\Big(e^{2\omega_\Theta}r''R^2\\
&+e^{2\omega_R}r\Big)-2e^{2\omega_R+4\omega_Z}R{r'}^2\Big[e^{2(\omega_Z+\omega_\Theta)}\Big(2W_{I_1I_5}{r'}^2+e^{2\omega_R}W_{I_1I_4}\Big)R^2+e^{2\omega_R}r^2\alpha^2W_{I_1I_3}\Big]\left(2e^{2\omega_\Theta}r''R^2+e^{2\omega_R}r\right)\\
&-2e^{4(\omega_R+\omega_Z)}RW_{I_1I_2}{r'}^2\Big(2e^{4\omega_\Theta}\alpha^2r''R^4+2e^{2(\omega_R+\omega_\Theta)}r\alpha^2R^2+2e^{2(\omega_Z+\omega_\Theta)}r^2r''R^2+e^{2(\omega_R+\omega_Z)}r^3\Big)\\
&-2e^{2(\omega_R+\omega_Z)}R{r'}^2\Big[e^{2(\omega_Z+\omega_\Theta)}\Big(2W_{I_2I_5}{r'}^2+e^{2\omega_R}W_{I_2I_4}\Big)R^2+e^{2\omega_R}r^2\alpha^2W_{I_2I_3}\Big]\left\{e^{2\omega_R}r\alpha^2+2\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)r''\right\}=0\,.
\end{aligned}
\tag{C.1}
$$