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RESEARCH PAPER

Closed form solutions for free vibrations of rectangular Mindlin plates

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Abstract A new two-eigenfunctions theory, using the amplitude deflection and the generalized curvature as two fundamental eigenfunctions, is proposed for the free vibration solutions of a rectangular Mindlin plate. The three classical eigenvalue differential equations of a Mindlin plate are reformulated to arrive at two new eigenvalue differential equations for the proposed theory. The closed form eigensolutions, which are solved from the two differential equations by means of the method of separation of variables are identical with those via Kirchhoff plate theory for thin plate, and can be employed to predict frequencies for any combinations of simply supported and clamped edge conditions. The free edges can also be dealt with if the other pair of opposite edges are simply supported. Some of the solutions were not available before. The frequency parameters agree closely with the available ones through *pb*-2 Rayleigh–Ritz method for different aspect ratios and relative thickness of plate.

Keywords Mindlin plate · Free vibration · Closed form solution · Separation of variable

1 Introduction

A plate is one of the most important structural elements, its theoretical descriptions were established by Chladni [1] and Kirchhoff [2]. Since then, there have been extensive investigations on the vibrations of plates with various shapes, supports, loading conditions, and complicating effects, as reported in Refs. [3–12]. However, most of them were based on thin plate theory, in which no account is taken for the effect of transverse shear deformation on the mechanical behavior of thick plates [13,14]. To allow for this shear effect, Mindlin [15] proposed the first order shear deformation theory for the motion of moderately thick plates and incorporated the effect of rotatory inertia. A shear correction factor, κ, was introduced in this theory to compensate the errors resulting from the assumption of uniform shear strain distribution in the thickness direction.

Accurate analytical results for free vibration of thin rectangular plates can be easily obtained [4], but it is more difficult to obtain the analytical solutions for the free vibrations of rectangular Mindlin plates due to more governing equations and kinetic parameters involved. For this reason many efforts were devoted to approximate solutions with a high level of accuracy. FEM, Rayleigh–Ritz method, finite strip method and collocation methods etc have been widely used to study the free vibrations of Mindlin rectangular plates. Liew et al. [13] presented a literature survey on the relevant works up to 1994. Thus only some representative works published after 1994 are reviewed below.

The methods for the free vibration analysis of rectangular Mindlin plates include analytical approaches [16–23] as well as numerical methods [24–31]. For simplifying the mechanical behavior analysis of Mindlin plates, two two-variable alternative formulations for Mindlin plate have been proposed, one used the bending defection as a fundamental variable instead of the rotation angle by Endo and Kimura [16], the other took the bending component and the shearing component of lateral deflections w as fundamental variables by Shimpi and Patel [17]. Hashemi and Arsanjani

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[18] presented the exact characteristic equations supposed that at least one pair of opposite plate edges is simply supported, for which, in point of fact, an easier solution method was given by Brunelle [19]. In Gorman's [20] superposition method the solution satisfied the differential equations exactly but approximated the boundary conditions. Wang [21] presented an explicit formula for the natural frequencies of simply supported Mindlin plates in terms of the corresponding thin plate frequencies. Xiang and Wei [22] and Xiang [23] employed the Levy solution approach in conjunction with the state space technique to derive the analytical eigensolutions of rectangular Mindlin plates with two opposite edges simply supported.

Liew et al. [24] obtained accurate natural frequencies of Mindlin plates via the *pb*-2 Rayleigh–Ritz method, Cheung and Zhou [25] studied the similar problems in terms of a set of static Timoshenko beam functions, and Shen et al. [26] developed a new set of admissible functions satisfying natural boundary conditions for plates with four free edges. A 2D differential quadrature element method (DQEM) [27] and a semi-analytical DQEM [28] were developed for the free vibrations of thick plates. Hou et al. [29] proposed a DSC-Ritz method taking advantages of both the local bases of the discrete singular convolution (DSC) algorithm and the *pb*-2 Ritz boundary functions to arrive at a new approach. Diaz-Contreras and Nomura [30] derived numerical Green's functions constructed via the eigenfunction expansion method to solve Mindlin plate problems. Sakiyama and Huang [31] proposed a Green function method for the free vibration analysis of thin and moderately thick rectangular plates with arbitrary variable thickness.

It is noteworthy that hitherto the exact solutions are available only for the free vibrations of rectangular Mindlin plates with four simply supported edges [32,33] and with at least two simply supported opposite edges [18,22]. It was believed that difficulties would encounter in solving the exact solutions for the free vibrations of rectangular Mindlin plates without two simply supported opposite edges. In this context, this paper introduces a few new closed form solutions for free vibrations of rectangular Mindlin plates on the basis of the proposed theory and the authors' previous work [34,35] pertaining to thin plates.

The outline of this paper is as follows. The amplitude deflection and the generalized curvature relevant to the spatial rotational angle are regarded as the two independent eigenfunctions to arrive at a new two-eigenfunction theory in Sect. 2. The general closed form eigenfunctions are solved from the two governing equations by means of the method of separation of variables in Sect. 3. In Sect. 4 the eigenvalue equations and eigenfunctions coefficients are determined for different boundary conditions. The extensive numerical experiments are conducted in Sect. 5. Finally, conclusions are outlined in Sect. 6.

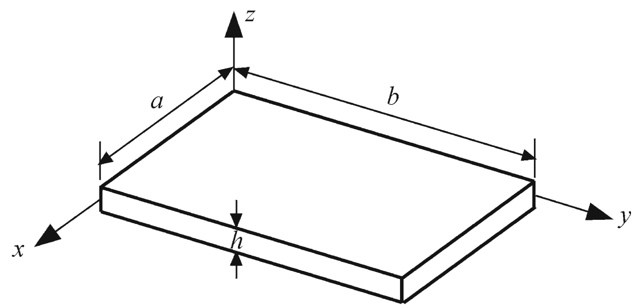

Fig. 1 Mindlin plate and coordinates

2 A new two-eigenfunction theory

In Mindlin plate theory (MPT), the normal straight line is extensible and not compressible. The shortest or original length of the normal line is the thickness of the plate. For an isotropic plate, the deformed (or the rotated and extended) normal line intersects at a point on the middle surface with the original undeformed normal line. Consider a thick rectangular plate of length a, width b and uniform thickness h, oriented so that its undeformed middle surface contains the x and y axes of a Cartesian coordinate system (x, y, z), as shown in Fig. 1. Three fundamental variables in classical MPT are the displacements along x, y and z directions, as

$$\begin{aligned} u &= -z\psi_x(x, y, z, t), \\ v &= -z\psi_y(x, y, z, t), \\ w &= w(x, y, z, t), \end{aligned} \tag{1}$$

where t is the time, w the deflection, and ψ_x and ψ_y are the angles of rotations of a normal line due to plate bending with respect to y and x coordinates, respectively, they can also be considered as the projections of the spatial rotation angle ψ of normal line on the plane xz and plane xy, respectively. It should be emphasized that w, ψ_x and ψ_y are regarded as fundamental variables in place of u, v, and w for classical MPT, it is reasonable and feasible due to the assumptions of the MPT, see also Eq. (1). The relations between the internal forces and displacements in MPT are

$$\begin{aligned} M_x &= -D\left(\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\right), \\ M_y &= -D\left(\frac{\partial\psi_y}{\partial y} + \nu\frac{\partial\psi_x}{\partial x}\right), \end{aligned} \tag{2a}$$

$$M_{xy} = -\frac{1}{2}(1-\nu)D\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right), \tag{2b}$$

$$Q_x = C\left(\frac{\partial w}{\partial x} - \psi_x\right), \quad Q_y = C\left(\frac{\partial w}{\partial y} - \psi_y\right), \tag{2c}$$

where ν is the Poisson's ratio, $D = Eh^3/12(1-\nu^2)$ the flexural rigidity, $C = \kappa Gh$ the shear rigidity, $G = E/2(1+\nu)$ the shear modulus, and $\kappa = 5/6$ the shear correction factor.

The equations of free motion in absence of the external loads are given by

$$-\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x - \rho J\frac{\partial^2 \psi_x}{\partial t^2} = 0, \quad (3a)$$

$$-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y - \rho J\frac{\partial^2 \psi_y}{\partial t^2} = 0, \quad (3b)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho h\frac{\partial^2 w}{\partial t^2} = 0, \quad (3c)$$

where $J = h^3/12$ is area axial moment of inertia of cross section. For principle or harmonic vibration, it is assumed that

$$\psi_x = \Psi_x(x, y)e^{i\omega t},\\ \psi_y = \Psi_y(x, y)e^{i\omega t},\\ w = W(x, y)e^{i\omega t}. \quad (4)$$

Substitutions of expressions (4) into Eq. (2) and then Eq. (2) into Eq. (3) lead to the eigenvalue partial differential equations in terms of displacements, as

$$\frac{\partial^2 \Psi_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \Psi_x}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \Psi_y}{\partial x \partial y} + \frac{C}{D}\left(\frac{\partial W}{\partial x} - \Psi_x\right) + \frac{\omega^2 \rho J}{D}\Psi_x = 0, \quad (5a)$$

$$\frac{\partial^2 \Psi_y}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 \Psi_y}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \Psi_x}{\partial x \partial y} + \frac{C}{D}\left(\frac{\partial W}{\partial y} - \Psi_y\right) + \frac{\omega^2 \rho J}{D}\Psi_y = 0, \quad (5b)$$

$$\nabla^2 W - \left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right) + \frac{\omega^2 \rho h}{C}W = 0, \quad (5c)$$

where ∇^2 is the Laplace operator. The sum of the differentiation of Eq. (5a) with respect to x and Eq. (5b) with respect to y results in

$$\nabla^2 \Psi + \frac{C}{D}\nabla^2 W + \frac{\rho J\omega^2 - C}{D}\Psi = 0, \quad (6)$$

where

$$\Psi = \frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}, \quad (7)$$

where Ψ is called the generalized curvature in this paper. Inserting Eq. (7) into Eq. (5c) yields

$$\nabla^2 W - \Psi + \frac{\omega^2 \rho h}{C}W = 0. \quad (8)$$

In Eqs. (6) and (8) there are two independent unknown functions, namely, the generalized curvature Ψ and the amplitude deflection W, therefore these two equations can be considered as the eigenvalue differential equations of Mindlin plate. Here, the theory based on Eqs. (6) and (8) involving two fundamental eigenfunctions Ψ and W is called the new two-eigenfunction theory for the free vibrations of Mindlin plate, for which we have following comments.

(1) It follows from Eq. (2a) that

$$M_x + M_y = -D(1+\nu)\Psi, \quad (9)$$

which is the relation between the "generalized curvature" Ψ and the "generalized internal moment " $M_x + M_y$.

(2) To replace ∇^2 by d^2/dx^2, D by EI, C by κGA, ρh by ρA, then Eqs. (6) and (8) will reduce to be the governing equations for the free vibrations of Timoshenko beam, wherein $\Psi = d\Psi_x/dx$. But Eq. (6) is one order higher than the corresponding equation in Timoshenko beam.

(3) It can been seen from following solution procedure that only two of the three classic boundary conditions in MPT are necessary for each edge, which is consistent with two fundamental variables and greatly simplifies the solution procedure of free vibration problem. Moreover, the closed form eigensolutions of Mindlin plate can be obtained for any combinations of simply supported and clamped edges, the free edges can also be dealt with when the other pair of opposite edges is simply supported. And some of closed form eigensolutions were not available before.

3 Closed form eigensolutions

One can obtain the closed form solutions of Eqs. (6) and (8) by using the separation of variables. Eliminating Ψ or W from Eqs. (6) and (8) gives the same differential equation as

$$\nabla^4 X + \left(\frac{\omega^2 \rho h}{C} + \frac{\omega^2 \rho J}{D}\right)\nabla^2 X - \left(1 - \frac{\rho J\omega^2}{C}\right)\frac{\omega^2 \rho h}{D}X = 0, \quad (10)$$

where $X = \Psi$ or W. It should be pointed out that different equations would be obtained if W and Ψ_x, or W and Ψ_y are eliminated from Eqs. (5) [36], which is the reason of regarding Eqs. (6) and (8) as the eigenvalue differential equations of Mindlin plate in this paper. For brevity, define following parameters

$$R^2 = \omega\sqrt{\frac{\rho h}{D}}, \quad c = \frac{D}{C} + \frac{J}{h}, \quad s = 1 - \frac{DJ}{Ch}R^4, \quad (11)$$

then Eq. (10) can be rewritten as

$$\nabla^4 X + cR^4\nabla^2 X - sR^4 X = 0. \quad (12)$$

It is apparent that if $c = 0$ and $s = 1$ or $C \to \infty$ and $\rho J \to 0$, Eq. (12) reduces to be the eigenvalue differential equation of Kirchhoff plate. To solve Eq. (12) by means of the separation of variables method [34–36], assume that

$$W(x, y) = e^{\mu x} e^{\lambda y}. \tag{13}$$

Substituting expression (13) into Eq. (12), one has

$$(\mu^2 + \lambda^2)^2 + cR^4(\mu^2 + \lambda^2) - sR^4 = 0, \tag{14}$$

which is the characteristic equation of Eqs. (6) and (8), and can be rewritten as

$$\mu^2 + \lambda^2 = -R_1^2 \quad \text{and} \quad \mu^2 + \lambda^2 = R_2^2, \tag{15}$$

where

$$R_1^2 = \frac{R^4}{2}\left(c + \sqrt{c^2 + \frac{4s}{R^4}}\right), \quad R_2^2 = \frac{R^4}{2}\left(-c + \sqrt{c^2 + \frac{4s}{R^4}}\right). \tag{16}$$

The roots of characteristic equation (15) are

$$\mu_{1,2} = \pm \mathrm{i}\sqrt{R_1^2 + \lambda^2} = \pm \mathrm{i}\beta_1, \quad \mu_{3,4} = \pm\sqrt{R_2^2 - \lambda^2} = \pm\beta_2, \tag{17a}$$

or

$$\lambda_{1,2} = \pm \mathrm{i}\sqrt{R_1^2 + \mu^2} = \pm \mathrm{i}\alpha_1, \quad \lambda_{3,4} = \pm\sqrt{R_2^2 - \mu^2} = \pm\alpha_2. \tag{17b}$$

So the eigenfunction W in separation of variable form can be expressed in terms of the eigenvalues as

$$W(x, y) = \phi(x)\psi(y), \tag{18}$$

where

$$\phi(x) = A_1 \cos\beta_1 x + B_1 \sin\beta_1 x + C_1 \cosh\beta_2 x + D_1 \sinh\beta_2 x, \tag{19a}$$

$$\psi(y) = E_1 \cos\alpha_1 y + F_1 \sin\alpha_1 y + G_1 \cosh\alpha_2 y + H_1 \sinh\alpha_2 y. \tag{19b}$$

Similarly, one can obtain the closed-form eigenfunction Ψ by using the eigenvalues in Eq. (17). Since it is more convenient to deal with boundary conditions by Ψ_x and Ψ_y than by Ψ, here Ψ_x and Ψ_y are solved instead of Ψ. Based on the form of Eq. (8), the eigenfunctions Ψ_x and Ψ_y can be assumed as

$$\Psi_x(x, y) = g(x)\psi(y), \quad \Psi_y(x, y) = \phi(x)h(y), \tag{20}$$

where

$$g(x) = g_1\beta_1(-A_1 \sin\beta_1 x + B_1 \cos\beta_1 x) + g_2\beta_2(C_1 \sinh\beta_2 x + D_1 \cosh\beta_2 x), \tag{21a}$$

$$h(y) = h_1\alpha_1(-E_1 \sin\alpha_1 y + F_1 \cos\alpha_1 y) + h_2\alpha_2(G_1 \sinh\alpha_2 y + H_1 \cosh\alpha_2 y). \tag{21b}$$

Substitution of expressions (18) and (20) into Eq. (8) leads to

$$g_1\beta_1^2 + h_1\alpha_1^2 = \beta_1^2 + \alpha_1^2 - DR^4/C, \quad g_1\beta_1^2 - h_2\alpha_2^2 = \beta_1^2 - \alpha_2^2 - DR^4/C, \tag{22a}$$

$$g_2\beta_2^2 - h_1\alpha_1^2 = \beta_2^2 - \alpha_1^2 + DR^4/C, \quad g_2\beta_2^2 + h_2\alpha_2^2 = \beta_2^2 + \alpha_2^2 + DR^4/C. \tag{22b}$$

The coefficient matrix of Eqs. (22) is singular, thus the solution can be determined, when $g_1 = h_1$, as

$$g_1 = h_1 = 1 - \gamma, \quad g_2 = 1 + \gamma\left(\frac{\beta_1}{\beta_2}\right)^2, \quad h_2 = 1 + \gamma\left(\frac{\alpha_1}{\alpha_2}\right)^2, \tag{23}$$

where the dimensionless parameter γ is given by

$$\gamma = \frac{DR^4}{CR_1^2}. \tag{24}$$

The closed form solutions given by expressions (18) and (20) satisfy the governing equation (8) or (5c), and Eq. (10) exactly, is accurate enough for the plate with moderate thickness, and are exact for the plate with four simply supported edges.

4 Eigenvalue equations and eigenfunctions

In amplitude eigenfunctions in expressions (18) and (20), there are eight unknown integral constants altogether, which can be determined by using eight boundary conditions. But all twelve classical boundary conditions for a rectangular Mindlin plate can be satisfied simultaneously by present eigensolutions, the reasons for different boundary conditions are as follows.

(1) Simply supported edge (S)

Three simple support boundary conditions are $W = 0$, $\Psi_s = 0$ (subscript s denotes the tangent of the edge) and $M_n = 0$ (subscript n denotes the normal of the plate edge). Since $W = 0$, we have $\Psi_s = 0$ from expressions (18) and (20), namely, $\Psi_s = 0$ is satisfied naturally. Thus the remaining two independent boundary conditions are

$$W = 0, \quad M_n = 0 \Rightarrow \frac{\partial\Psi_n}{\partial n} + \nu\frac{\partial\Psi_s}{\partial s} = 0 \Rightarrow \frac{\partial\Psi_n}{\partial n} = 0. \tag{25}$$

(2) Clamped edge (C)

Three clamp boundary conditions are $W = 0$, $\Psi_s = 0$ and $\Psi_n = 0$. Similarly, $\Psi_s = 0$ is satisfied naturally as aforementioned. Thus, the two independent boundary conditions for clamped edge are

$$W = 0, \quad \Psi_n = 0. \tag{26}$$

(3) Free edge (F)

Three free boundary conditions are $Q_n = 0$, $M_n = 0$ and $M_{ns} = 0$. Since none of them is satisfied naturally, thus the normal bending moment and total shear force are assumed to be zeros

$$M_n = 0, \quad Q_n + \frac{\partial M_{ns}}{\partial s} = 0. \tag{27}$$

This approach is the same as that in thin plate, and is more reasonable and more accurate than the original three conditions for practical problems. Equation (27) can be rewritten in terms of displacements by substituting Eq. (2) into Eq. (27), as

$$\begin{aligned} &\frac{\partial \Psi_n}{\partial n} + \nu \frac{\partial \Psi_s}{\partial s} = 0, \\ &\frac{\partial^2 \Psi_n}{\partial n^2} + (1-\nu)\frac{\partial^2 \Psi_n}{\partial s^2} + \frac{\partial^2 \Psi_s}{\partial n \partial s} + \frac{\omega^2 \rho J}{D}\Psi_s = 0. \end{aligned} \tag{28}$$

It should be pointed out again that the closed form solutions can be obtained for any combinations of simply supported and clamped edges, which is done below. The free edges can also be dealt with if another two opposite edges are simply supported. Regardless of the two opposite edges being S-C or others, the eigenfunctions and eigenvalue equations can be derived in the same way, so only the case S-C is solved. Assume the edge $x = 0$ is simply supported and the edge $x = a$ is clamped (S-C), the boundary conditions are

$$\begin{aligned} &W(0, y) = \phi(0)\psi(y) = 0 \Rightarrow \phi(0) = 0, \\ &\frac{\partial \Psi_x(0, y)}{\partial x} = \frac{\partial g(0)}{\partial x}\psi(y) = 0 \Rightarrow \frac{\partial g(0)}{\partial x} = 0, \end{aligned} \quad \text{at } x = 0, \tag{29a}$$

$$\begin{aligned} &W(a, y) = \phi(a)\psi(y) = 0 \Rightarrow \phi(a) = 0, \\ &\Psi_x(a, y) = g(a)\psi(y) = 0 \Rightarrow g(a) = 0, \end{aligned} \quad \text{at } x = a. \tag{29b}$$

Substitution of expressions (18) and (20) into Eq. (29) yields

$$A_1 = C_1 = 0, \tag{30a}$$

$$\begin{bmatrix} \sin\beta_1 a & \sinh\beta_2 a \\ g_1\beta_1\cos\beta_1 a & g_2\beta_2\cosh\beta_2 a \end{bmatrix} \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{30b}$$

The condition of nontrivial solutions of Eqs. (30b) results in the eigenvalue equation as

$$g_2\beta_2 \tan\beta_1 a = g_1\beta_1 \tanh\beta_2 a. \tag{31}$$

The corresponding normal eigenfunction is

$$\phi(x) = \sin\beta_1 x - \frac{\sin\beta_1 a}{\sinh\beta_2 a}\sinh\beta_2 x. \tag{32}$$

The closed form eigenfunctions and eigenvalue equations for six cases SSCC, SCCC, CCCC, SSSF, SCSF, and SFSF are presented in Table 1, other available eigensolutions can be readily derived similarly. In any two eigenvalue equations in Table 1, there are five unknowns ω, β_1, β_2, α_1 and α_2, so three additional relations are needed. From Eq. (17a), one can obtain

$$\beta_1^2 + \beta_2^2 = R_1^2 + R_2^2. \tag{33}$$

Substituting $\mu = \mathrm{i}\beta_1$ into Eq. (17b), one can have another two relations as

$$\alpha_1 = \sqrt{R_1^2 - \beta_1^2}, \quad \alpha_2 = \sqrt{R_2^2 + \beta_1^2}. \tag{34}$$

Solving Eqs. (33) and (34) together with the two eigenvalue equations in Table 1, one can obtain the natural frequencies and the eigenvalues, then the normal eigenfunctions are determined accordingly. The solutions for cases SSCC, SCCC and CCCC were not available before.

5 Numerical comparisons

Numerical calculations have been performed for 9 different combinations of clamped, simply supported and free edge conditions. Poisson's ratio $\nu = 0.3$, shear correction factor $\kappa = 5/6$. All numerical results are compared with Liew's [24] results obtained by using the *pb*-2 Rayleigh–Ritz method. The non-dimensional frequencies $\bar{\omega} = (\omega^2 b^2/\pi^2)\sqrt{\rho h/D}$ are shown in Tables 2, 3, 4, 5, 6, 7, 8, 9 and 10 for relative thickness $h/b = 0.001$, 0.1 and 0.2, aspect ratios $a/b = 0.4$ and 0.6. MP denotes the present closed form results.

It is apparent from Table 2 that present results are exact for simply supported plates. For all other cases, the results agree closely with those by Liew. For the cases CCCS, CCCC and SFSF, the relative differences denoted by percentage between the results of two methods seem to be larger for $a/b = 0.4$ and $h/b = 0.2$ (a thick plate due to $h/a = 0.5$), and are presented in Tables 6, 7 and 10, from which it follows that the largest relative difference is less than 3% for the first 8 frequencies, such accuracy is adequate for practical predictions of frequencies of moderately thick plate.

6 Conclusions

In present study a new two-eigenfunction theory was developed, in which the amplitude deflection and the generalized curvature are the independent generalized displacements, two new eigenvalue differential equations were derived from three classical eigenvalue differential equations of Mindlin plate. The closed form eigensolutions were obtained from two eigenvalue differential equations by using separation

Table 1 Eigensolutions for cases SSCC, SCCC, CCCC, SSSF, SCSF, and SFSF

SSCC

Eigenvalue equations

$g_2\beta_2 \tan\beta_1 a \coth\beta_2 a = g_1\beta_1,\ h_2\alpha_2 \tan\alpha_1 b \coth\alpha_2 b = h_1\alpha_1$

Normal eigenfunctions

$\phi(x) = \sin\beta_1 x - \dfrac{\sin\beta_1 a}{\sinh\beta_2 a}\sinh\beta_2 x,\ \psi(y) = \sin\alpha_1 y - \dfrac{\sin\alpha_1 b}{\sinh\alpha_2 b}\sinh\alpha_2 y$

SCCC

Eigenvalue equations

$g_2\beta_2 \tan\beta_1 a \coth\beta_2 a = g_1\beta_1$

$2(h_2\alpha_2)(h_1\alpha_1)(\cos\alpha_1 b \coth\alpha_2 b - \sinh^{-1}\alpha_2 b) = [(h_2\alpha_2)^2 - (h_1\alpha_1)^2]\sin\alpha_1 b$

Normal eigenfunctions

$\phi(x) = \sin\beta_1 x - (\sin\beta_1 a/\sinh\beta_2 a)\sinh\beta_2 x$

$\psi(y) = -\cos\alpha_1 y + k\Pi\sin\alpha_1 y + \cosh\alpha_2 y - \Pi\sinh\alpha_2 y$

$\Pi = (\cos\alpha_1 a - \cosh\alpha_2 a)/(k\sin\alpha_1 a - \sinh\alpha_2 a),\ k = (h_2\alpha_2)/(h_1\alpha_1)$

CCCC

Eigenvalue equations

$2(g_2\beta_2)(g_1\beta_1)(\cos\beta_1 a \coth\beta_2 a - \sinh^{-1}\beta_2 a) = [(g_2\beta_2)^2 - (g_1\beta_1)^2]\sin\beta_1 a$

$2(h_2\alpha_2)(h_1\alpha_1)(\cos\alpha_1 b \coth\alpha_2 b - \sinh^{-1}\alpha_2 b) = [(h_2\alpha_2)^2 - (h_1\alpha_1)^2]\sin\alpha_1 b$

Normal eigenfunctions

$\phi(x) = -\cos\beta_1 x + \chi\Xi\sin\beta_1 x + \cosh\beta_2 x - \Xi\sinh\beta_2 x$

$\Xi = (\cos\beta_1 a - \cosh\beta_2 a)/(\chi\sin\beta_1 a - \sinh\beta_2 a),\ \chi = (g_2\beta_2)/(g_1\beta_1)$

$\psi(y) = -\cos\alpha_1 y + k\Pi\sin\alpha_1 y + \cosh\alpha_2 y - \Pi\sinh\alpha_2 y$

$\Pi = (\cos\alpha_1 a - \cosh\alpha_2 a)/(k\sin\alpha_1 a - \sinh\alpha_2 a)$

SSSF

Eigenvalue equations

$(b_2/a_2)\tan\alpha_1 b = (b_1/a_1)\tanh\alpha_2 b,\ a_1 = g_1\alpha_1^2 + \nu g_1\beta_1^2,\ a_2 = h_2\alpha_2^2 - \nu h_1\beta_1^2$

$b_1 = g_1\alpha_1[\alpha_1^2 + (1-\nu)\beta_1^2 - JR^4/h] + g_1\alpha_1\beta_1^2$

$b_2 = h_2\alpha_2[\alpha_2^2 - (1-\nu)\beta_1^2 + JR^4/h] - h_1\alpha_2\beta_1^2$

Normal eigenfunctions

$\psi(y) = \sin\alpha_1 y + [(h_2 a_1\sin\alpha_1 b)/(h_1 a_2\sinh\alpha_2 b)]\sinh\alpha_2 y$

SCSF

Eigenvalue equations

$(a_2b_2 + ka_1b_1)\sinh^{-1}\alpha_2 b + (ka_2b_1 + a_1b_2)\cos\alpha_1 b\coth\alpha_2 b = (ka_1b_2 - a_2b_1)\sin\alpha_1 b$

Normal eigenfunctions

$\psi(y) = \theta\cos\alpha_1 y - k\sin\alpha_1 y - \theta\cosh\alpha_2 y + \sinh\alpha_2 y$

$\theta = (a_2\sinh\alpha_2 b + ka_1\sin\alpha_1 b)/(a_2\cosh\alpha_2 b + a_1\cos\alpha_1 b)$

SFSF

Eigenvalue equations

$2a_2b_2(\sinh^{-1}\alpha_2 b - \coth\alpha_2 b\cos\alpha_1 b) = (a_2^2 b_1/a_1 - b_2^2 a_1/b_1)\sin\alpha_1 b$

Normal eigenfunctions

$\psi(y) = -k_1\gamma\cos\alpha_1 y + k_2\sin\alpha_1 y - \gamma\cosh\alpha_2 y + \sinh\alpha_2 y$

$\gamma = (-a_1k_2\sin\alpha_1 b + a_2\sinh\alpha_2 b)/(-a_2\cos\alpha_1 b + a_2\cosh\alpha_2 b)$

$k_1 = a_2/a_1,\ k_2 = b_2/b_1$

of variables, the eigenvalues and the eigenfunction coefficients were determined through the two necessary boundary conditions of each edge.

It deserves to be emphasized that the closed form eigensolutions satisfy one of the newly derived two governing equations and all boundary conditions, and are accurate for a moderately thick plate. These solutions can be employed to predict the natural frequencies of a rectangular Mindlin plate with any combinations of simply supported and clamped edge conditions, the free edges can also be taken into account when the other pair of opposite edges is simply supported. And some of eigensolutions were not available before.

Table 2 The dimensionless frequencies $\bar{\omega}$ for case SSSS

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	7.2500	10.250	15.250	22.249	25.999	26.999	31.248	33.998
		MP	7.2500	10.250	15.250	22.249	25.999	28.998	31.248	33.998
	0.100	Liew	6.4733	8.8043	12.370	16.845	19.050	20.732	21.952	23.399
		MP	6.4733	8.8043	12.370	16.845	19.050	20.732	21.952	23.399
	0.200	Liew	5.1831	6.7212	8.9137	11.487	12.703	13.614	14.267	15.034
		MP	5.1831	6.7212	8.9137	11.487	12.703	13.614	14.267	15.033
0.6	0.001	Liew	3.7777	6.7777	11.778	12.111	16.111	18.777	20.110	25.598
		MP	3.7778	6.7777	11.778	12.111	15.111	18.777	20.110	25.999
	0.100	Liew	3.5465	6.0909	9.9324	10.174	12.275	14.690	15.532	19.050
		MP	3.5465	6.0909	9.9324	10.174	12.275	14.690	15.532	19.050
	0.200	Liew	3.0688	4.9262	7.4327	7.5825	8.8576	10.268	10.748	12.703
		MP	3.0688	4.9202	7.4327	7.5825	8.8576	10.268	10.748	12.703

Table 3 The dimensionless frequencies $\bar{\omega}$ for case SCSS

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	7.4408	10.884	16.412	23.956	26.093	29.358	33.498	34.746
		MP	7.4408	10.884	16.412	23.956	26.093	29.358	33.498	34.746
	0.100	Liew	6.5903	9.1430	12.876	17.424	19.077	20.827	22.525	23.571
		MP	6.6061	9.1887	12.945	17.504	19.087	20.862	22.607	23.634
	0.200	Liew	5.2319	6.8380	9.0478	11.598	12.709	13.634	14.340	15.064
		MP	5.2500	6.8835	9.1056	11.655	12.714	13.651	14.393	15.095
0.6	0.001	Liew	4.0543	7.5748	12.254	13.120	15.622	20.657	21.107	26.093
		MP	4.0544	7.5748	12.254	13.120	15.622	20.657	21.107	26.093
	0.100	Liew	3.7546	6.6037	10.243	10.621	12.495	15.424	15.889	19.077
		MP	3.7696	6.6396	10.257	10.669	12.540	15.475	15.964	19.087
	0.200	Liew	3.1829	5.1382	7.6052	7.6444	8.9198	10.424	10.829	12.709
		MP	3.2076	5.1858	7.6171	7.6944	8.9546	10.468	10.880	12.714

Table 4 The dimensionless frequencies $\bar{\omega}$ for case SCSC

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	7.6843	11.629	17.709	25.804	26.202	29.762	35.573	35.888
		MP	7.6842	11.629	17.709	25.804	26.202	29.762	35.573	35.888
	0.100	Liew	6.7300	9.5130	13.397	17.989	19.106	20.927	23.086	23.748
		MP	6.7683	9.6133	13.539	18.159	19.127	21.000	23.246	23.880
	0.200	Liew	5.2358	6.9578	9.1799	11.705	12.716	13.654	14.410	15.095
		MP	5.3265	7.0520	9.2945	11.818	12.726	13.690	14.514	15.156
0.6	0.001	Liew	4.4270	8.5095	12.428	14.605	16.217	22.221	22.676	26.202
		MP	4.4269	8.5095	12.429	14.605	16.217	22.221	22.676	26.202
	0.100	Liew	4.0188	7.1608	10.322	11.319	12.732	16.144	16.258	19.106
		MP	4.0570	7.2385	10.354	11.415	12.831	16.244	16.414	19.127
	0.200	Liew	3.3571	5.3571	7.6293	7.8482	8.9836	10.573	10.911	12.716
		MP	3.3736	5.4534	7.6553	7.9450	9.0550	10.657	11.013	12.726

Table 5 The dimensionless frequencies $\bar{\omega}$ for case CCSS

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	10.669	13.524	18.507	25.641	32.579	34.888	35.436	40.279
		MP	10.651	13.484	18.462	25.603	32.573	34.858	35.416	40.248
	0.100	Liew	8.5213	10.504	13.752	17.978	20.950	22.412	22.883	24.818
		MP	8.5673	10.625	13.918	18.156	21.000	22.572	23.049	25.068
	0.200	Liew	5.9939	7.3080	9.3061	11.736	12.973	13.846	14.414	15.222
		MP	6.0574	7.4369	9.4445	11.859	13.019	13.965	14.517	15.378
0.6	0.001	Liew	5.3408	8.4789	13.779	15.049	18.061	21.167	23.164	30.324
		MP	5.3187	8.4489	13.756	15.041	18.039	21.151	23.136	30.323
	0.100	Liew	4.7158	7.1714	10.948	11.640	13.560	15.620	16.633	20.346
		MP	4.7335	7.2260	11.026	11.679	13.673	15.701	16.798	20.385
	0.200	Liew	3.7121	5.3972	7.7639	8.0252	9.2073	10.483	11.006	12.892
		MP	3.7542	5.4737	7.8435	8.0673	9.3090	10.549	11.128	12.925

Table 6 The dimensionless frequencies $\bar{\omega}$ for case CCCS

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	14.842	17.107	21.394	27.930	36.731	39.868	42.376	46.706
		MP	14.817	17.035	21.298	27.840	36.652	39.860	42.345	46.649
	0.100	Liew	10.608	12.048	14.763	18.612	22.596	23.278	23.962	25.998
		MP	10.689	12.268	15.060	18.917	22.682	23.557	24.133	26.434
	0.200	Liew	6.8374	7.8352	9.5938	11.887	13.119	13.985	14.495	15.341
		MP	6.9265	8.0307	9.8097	12.077	13.201	14.202	14.649	15.626
		%	1.29	2.43	2.20	1.57	0.62	1.53	1.05	1.82
0.6	0.001	Liew	7.0598	9.7309	14.663	18.217	20.899	21.821	25.593	31.114
		MP	7.0239	9.6657	14.608	18.203	20.859	21.782	25.537	31.081
	0.100	Liew	5.8624	7.8731	11.342	12.982	14.629	15.845	17.400	21.001
		MP	5.8963	7.9632	11.462	13.048	14.817	15.966	17.669	21.113
	0.200	Liew	4.2822	5.6866	7.8996	8.3729	9.4678	10.546	11.175	13.089
		MP	4.3442	5.7985	8.0097	8.4476	9.6362	10.637	11.371	13.145

Table 7 The dimensionless frequencies $\bar{\omega}$ for case CCCC

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	14.972	17.608	22.427	29.553	38.951	39.943	42.671	47.349
		MP	14.910	17.445	22.228	29.374	38.795	39.923	42.602	47.226
	0.100	Liew	10.702	12.352	15.257	19.195	22.627	23.861	23.972	26.198
		MP	10.748	12.494	15.478	19.457	22.710	24.130	24.230	26.615
	0.200	Liew	6.9109	7.9963	9.7716	12.028	13.128	14.013	14.586	15.384
		MP	6.9598	8.1342	9.9572	12.222	13.214	14.238	14.764	15.682
		%	0.70	1.70	1.86	1.59	0.65	1.58	1.21	1.90
0.6	0.001	Liew	7.2864	10.488	16.018	18.340	21.362	23.751	26.541	33.606
		MP	7.1955	10.352	15.914	18.308	21.274	23.680	26.422	33.541
	0.100	Liew	6.0364	8.3527	12.010	13.049	14.848	16.560	17.764	21.542
		MP	6.0225	8.3904	12.107	13.107	15.017	16.683	18.021	21.629
	0.200	Liew	4.4084	5.9199	8.1258	8.4053	9.5518	10.710	11.280	13.100
		MP	4.4211	5.9969	8.2329	8.4773	9.7174	10.818	11.485	13.156

Table 8 The dimensionless frequencies $\bar{\omega}$ for case SSSF

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	6.4124	8.2685	11.921	17.451	24.910	25.083	27.036	30.886
		MP	6.4122	8.2683	11.930	17.451	24.910	25.083	27.036	30.885
	0.100	Liew	5.7705	7.2381	9.9582	13.726	18.212	18.471	19.589	21.672
		MP	5.7872	7.2771	10.037	13.835	18.433	18.501	19.622	21.738
	0.200	Liew	4.6886	5.6837	7.4226	9.6690	12.211	12.382	12.991	14.110
		MP	4.7028	5.7198	7.4990	9.7879	12.387	12.390	13.006	14.148
0.6	0.001	Liew	2.9585	4.7249	8.2605	11.241	13.152	13.713	16.904	21.158
		MP	2.9581	4.7247	8.2604	11.250	13.152	13.713	16.904	21.158
	0.100	Liew	2.8006	4.3315	7.2082	9.5156	10.870	11.241	13.390	16.122
		MP	2.8113	4.3697	7.2762	9.5380	10.908	11.318	13.465	16.197
	0.200	Liew	2.4760	3.6431	5.6551	7.1634	7.9991	8.2054	9.4935	11.014
		MP	2.4887	3.6871	5.7298	7.1770	8.0266	8.2970	9.5572	11.131

Table 9 The dimensionless frequencies $\bar{\omega}$ for case SCSF

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	6.4520	8.6144	12.726	18.757	25.100	26.749	27.228	31.391
		MP	6.4520	8.6144	12.726	18.758	25.100	26.749	27.228	31.391
	0.100	Liew	5.7941	7.4351	10.350	14.250	18.475	18.884	19.641	21.804
		MP	5.8157	7.5063	10.488	14.432	18.507	19.085	19.694	21.911
	0.200	Liew	4.6987	5.7600	7.5460	9.7941	12.309	12.383	13.003	14.134
		MP	4.7177	5.8283	7.6751	9.9708	12.391	12.539	13.027	14.196
0.6	0.001	Liew	3.0203	5.1663	9.1995	11.278	13.429	15.184	17.583	23.162
		MP	3.0203	6.1664	9.1996	11.278	13.429	15.184	17.583	23.162
	0.100	Liew	2.8474	4.6408	7.7697	9.5283	10.995	11.940	13.658	16.844
		MP	2.8638	4.7080	7.8801	9.5546	11.062	12.065	13.794	16.967
	0.200	Liew	2.5032	3.7976	5.8720	7.1676	8.0375	8.3938	9.5634	11.150
		MP	2.5240	3.8806	5.9955	7.1838	8.0877	8.5365	9.6706	11.307

Table 10 The dimensionless frequencies $\bar{\omega}$ for case SFSF

a/b	h/b		1	2	3	4	5	6	7	8
0.4	0.001	Liew	6.1807	6.9892	9.5383	13.745	19.742	24.847	25.648	27.657
		MP	6.2499	6.9891	9.5383	13.745	19.742	24.999	25.648	27.653
	0.100	Liew	5.5840	6.2084	8.1490	11.150	15.086	18.340	18.755	19.774
		MP	5.6580	6.2529	8.2534	11.326	15.309	18.473	18.811	20.017
	0.200	Liew	4.5595	4.9810	6.2614	8.1230	10.396	12.310	12.533	12.881
		MP	4.6183	5.0201	6.3594	8.2989	10.655	12.388	12.551	13.267
		%	1.27	0.78	1.54	2.12	2.43	0.63	0.14	2.91
0.6	0.001	Liew	2.7342	3.4998	5.8305	9.8322	11.015	11.832	14.510	15.735
		MP	2.7778	3.4995	5.8303	9.8322	11.111	11.832	14.510	15.735
	0.100	Liew	2.6011	3.2590	5.2083	8.3506	9.3476	9.9040	11.744	12.564
		MP	2.6491	3.2953	5.3047	8.4916	9.4447	9.9547	11.843	12.716
	0.200	Liew	2.3178	2.8243	4.2593	6.3805	7.0589	7.3974	8.5104	8.9583
		MP	2.3612	2.8674	4.3674	6.5369	7.1273	7.4303	8.5883	9.1476

The good agreements of present results with existed results validate present work. The present solutions would be useful to the initial parametric design and analyses in practice.

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