Damping Matrix of the Constant Strain Tetrahedron

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Abstract : The objective of this analysis is to present an efficient method to compute the exact damping matrix of the constant strain tetrahedron finite element and the intervening damping coefficient of a steel material from the information of a specimen in tension.

Keywords : damping matrix, tetrahedron, constant strain, damping coefficient, critical damping coefficient, damping factor, tension, specimen.

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Mathematical formulations

Using barycentric coordinates of a regular arbitrary shaped tetrahedron Ω defined by its vertices' cartesian coordinates, we have:

$r_i = \frac{V_i}{V}$ $for\ i = 1\ to\ 4$,

$V = \sum_{i=1}^{4} V_i$ at a $P(x, y, z)$ point ∈ tetrahedron,

$V_1 = V_{P234}$, $V_2 = V_{P134}$, $V_3 = V_{P124}$ and $V_4 = V_{P123}$.

The degrees of freedom are $\{d\}_{(12\times1)} = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}_{(12\times1)}$

and the displacements inside this element are given with

$u = \sum_{i=1}^{4} N_i u_i = \sum_{i=1}^{4} r_i u_i$,

$v = \sum_{i=1}^{4} r_i v_i$,

$w = \sum_{i=1}^{4} r_i w_i$ at the corresponding $P(x, y, z)$ point.

Therefore, the displacements of P become

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(3\times1)} = [N]_{(3\times12)}\{d\}_{(12\times1)}$$

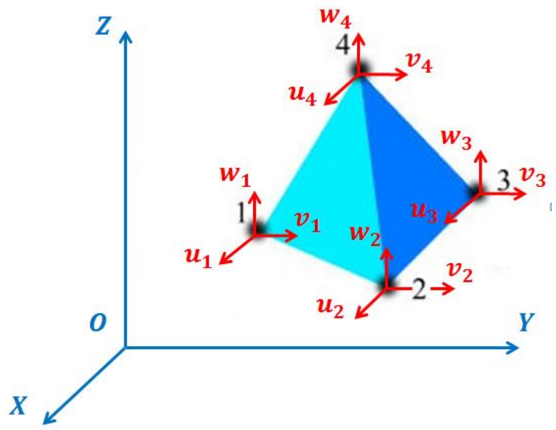

Figure 1: Nodal displacements in cartesian coordinates of a tetrahedron

The tetrahedron degrees of freedom shown in Fig. 1 are

$\{d\}_{(12\times1)} = \lfloor u_1 \quad v_1 \quad w_1 \quad u_2 \quad v_2 \quad w_2 \quad u_3 \quad v_3 \quad w_3 \quad u_4 \quad v_4 \quad w_4 \rfloor^T$

$[N]$ is the matrix of the shape functions

$$[N]_{(6\times12)} = \begin{bmatrix} r_1 & 0 & 0 & r_2 & 0 & 0 & r_3 & 0 & 0 & r_4 & 0 & 0 \\ 0 & r_1 & 0 & 0 & r_2 & 0 & 0 & r_3 & 0 & 0 & r_4 & 0 \\ 0 & 0 & r_1 & 0 & 0 & r_2 & 0 & 0 & r_3 & 0 & 0 & r_4 \end{bmatrix}$$

If we consider that $\begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$

We get the r_i coordinates without using the constraint equation $\sum_{i=1}^{4} r_i = 1$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \\ (y_1 - y_4) & (y_2 - y_4) & (y_3 - y_4) \\ (z_1 - z_4) & (z_2 - z_4) & (z_3 - z_4) \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} + \begin{Bmatrix} x_4 \\ y_4 \\ z_4 \end{Bmatrix}$$

Such that $r_4 = 1 - r_1 - r_2 - r_3$

Then $$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{bmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \\ (y_1 - y_4) & (y_2 - y_4) & (y_3 - y_4) \\ (z_1 - z_4) & (z_2 - z_4) & (z_3 - z_4) \end{bmatrix} \begin{Bmatrix} dr_1 \\ dr_2 \\ dr_3 \end{Bmatrix} = [J] \begin{Bmatrix} dr_1 \\ dr_2 \\ dr_3 \end{Bmatrix}$$

Where $|J|$ is the $Jacobian$ determinant

And $dV = abs(|J|)\, dr_1 dr_2 dr_3$

$V^e = \int_\Omega dV$ is the volume of the tetrahedron finite element

The volume of the 4 vertices tetrahedron finite element is

$$V^e = \frac{1}{6} abs\left(\begin{vmatrix} (x_2 - x_1) & (x_3 - x_2) & (x_4 - x_3) \\ (y_2 - y_1) & (y_3 - y_2) & (y_4 - y_3) \\ (z_2 - z_1) & (z_3 - z_2) & (z_4 - z_3) \end{vmatrix}\right)$$

And $abs(|\,J\,|) = 6\,V^e$

Such that $\int_0^1 \int_0^{1-r_1} \int_0^{1-r_1-r_2} dr_1 dr_2 dr_3 = \frac{1}{6}$

We have then $\mathrm{d}V = abs(|\,J\,|)\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}r_3$. Using the plane's equation as shown in [1,2,3], one can solve the four following sets of equations:

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} \begin{Bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \qquad i = 1 \; to \; 4$$

such that

$$\begin{aligned} &\alpha_1 = 1 \;\; and \;\; \alpha_2 = \alpha_3 = \alpha_4 = 0 \;\; for \;\; i = 1 \\ &\alpha_2 = 1 \; and \;\; \alpha_1 = \alpha_3 = \alpha_4 = 0 \;\; for \;\; i = 2 \\ &\alpha_3 = 1 \; and \;\; \alpha_1 = \alpha_2 = \alpha_4 = 0 \;\; for \;\; i = 3 \\ &\alpha_4 = 1 \; and \;\; \alpha_1 = \alpha_2 = \alpha_3 = 0 \;\; for \;\; i = 4 \end{aligned}$$

The Gauss LU method with partial pivoting allows to obtain the barycentric coordinates versus cartesian coordinates with analytical geometry and without inverting the matrix of the initial sets of equations. This is an original new and efficient method for determining the shape functions of the constant strain tetrahedron. Here, the inversion is not numerically recommended because it is a source of roundoff errors and ill-conditioning problems for arbitrary shaped elements.

The strains inside the tetrahedron are given by:

$$\{\varepsilon\} = \lfloor \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}, \varepsilon_{zx}, \varepsilon_{xy} \rfloor^T = [B]_{(6\times12)}\{d\}_{(12\times1)}$$

$$[B]_{(6\times12)} = \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\ 0 & \frac{c_1}{2} & \frac{b_1}{2} & 0 & \frac{c_2}{2} & \frac{b_2}{2} & 0 & \frac{c_3}{2} & \frac{b_3}{2} & 0 & \frac{c_4}{2} & \frac{b_4}{2} \\ \frac{c_1}{2} & 0 & \frac{a_1}{2} & \frac{c_2}{2} & 0 & \frac{a_2}{2} & \frac{c_3}{2} & 0 & \frac{a_3}{2} & \frac{c_4}{2} & 0 & \frac{a_4}{2} \\ \frac{b_1}{2} & \frac{a_1}{2} & 0 & \frac{b_2}{2} & \frac{a_2}{2} & 0 & \frac{b_3}{2} & \frac{a_3}{2} & 0 & \frac{b_4}{2} & \frac{a_4}{2} & 0 \end{bmatrix}$$

Where $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ in linear elasticity

And : $\frac{\partial r_i}{\partial x} = a_i$, $\frac{\partial r_i}{\partial y} = b_i$, $\frac{\partial r_i}{\partial z} = c_i$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial u}{\partial r_2}\frac{\partial r_2}{\partial x} + \frac{\partial u}{\partial r_3}\frac{\partial r_3}{\partial x} + \frac{\partial u}{\partial r_4}\frac{\partial r_4}{\partial x} = \sum_{i=1}^{4}(u_i a_i)$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial v}{\partial r_2}\frac{\partial r_2}{\partial y} + \frac{\partial v}{\partial r_3}\frac{\partial r_3}{\partial y} + \frac{\partial v}{\partial r_4}\frac{\partial r_4}{\partial y} = \sum_{i=1}^{4}(v_i b_i)$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \frac{\partial w}{\partial r_1}\frac{\partial r_1}{\partial z} + \frac{\partial w}{\partial r_2}\frac{\partial r_2}{\partial z} + \frac{\partial w}{\partial r_3}\frac{\partial r_3}{\partial z} + \frac{\partial w}{\partial r_4}\frac{\partial r_4}{\partial z} = \sum_{i=1}^{4}(w_i c_i)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\sum_{i=1}^{4}(v_i a_i + u_i b_i)$$
$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{1}{2}\sum_{i=1}^{4}(w_i b_i + v_i c_i)$$
$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\sum_{i=1}^{4}(u_i c_i + w_i a_i)$$

The stiffness matrix of the tetrahedron finite element is

$$[K^{e}]_{(12\times12)} = \int_{\Omega} [B]^T[D][B]\, dV = [B]^T[D][B]\, V^e$$

Where $[B(r_1, r_2, r_3)]$ the derivatives shape functions is constant.

The mass matrix of the tetrahedron finite element is

$$[M^{e}]_{(12\times12)} = \int_{\Omega} \rho\, [N]^T[N]\, dV = \rho.abs(|\,J\,|) \int_{\Omega} [N]^T[N]\ dr_1 dr_2 dr_3$$

Where ρ the mass per unit volume

The damping matrix of the tetrahedron finite element is

$$[C^{e}]_{(12\times12)} = \int_{\Omega} \xi\, [N]^T[N]\, dV$$

Where ξ is the damping coefficient of the structure per unit volume

We get

$$\int_0^1\int_0^{1-r_1}\int_0^{1-r_1-r_2} \xi.abs(|\,J\,|)\, r_1^2\ dr_1 dr_2 dr_3 = \frac{\xi\, V^e}{10}$$
$$\int_0^1\int_0^{1-r_1}\int_0^{1-r_1-r_2} \xi.abs(|\,J\,|)\, r_1 r_2\ dr_1 dr_2 dr_3 = \frac{\xi\, V^e}{20}$$

And

$$\int_0^1\int_0^{1-r_1}\int_0^{1-r_1-r_2} \xi.abs(|\,J\,|)\, r_4^2\ dr_1 dr_2 dr_3$$
$$= \int_0^1\int_0^{1-r_1}\int_0^{1-r_1-r_2} \xi.abs(|\,J\,|)\,(1 - r_1 - r_2 - r_3)^2\ dr_1 dr_2 dr_3 = \frac{\xi\, V^e}{20}$$

The damping matrix of this element is then

$$[C^{e}] = \xi . abs(|J|) *$$

$$\int_{\Omega} \begin{bmatrix} r_1^2 & 0 & 0 & r_1r_2 & 0 & 0 & r_1r_3 & 0 & 0 & r_1r_4 & 0 & 0 \\ 0 & r_1^2 & 0 & 0 & r_1r_2 & 0 & 0 & r_1r_3 & 0 & 0 & r_1r_4 & 0 \\ 0 & 0 & r_1^2 & 0 & 0 & r_1r_2 & 0 & 0 & r_1r_3 & 0 & 0 & r_1r_4 \\ r_1r_2 & 0 & 0 & r_2^2 & 0 & 0 & r_2r_3 & 0 & 0 & r_2r_4 & 0 & 0 \\ 0 & r_1r_2 & 0 & 0 & r_2^2 & 0 & 0 & r_2r_3 & 0 & 0 & r_2r_4 & 0 \\ 0 & 0 & r_1r_2 & 0 & 0 & r_2^2 & 0 & 0 & r_2r_3 & 0 & 0 & r_2r_4 \\ r_1r_3 & 0 & 0 & r_2r_3 & 0 & 0 & r_3^2 & 0 & 0 & r_3r_4 & 0 & 0 \\ 0 & r_1r_3 & 0 & 0 & r_2r_3 & 0 & 0 & r_3^2 & 0 & 0 & r_3r_4 & 0 \\ 0 & 0 & r_1r_3 & 0 & 0 & r_2r_3 & 0 & 0 & r_3^2 & 0 & 0 & r_3r_4 \\ r_1r_4 & 0 & 0 & r_2r_4 & 0 & 0 & r_3r_4 & 0 & 0 & r_4^2 & 0 & 0 \\ 0 & r_1r_4 & 0 & 0 & r_2r_4 & 0 & 0 & r_3r_4 & 0 & 0 & r_4^2 & 0 \\ 0 & 0 & r_1r_4 & 0 & 0 & r_2r_4 & 0 & 0 & r_3r_4 & 0 & 0 & r_4^2 \end{bmatrix} dr_1 dr_2 dr_3$$

The exact damping matrix is then

$$[C^{e}] = \begin{bmatrix} \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 \\ 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 \\ 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} \\ \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 \\ 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 \\ 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} \\ \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 \\ 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} & 0 \\ 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 & \frac{\xi V^e}{20} \\ \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 & 0 \\ 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} & 0 \\ 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{20} & 0 & 0 & \frac{\xi V^e}{10} \end{bmatrix}$$

I

If the the damping factor is $\beta = \frac{\xi}{\xi_{cr}}$

- $\beta = 0.04$ for Bolted and Riveted Steel
- $\beta = 0.05$ for Reinforced-Concrete

ξ : is the damping coefficient per unit volume

ξ_{cr} : is the critical damping coefficient per unit volume

$\psi = \xi.V$: is the damping coefficient

ψ_{cr} : is the critical damping coefficient

Numerical application

In practice, the steel specimen in tension allows to calculate the critical damping coefficient of steel with its , E , A and L characteristics such that

$$d = 0.01\ m$$
$$A = \frac{\pi d^2}{4}$$
$$E = 2.1 * 10^{11}\ N/m^2$$
$$\rho = 7850\ kg/m^3$$

The stiffness matrix of a bar element is given in general with :

$$[K] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

If this element is fixed at one end, its stiffness is then :

$$k = \frac{E.A}{L}$$
$$m = \rho.A.L$$

For a one degree of freedom case, the critical damping coefficient is calculated with

$\psi_{cr} = 2\sqrt{k.m} = 2\sqrt{\rho.E.A^2} = 6377.719\ N.s/m$ which is also in kg/s

from its $1D$ equation : $m.\ddot{x} + \psi.\dot{x} + k.x = 0$

Then the damping coefficient becomes

$$\psi = \beta.\psi_{cr} = 255.109\ N.s/m$$

The crucial problem is how to compare a $1D$ steel specimen with a $3D$ steel block specimen.
If one uses the developed damping matrix, one finds that there are several coefficients for each direction and the off-diagonal coupled ones between two directions.
We need to keep the volume characteristics of the $3D$ element such that we assume that only the u_1 degree of freedom exist and all the other degrees of freedom are fixed.
Then we get the damping force in the motion equation with $c_{11}.\dot{u}_1$

Assuming that the c_{11} value is equivalent to a $10\,mm$ steel specimen value and this represents the same behavior, we use $\psi = 255.109\,N.s/m$

We can then bound the specimen thicknesses with selecting the desired thickness.

We obtain $c_{11} = \frac{\xi\, V^e}{10}$

Let be the tetrahedron defined with the following $1, 2, 3$ and 4 vertices :
$(0, 0, 0)$, $(0.01, 0, 0)$,$(0, 0.01, 0)$ and $(0, 0, 0.01)$ then the volume of a cube is

$$6\,V^e = abs\left(\begin{vmatrix} 0.01 & -0.01 & 0 \\ 0 & 0.01 & -0.01 \\ 0 & 0 & 0.01 \end{vmatrix}\right) = (0.01)^3 = 10^{-6}\,m^3$$

The volume of a $10\,mm$ sided tetrahedron is $\frac{10^{-6}}{6}$

One has ξ for $1\,m^3$ and ψ for $V^e\ m^3$

Then $\psi = \xi\, V^e$

And $c_{11} = \frac{\xi\, V^e}{10} = \frac{\psi}{10} = 25.511\,N.s/m$

The finite element user is advised to use the same selected size element in his mesh data computations.

The damping matrix of this specific tetrahedron becomes

$[C^{\,e}] =$

$$\begin{bmatrix}
25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 \\
0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 \\
0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 \\
12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 \\
0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 \\
0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 & 12.755 \\
12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 & 0 \\
0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 & 0 \\
0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 & 12.755 \\
12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 & 0 \\
0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511 & 0 \\
0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 12.755 & 0 & 0 & 25.511
\end{bmatrix}$$

Conclusion

This analysis allowed to present an efficient method to compute the exact damping matrix of the four nodes constant strain tetrahedron finite element. We reported that the damping coefficient is an important parameter in this matrix. We shown that the application of a specimen in tension allowed to get the numerical value of the steel material critical damping coefficient.

References

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