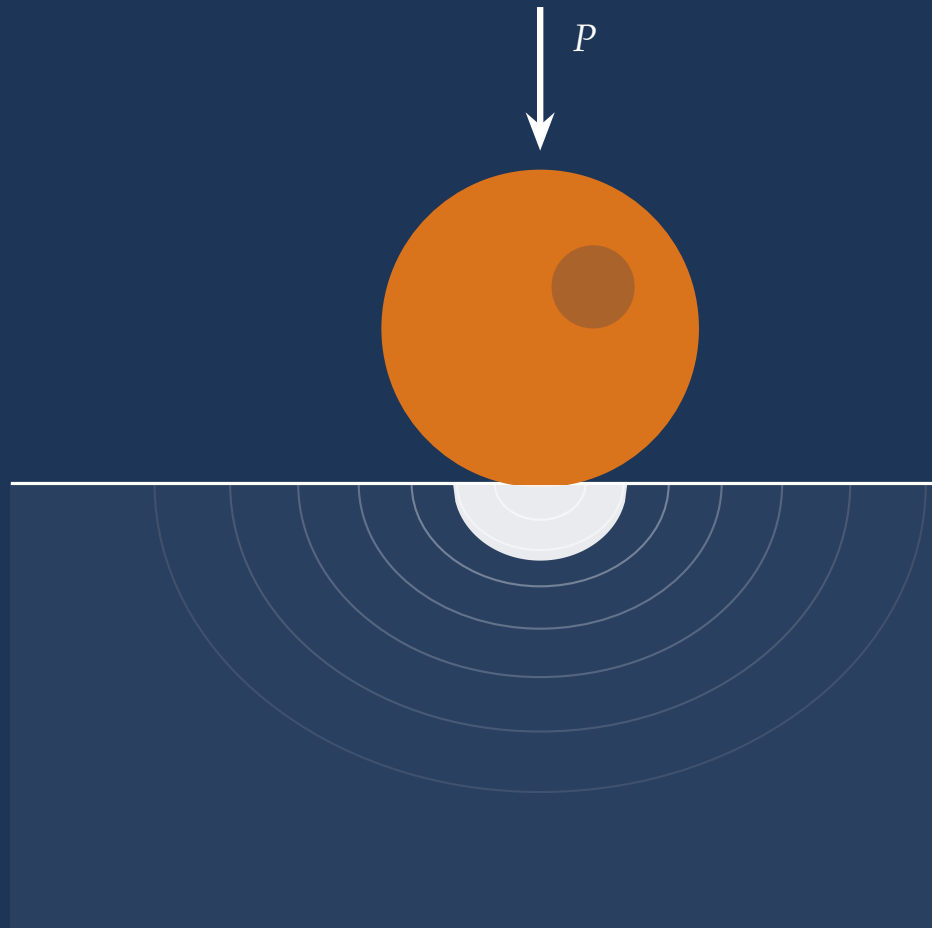


Contact Mechanics for Beginners



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Preface

Contact mechanics is well served by excellent books. Johnson's *Contact Mechanics* [1] remains the reference of the field; Barber's *Contact Mechanics* [2], Popov's *Contact Mechanics and Friction* [3], and Hills, Nowell and Sackfield's *Mechanics of Elastic Contacts* [4] cover the subject with rigor and depth. These books were written mainly for graduate students, researchers and specialists. They assume a confident command of elasticity theory, complex variables, integral equations and special functions, and an undergraduate student who opens them for the first time often finds the mathematics, rather than the physics, to be the main obstacle.

This book is meant to fill that gap. It is written for engineering students at the undergraduate level, typically in the second or third year of a Bachelor's degree, who have taken a first course in mechanics of solids and calculus, but not yet advanced elasticity. The mathematics has been kept deliberately to an elementary minimum: superposition of simple solutions, dimensional arguments, energy balances and a few closed-form results that can be checked by hand. Where a full derivation would require tools beyond this level, we state the result, explain where it comes from and show how to use it, rather than prove it. In exchange, we try to explain clearly why contacts behave as they do, and how simple estimates can be trusted, or not.

Three ideas recur throughout the book: superposition, the analogy between the edge of a contact and the tip of a crack, and the joining of two known limits by the simplest consistent formula. With these, the reader can go from Hertzian contact to partial slip, rough surfaces, adhesion, viscoelasticity, friction, fretting, rolling, wear and friction-induced vibrations without losing the thread. Each chapter includes short boxes stating the physical idea, worked examples with realistic numbers, and figures computed from the equations in the text; a formula sheet and a table of typical material data are collected in the appendices.

Because the mathematical prerequisites are modest, we hope the book will also reach a wider audience than the existing texts: students of mechanical, civil, materials and biomedical engineering, physicists and engineers in industry who need to estimate a contact pressure, a friction force or a fatigue risk, and anyone who wishes to approach the specialist literature with a sound physical picture first. Readers who want to go further will find in the books cited above the rigorous developments that are only sketched here.

This text grew out of lecture notes for courses taught at the Politecnico di Bari. We are grateful to the students whose questions shaped it, and we will welcome comments and corrections from readers.

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Introduction

Every machine is held together, and does its work, through surfaces that touch. A shaft transmits torque to a gear through a press fit, a ball carries the load of a bearing through a spot smaller than a pin head, a brake stops a car through a pad pressed on a disc. The strength, the life and the efficiency of all these parts are decided in a very thin layer near the contact. This chapter explains what the subject is about, gives the orders of magnitude, and shows how far a simple estimate can take us before any theory is needed.

In this chapter you will learn

- what contact mechanics and tribology study, and why contact problems are nonlinear;
- how to classify a contact (conforming, non-conforming, rough) and why it matters;
- the typical sizes, pressures and forces in real machine contacts;
- how to make an engineering estimate within a few tens of percent;

1.1 What happens when two solids touch

Press a ball on a flat. The load P cannot pass through a single point, so both bodies deform and touch over a small *contact area*. *Contact mechanics* answers three questions: how large is this area, how is the pressure distributed inside it, and how large are the stresses in the material below it. When the bodies also slide or roll, new questions appear: what is the friction force, how much material is lost by wear, how much heat is produced, can a lubricant keep the surfaces apart? This wider field is called *tribology*.

A first, surprising feature is that a contact problem is *nonlinear even when the material is linear elastic*. In a normal strength-of-materials problem we know where the load is applied. In a contact problem the region where the load is transmitted is itself an unknown, and it changes with the load: double the force on the ball and the contact area grows, but not by a factor of two. Two simple inequalities rule every contact: the bodies cannot overlap (the gap $g \geq 0$), and, without adhesion, they can only push on each other (the contact pressure $p \geq 0$). Where $g > 0$ we must have $p = 0$, and where $p > 0$ we must have $g = 0$.

Physical idea

The boundary of the contact is unknown. Finding it is the core of most problems in this book, and the reason why contact mechanics needs tools of its own.

Table 1.1 lists some contacts met in mechanical design. Note the enormous spread: the contact width goes from tens of nanometers to tens of centimeters, the pressure from a fraction of a megapascal to several gigapascals.

1.2 Types of contact

The geometry before loading decides which tools we need (Fig. 1.1).

Table 1.1: Typical contacts in machines (orders of magnitude only).

Contact	Size of contact	Peak pressure	Main concern
Ball on race (rolling bearing)	0.1 mm to 1 mm	1 GPa to 3 GPa	rolling contact fatigue
Spur gear teeth	0.2 mm to 1 mm wide	0.5 GPa to 1.5 GPa	pitting, scuffing
Wheel on rail	10 mm to 15 mm	0.8 GPa to 1.5 GPa	wear, cracks, corrugation
Press fit, bolted flange	whole surface	20 MPa to 200 MPa	fretting fatigue
Brake pad on disc	50 × 100 mm	1 MPa to 5 MPa	heat, wear, squeal
Tire on road	150 × 200 mm	0.2 MPa to 0.4 MPa	grip, rolling loss
Lip seal on shaft	0.1 mm to 0.3 mm wide	0.5 MPa to 2 MPa	leakage, wear
MEMS switch, probe tip	10 nm to 100 nm	GPa	adhesion (stiction)

Non-conforming contacts. A ball on a race or a wheel on a rail first touch at a point or along a line. Under load the contact stays small compared with the size of the bodies, so near the contact each body behaves like an infinite elastic *half-space*. The size of the contact grows with the load and is an unknown. This is the world of Hertz (Chapters 5 and 6).

Conforming and complete contacts. A shaft in a hub, a flat pad on a flat, a blade root in its slot: the extent of the contact is fixed mainly by the shape, not by the load. If the edges are sharp, the pressure grows without bound near them, as at the tip of a crack. These contacts are the typical site of fretting damage (Chapters 7 and 16).

Rough contacts. Real surfaces are never smooth. At the micrometer scale they touch only at the tips of the highest asperities, so the *real* contact area A is a tiny fraction of the *nominal* area A_0 . Friction, wear, adhesion and the flow of heat and current all depend on A (Chapter 8).

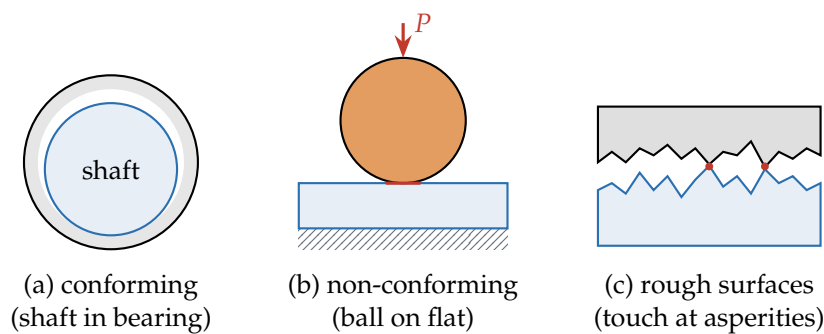


Figure 1.1: Three types of contact. (a) Conforming: the surfaces fit before loading. (b) Non-conforming: first touch at a point, the contact grows with P . (c) Rough: touch only at the highest asperities (red dots).

1.3 One interface, many scales

A tribological interface must be looked at on several scales (Fig. 1.2). At the *component* scale, from millimeters to meters, we see loads, shapes and nominal pressures; the finite element method is the usual tool. At the *asperity* scale, micrometers, the nominal pressure is carried by many small spots; here half-space methods and statistical models work best (Chapters 5 and 8). At the *molecular* scale, nanometers, bonds form and break, and this is where friction and adhesion are born (Chapter 4). Each scale passes its state to the smaller one (the local pressure, the sliding speed, the temperature) and receives back an *interface law*: a contact stiffness, a real

area, a friction coefficient, a wear rate. When the scales are well separated this chain is enough. Roughness, which covers many decades of length without a clear gap, is the main exception, and the reason why it gets a full chapter.

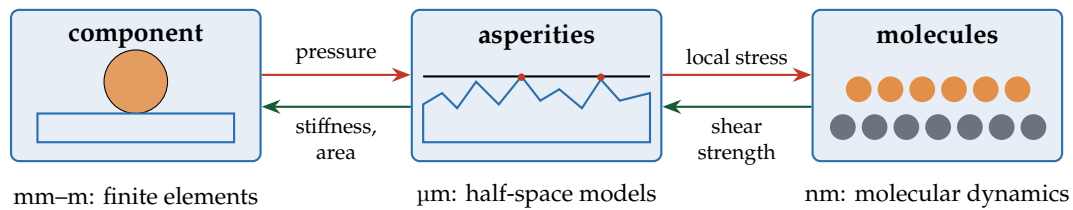


Figure 1.2: The scales of a tribological interface. Red arrows pass the state down to the smaller scale; green arrows pass interface laws back up.

1.4 Engineering estimates

Exact solutions in contact mechanics are few, and real data (loads, roughness, friction) are rarely known better than to 20 or 30 %. The skill we want to build is the *estimate*: a few lines of calculation that give the right order of magnitude and show which parameter matters. Three habits help.

1. **Find the dominant effect.** A part usually sees several loads at once, but one of them almost always dominates by a large factor. A rough estimate of each is enough to rank them; only the winner deserves a careful calculation.
2. **Look at the exponents.** Contact results are power laws. If the peak pressure grows as $P^{1/3}$, doubling the load raises it only by $2^{1/3} - 1 = 26\%$; to double the pressure you need eight times the load.
3. **Use dimensions.** The answer must be a combination of the data with the right units. Often this fixes the formula up to a number of order one.

Worked example: the footprint of a tire

A car wheel carries $P = 4 \text{ kN}$ and the tire is inflated to $p_i = 0.22 \text{ MPa}$. The carcass is thin and flexible, so the pressure on the road is close to p_i . Equilibrium alone gives the footprint area

$$A \approx \frac{P}{p_i} = \frac{4000}{0.22 \times 10^6} = 0.018 \text{ m}^2 \approx 180 \text{ cm}^2,$$

for instance a patch of about $12 \times 15 \text{ cm}$. No elasticity was needed: the stiffness that matters is that of the air.

Worked example: the real area under a steel vise

A 10 kg steel vise rests on a steel bench; the nominal area is $100 \times 150 \text{ mm}$. The asperity tips yield, so each spot carries a pressure close to the hardness $H_w \approx 2 \text{ GPa}$ (Chapter 6). The real area is

$$A \approx \frac{P}{H_w} = \frac{98.1}{2 \times 10^9} = 4.9 \times 10^{-8} \text{ m}^2 = 0.049 \text{ mm}^2, \quad \frac{A}{A_0} = \frac{0.049}{15000} \approx 3 \times 10^{-6}.$$

Only three parts per million of the surface are in contact. This explains why the friction force does not depend on the nominal area (Chapter 12).

1.5 Map of the book

The book has five parts (Fig. 1.3). *Part I* collects the foundations: solid mechanics (Chapter 2), fracture (Chapter 3) and surface forces (Chapter 4). *Part II* solves the elastic contact of smooth bodies. *Part III* adds friction, roughness and adhesion. *Part IV* treats friction, lubrication, heat and wear, and *Part V* the machine elements and their dynamics. A closing chapter (Chapter 18) collects the simple models and their limits. Each chapter ends with problems; the answers are in Chapter D.

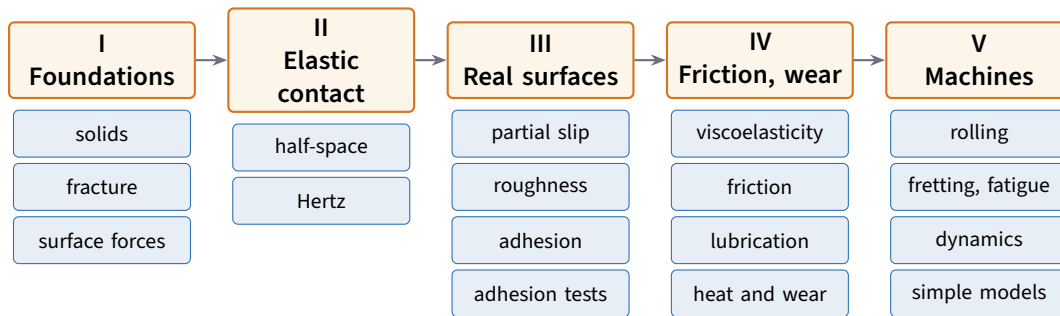


Figure 1.3: Map of the book: the five parts and their chapters, in reading order.

Summary

- Contact mechanics finds the contact size, the pressure and the stresses; tribology adds friction, wear, heat and lubrication.
- The contact region is unknown in advance ($g \geq 0$, $p \geq 0$, $gp = 0$), so contact problems are nonlinear even for linear elastic solids.
- Non-conforming contacts are small (half-space model); complete contacts have crack-like edges; rough surfaces touch over a real area $A \ll A_0$, roughly $A \approx P/H_w$.
- Interfaces are multiscale: each scale passes the state down and an interface law up.
- Estimates: find the dominant effect, read the exponents, check the dimensions.
- Metal contacts are fractions of a millimeter wide, with peak pressures of order GPa.

Problems

- 1.1 A truck wheel carries 30 kN and its tire is inflated to 0.8 MPa. Estimate the footprint area.
- 1.2 A 5 kg aluminum-alloy block (hardness $H_w \approx 1$ GPa) rests on a steel table. Estimate the real contact area. Does it change if the block is turned onto a smaller face?
- 1.3 In a Hertz contact the contact radius and the peak pressure both grow as $P^{1/3}$. By how much does the contact radius grow if the load rises by 50 %? By what factor must the load grow to double the peak pressure?
- 1.4 A glass dust particle of radius $10 \mu\text{m}$ (density 2500 kg/m^3) sticks to a wall. Its pull-off force is about $\frac{3}{2}\pi wR$ with $w = 0.1 \text{ J/m}^2$. Compare it with the weight of the particle.

- 1.5 A gear of radius 0.1 m turns at 6000 rpm. Estimate the stress due to rotation, $\sigma \approx \rho \omega^2 r^2$ (steel, $\rho = 7800 \text{ kg/m}^3$), and compare it with a tooth contact pressure of 1 GPa. Which effect dominates?
- 1.6 The contact of a bolted joint has a normal stiffness of 50 N/ μm and the bolt and plates together 200 N/ μm . The two act as springs in series, so $1/k = 1/k_1 + 1/k_2$. Find the total stiffness k . Which part controls it?
- 1.7 * The contact radius a of a sphere on a flat can depend only on P , R and E^* . Show by dimensional analysis that $a = R \phi(P/(E^* R^2))$. If experiments show $a \propto P^{1/3}$, show that $a \propto (PR/E^*)^{1/3}$.
- 1.8 (Open) A sliding door runs on four nylon rollers along a steel track. List the questions that contact mechanics and tribology should answer in its design, and say for each one which part of the book deals with it.