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Part BGeneral Continuum Mechanics

In this part, we develop those aspects of mechanics that hold for all continuous bodies. For the most part, considerations appropriate to special bodies (elastic bodies, linearly viscous fluids, etc.) are treated in later chapters as are notions outside the realm of pure mechanics (heat flux, temperature, etc.) For discussion of the concept of continuous body, the reader is referred to Part A of the treatise by TRUESDALL and TOUPIN listed in the Supplementary Reading for § B1.1.

Office Hrs - Fall 1991.

Wed 1-3

Thurs 1-2.

TL-212.

Chapter B1

Kinematics

Here we consider those aspects of kinematics (i.e., considerations of space and time with regard to the deformations and motions of bodies) that are essential to an understanding of continuum mechanics. There are additional topics which are important to particular branches of continuum mechanics (elasticity, hydrodynamics, etc.) that are left for courses on those subjects.

B1.1. Deformations

As a preliminary step to the formal introduction of the notion of body, we observe that an obvious property of a body is that it is made up of material that occupies a region of physical space, which we model as the three dimensional Euclidean point-space E_3 .⁰ Furthermore, when a body is acted on by forces, then it is deformed into another region of E_3 . This leads us to

Definition B1.1.1. A deformation $\mathbb{F}$ is a smooth, one-to-one mapping of a compact regular region $R \subset E_3$ onto a compact regular region $\mathbb{F}(R) \subset E_3$ that meets the additional requirement that

$$\det D\mathbb{F}(X) \neq 0 \quad \forall X \in R,$$

Some comments on this definition are in order. First of all, the domain R is required to be a compact regular region. As noted earlier, we use region as a synonym for pathwise connected point set.¹ In particular, our bodies will be "in one piece"; and since $\mathbb{F}(R)$ is also a region, they stay in one piece. Thus, unless it is suitably modified later, our model will not allow for the complete fracturing of a body. As always, we use regular in the sense of Kellogg.² This allows for the regions of E_3 occupied by bodies to be quite general;

¹ See Definition A6.3.10.

² See Definition A6.10.1.

⁰ The translation space of E_3 is denoted by V_3 .

their boundaries can have edges, corners, and vertices. However, they are sufficiently special to permit application of the divergence theorem. Since the underlying space is E_3 , compact simply means closed and bounded. Generally, we only have access to a body at the boundary of the region that it occupies; thus, it is appropriate to include the boundary in the domain of definition of a deformation by requiring that the domain be closed. Bodies are always of finite extent and necessarily occupy bounded regions. Occasionally in practice it is a useful fiction to relax this requirement, but then special technical assumptions about the behavior of the deformation are needed.

The assumption that a deformation $\#$ is one-to-one is made so that two distinct points of $\mathcal{Q}$ cannot be deformed into the same place in $\#(\mathcal{Q})$; i.e., so that the body cannot penetrate itself. In the same vein, that a deformation is a function means that a ^(single) point cannot be deformed into more than one point. Thus, without some adjustment, tearing, cracking, cleaving, splitting, cavitating, etc. are ruled out of the present theory.

To assume that a deformation is smooth¹ permits us to use many of the standard theorems of advanced calculus in our study. In particular, the scalar field $\det D\#$ is well-defined. The requirement that $\det D\#(\mathbb{X}) \neq 0 \quad \forall \mathbb{X} \in \mathcal{Q}$ is made because,

¹I.e., $\# \in C^1(\mathcal{Q}, E_3)$; cf. Definition A5.10.2.

as will be made evident later, $|\det D\#(\mathbb{X})|$ is a local measure of the ratio of deformed volume to original volume;¹ and we wish to rule out the possibility of any subregion of $\mathbb{R}$ being so severely deformed that its image has zero volume. Many expositors require that

$$\det D\#(\mathbb{X}) > 0 \quad \forall \mathbb{X} \in \mathbb{R}.$$

There is a good reason for insisting on this, which will be easier to make clear later in our consideration of motions if we are less restrictive at this point.

As usual,² to say that $\# \in C^1(\mathbb{Q}, \mathbb{E}_3)$ means that $\# \in C^0(\mathbb{Q}, \mathbb{E}_3)$, $\# \in C^1(\mathring{\mathbb{Q}}, \mathbb{E}_3)$, and the derivative $D\#$ has a continuous extension from $\mathring{\mathbb{Q}}$ to $\mathbb{Q}$.

There are, of course, several mathematical consequences of the above definition of deformations that are important to the

¹G. the introduction to the definition of the determinant of a second-order tensor (Theorem A3.8.1),

²G. the footnote on p. A6.10.2.

physical (here merely geometrical) interpretation of deformations.
We gather some of these together as

Theorem B1.1.1. Let $f: R \rightarrow f(R)$ be a deformation
of a compact regular region $R \subset E_3$. Then

- (i) The image of a regular arc in R is a regular arc in $f(R)$.
- (ii) " " " " surface " " " " surface " " .
- (iii) " " " " subregion " " " " subregion " " .

Here, regular is used in the sense of Kellogg.

(iv) The image of the interior of R is the interior of the
image of R ; i.e.,

$$f(\overset{\circ}{R}) = \overset{\circ}{f(R)},$$

(v) The image of the boundary of R is the boundary of the
image of R ; i.e.,

$$f(\partial R) = \partial f(R).$$

(vi) The inverse mapping f^{-1} is also smooth,¹ and

$$Df^{-1}(f(x)) = (Df(x))^{-1}.$$

¹Thus, deformations are diffeomorphisms.

We shall have repeated need of

Theorem B1.1.2. The composition of any two deformations is a deformation.

Proof. Let $\#_1$ be a deformation on $\mathbb{R}$ and $\#_2$ be a deformation on $\#_1(\mathbb{R})$, and consider the composition $\#_2 \circ \#_1 : \mathbb{R} \rightarrow \#_2(\#_1(\mathbb{R}))$. By the chain rule (Theorem A5.7.1), $\#_2 \circ \#_1$ is differentiable, and

$$D(\#_2 \circ \#_1)(x) = (D\#_2(\#_1(x))) (D\#_1(x)). \quad (*)$$

$\therefore$

$$\det(D\#_2 \circ \#_1)(x) = \det(D\#_2(\#_1(x))) \det(D\#_1(x)) \neq 0 \quad \forall x \in \mathbb{R}.$$

(*) also provides the fact that $\#_2 \circ \#_1 \in C^1(\mathbb{R}, \mathcal{E}_3)$. For this, it suffices to show that $D(\#_2 \circ \#_1) \in C^0(\mathbb{R}, \text{Lin } \mathcal{V}_3)$. We get this from the observation that $D(\#_2 \circ \#_1)(x)$ is simply the product

$$D(\#_2 \circ \#_1)(x) =: \underline{C}(x) = \underline{A}(x) \underline{B}(x),$$

where $\underline{A} : \mathbb{R} \rightarrow \text{Lin } \mathcal{V}_3$ and $\underline{B} : \mathbb{R} \rightarrow \text{Lin } \mathcal{V}_3$ are defined by

$$\underline{A}(x) = D\#_2(\#_1(x)) \quad \text{and} \quad \underline{B}(x) = D\#_1(x).$$

The function $\underline{B}$ is continuous since $\#_1 \in C^1(\mathbb{R}, \mathcal{E}_3) \Rightarrow D\#_1 \in C^0(\mathbb{R}, \text{Lin } \mathcal{V}_3)$. The continuity of $\underline{A}$ is a bit more difficult to come by. It is the composition of two continuous functions; $D\#_2$ is

continuous because $f_2 \in C^1(f_1(\mathbb{R}), \mathbb{E}_3)$. Now, it is a standard and easy result that the composition of continuous functions is continuous.¹ $\therefore$ the function $\underline{A}$ is continuous. Finally, the product function $\underline{C}$ defined on $\mathbb{R}$ by

$$\underline{C}(x) = \underline{A}(x) \underline{B}(x)$$

is continuous because the factor functions $\underline{A}$ and $\underline{B}$ are continuous. We can establish this by following the analysis used in elementary calculus to demonstrate the corresponding result for the product of real-valued functions. Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then

$$\begin{aligned} |\underline{C}(x) - \underline{C}(x_0)| &= |\underline{A}(x)\underline{B}(x) - \underline{A}(x_0)\underline{B}(x_0)| && \text{(substitution)} \\ &= |\underline{A}(x)\underline{B}(x) - \underline{A}(x_0)\underline{B}(x) + \underline{A}(x_0)\underline{B}(x) - \underline{A}(x_0)\underline{B}(x_0)| && \text{(add and subtract same term)} \\ &= |[\underline{A}(x) - \underline{A}(x_0)]\underline{B}(x) + \underline{A}(x_0)[\underline{B}(x) - \underline{B}(x_0)]| && \text{(distributivity)} \\ &\leq |[\underline{A}(x) - \underline{A}(x_0)]\underline{B}(x)| + |\underline{A}(x_0)[\underline{B}(x) - \underline{B}(x_0)]| && \text{(triangle inequality)} \\ &\leq |\underline{A}(x) - \underline{A}(x_0)| |\underline{B}(x)| + |\underline{A}(x_0)| |\underline{B}(x) - \underline{B}(x_0)| && \text{(inner product - magnitude inequality).} \\ &&& \text{Thm A3.7.9.} \end{aligned}$$

The only difficult term in the above expression is $|\underline{B}(x)|$; we need to show that it is bounded in some neighborhood of x_0 .

¹ See, e.g., p. 154 of BARTLE's The Elements of Real Analysis.

Since $\underline{B}$ is continuous at $\overset{\circ}{x}$, $\exists \delta_1 > 0 \ni$ for $x \in B_{\delta_1}(\overset{\circ}{x}) \cap \mathbb{R}$

$$|\underline{B}(x) - \underline{B}(\overset{\circ}{x})| < \varepsilon$$

$\Rightarrow$

$$||\underline{B}(x)| - |\underline{B}(\overset{\circ}{x})|| < \varepsilon \quad (\text{Theorem A2.3.10. (ii)}),$$

Thus, the continuity of $\underline{B} \Rightarrow$ the continuity of $|\underline{B}(\cdot)|$. The last inequality above is equivalent to

$$-\varepsilon < |\underline{B}(x)| - |\underline{B}(\overset{\circ}{x})| < \varepsilon \quad)^1$$

$\therefore$

$$|\underline{B}(x)| < \varepsilon + |\underline{B}(\overset{\circ}{x})| =: M \quad \forall x \in B_{\delta_1}(\overset{\circ}{x}) \cap \mathbb{R}.$$

Hence, $|\underline{B}(\cdot)|$ is bounded by $M > 0$ on $B_{\delta_1}(\overset{\circ}{x}) \cap \mathbb{R}$. Now we are in position to return to the inequality

$$|\underline{C}(x) - \underline{C}(\overset{\circ}{x})| \leq |\underline{A}(x) - \underline{A}(\overset{\circ}{x})| |\underline{B}(x)| + |\underline{A}(\overset{\circ}{x})| |\underline{B}(x) - \underline{B}(\overset{\circ}{x})|. \quad (*)$$

Since $\underline{A}$ is continuous at $\overset{\circ}{x}$, $\exists \delta_2 > 0 \ni$ for $x \in B_{\delta_2}(\overset{\circ}{x}) \cap \mathbb{R}$

$$|\underline{A}(x) - \underline{A}(\overset{\circ}{x})| < \frac{\varepsilon}{2M}.$$

Similarly, the continuity of $\underline{B}$ at $\overset{\circ}{x} \Rightarrow \exists \delta_3 > 0 \ni$

¹ Use the definition of absolute value to show as Exercise B1.1.1 that if $\alpha, \gamma \in \mathbb{R}$, then

$$|\alpha| < \gamma \iff -\gamma < \alpha < \gamma.$$

for $x \in B_{\delta_3}(x_0) \cap \mathbb{R}$

$$|\underline{B}(x) - \underline{B}(x_0)| < \frac{\varepsilon}{2|\underline{A}(x_0)|} \quad (*)$$

Placing these estimates back into (*), we see that if $\delta = \min \{\delta_1, \delta_2, \delta_3\}$, then

$$|\underline{C}(x) - \underline{C}(x_0)| < \varepsilon \quad \forall x \in B_{\delta}(x_0) \cap \mathbb{R};$$

i.e., the product function $\underline{C}$ is continuous at x_0 . Consequently,

$$f_2 \circ f_1 \in C'(\mathbb{R}, \mathbb{R}).$$

It only remains to show that $f_2 \circ f_1$ is one-to-one on $\mathbb{R}$. To this end suppose that for some $x, y \in \mathbb{R}$

$$(f_2 \circ f_1)(x) = (f_2 \circ f_1)(y)$$

$$\Rightarrow f_2(f_1(x)) = f_2(f_1(y)) \quad (\text{definition of composition})$$

$$\Rightarrow f_1(x) = f_1(y) \quad (f_2 \text{ is 1-1})$$

$$\Rightarrow x = y \quad (f_1 \text{ is 1-1}). \quad \square$$

¹In the event that $|\underline{A}(x_0)| = 0$, we see from (*) that this estimate is not needed.

Throughout, we shall use the notation

$$\mathbf{x} = \mathbf{f}(\mathbf{X}).$$

Thus, the deformation $\mathbf{f}$ carries the point $\mathbf{X} \in \mathcal{Q}$ into the point $\mathbf{x} \in \mathbf{f}(\mathcal{Q})$.⁹ The following two derived concepts are often useful.

Definition B1.1.2. Let $\mathbf{f}: \mathcal{Q} \rightarrow \mathcal{E}_3$ be a deformation. Then
the vector

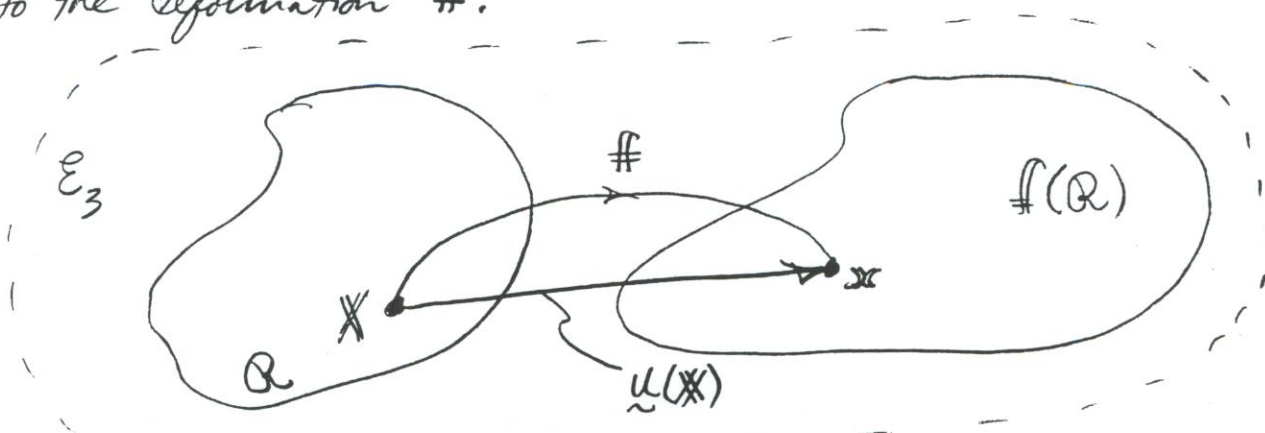
$$\underline{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X} = \mathbf{f}(\mathbf{X}) - \mathbf{X}$$

is the displacement of the point $\mathbf{X}$ under the deformation $\mathbf{f}$, and
the second-order tensor

$$\underline{F}(\mathbf{X}) = D\mathbf{f}(\mathbf{X})$$

is the deformation gradient at the point $\mathbf{X}$,

Of course, this defines the displacement field $\underline{u}: \mathcal{Q} \rightarrow \mathcal{V}_3$
 and the deformation gradient field $\underline{F}: \mathcal{Q} \rightarrow \text{Inv } \mathcal{V}_3$ corresponding
 to the deformation $\mathbf{f}$.



Actually, the concept of displacement does not play much of a role in the general theory; it is especially useful for "solid" bodies undergoing "small" deformations.

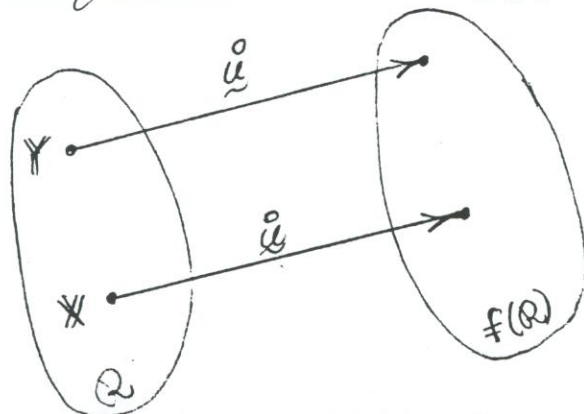
Unfortunately, the terminology deformation gradient is so entrenched in the continuum mechanics literature that it would be a disservice to the student not to use it here, even though it does not square with the usage of §A5.4. To make matters worse, this same literature almost always writes $\nabla \mathbf{f}$ for our $D\mathbf{f}$; on this we shall not acquiesce.

When each point of $\mathcal{Q}$ experiences the same displacement, we have an extremely special but still important situation, which we formalize in

Definition B1.1.3. Let $\underline{u}$ be the displacement field
corresponding to a deformation $\mathbf{f}: \mathcal{Q} \rightarrow \mathcal{E}_3$. If

$$\underline{u}(\mathbf{x}) = \underline{u}(\mathbf{y}) = \underline{\underline{u}} \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{Q},$$

then the deformation is a translation of amount $\underline{\underline{u}}$.



When each point of $\mathcal{Q}$ has the same deformation gradient, we get a larger and much more important class of deformations, called homogeneous deformations, to which § B1.4 is devoted.

Supplementary Reading

Linear
elasticity
non-linear elasticity

ATKIN and FOX, An Introduction to the Theory of Elasticity

BOWEN, Introduction to Continuum Mechanics for Engineers

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CIARLET, Mathematical Elasticity, Vol. I: Three-Dimensional Elasticity

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WANG, Mathematical Principles of Mechanics and Electro-
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B1.2. Rigid Deformations. The Extension Theorem

The word deformation suggests change of shape, and consequently "rigid deformation" sounds like a contradiction of terms. Technically, however, a deformation is a fairly general mapping, and we have

Definition B1.2.1. A deformation f on a region $\mathcal{Q}$ is rigid if it preserves distance; i.e.,

$$|f(X) - f(Y)| = |X - Y| \quad \forall X, Y \in \mathcal{Q}.$$

The following result provides two useful criteria that are each necessary and sufficient for a deformation to be rigid.

Theorem B1.2.1. Let f be a deformation on a region $\mathcal{Q}$. Then the following statements are equivalent:

- (i) f is rigid;
- (ii) f has the form

$$f(X) = f(\bar{X}) + Q(X - \bar{X}), \quad X \in \mathcal{Q},$$

where $\bar{X}$ is an arbitrary point of $\mathcal{Q}$ and Q is a uniform¹ orthogonal

¹Generally, we shall use the adjective uniform to mean "independent of position" and save constant for "independent of time".

tensor on $\mathcal{V}_3$; ⁰

$$(iii) \quad \underline{\underline{C}}(\underline{\underline{X}}) := \underline{\underline{F}}^T(\underline{\underline{X}}) \underline{\underline{F}}(\underline{\underline{X}}) = \underline{\underline{I}} \quad \forall \underline{\underline{X}} \in \mathcal{Q},$$

where $\underline{\underline{F}}(\underline{\underline{X}}) = D\#(\underline{\underline{X}})$.

Proof. (i) $\Rightarrow$ (ii): ¹ Suppose that $\#$ is rigid. Then we have the identity

$$[\#(\underline{\underline{X}}) - \#(\underline{\underline{Y}})] \cdot [\#(\underline{\underline{X}}) - \#(\underline{\underline{Y}})] = (\underline{\underline{X}} - \underline{\underline{Y}}) \cdot (\underline{\underline{X}} - \underline{\underline{Y}}). \quad (*)$$

For any fixed $\underline{\underline{X}} \in \mathcal{Q}$, restrict $\underline{\underline{Y}}$ to the interior $\mathring{\mathcal{Q}}$, and differentiate (*) w.r.t. $\underline{\underline{Y}}$ to get

$$-2(D\#(\underline{\underline{Y}})[\underline{\underline{u}}]) \cdot [\#(\underline{\underline{X}}) - \#(\underline{\underline{Y}})] = -2\underline{\underline{u}} \cdot (\underline{\underline{X}} - \underline{\underline{Y}})$$

(and Theorem A5.5.4 (ii))

by the product rule Theorem A5.6.5 (i). But $D\#(\underline{\underline{Y}}) \in \text{Lin } \mathcal{V}_3$ has a transpose by the finite dimensionality, and $\therefore$

$$\underline{\underline{u}} \cdot (D\#(\underline{\underline{Y}}))^T [\#(\underline{\underline{X}}) - \#(\underline{\underline{Y}})] = \underline{\underline{u}} \cdot (\underline{\underline{X}} - \underline{\underline{Y}}).$$

Since $\underline{\underline{u}}$ is arbitrary, we have

¹Actually, the proof of Theorem A3.10.2 can be adjusted (to accommodate the fact $\mathcal{Q}$ is not all of $\mathcal{E}_3$) to show that (i) $\Rightarrow$ (ii) without the extra assumption that $\#$ is smooth or even continuous (See p. 52 of CHADWICK's Continuum Mechanics).

We shall see in Theorem B1.2.3 that $\mathcal{Q}$ is uniquely determined by the rigid deformation.

$$(Df(Y))^T [f(X) - f(Y)] = (X - Y), \quad (**)$$

Now fix $Y \in \overset{\circ}{Q}$, restrict X to the interior $\overset{\circ}{Q}$, and differentiate $(**)$ w.r.t. X to get

$$(Df(Y))^T Df(X) = \underline{I} \quad (***)$$

by the product rule Theorem A5.6.6 and Theorems A5.5.3 and A5.5.4 (ii).

Taking $Y = X$ in $(***)$, we see that $Df(X) \in \text{Orth } V_3$ ¹ $\forall X \in \overset{\circ}{Q}$. In particular, $Df(Y) \in \text{Orth } V_3$, and multiplication of $(***)$ on the left by $Df(Y) \Rightarrow$

$$Df(X) = Df(Y).$$

Thus, the orthogonal tensor $Df(X)$ is independent of X , and we write

$$Df(X) = Df(Y) = \underline{Q}.$$

Multiplication of $(**)$ on the left by $\underline{Q}$ yields the desired form

$$f(X) - f(Y) = \underline{Q}(X - Y).$$

Now this equation has been derived on the assumption that both X and Y belong to the interior $\overset{\circ}{Q}$. However, both sides

¹ Cf. Theorem A3.10.4.

of the equation are continuous functions of $\mathbb{X}$ and $\mathbb{Y}$ on the closed region $\mathcal{Q}$, and we conclude by continuity that the equation is valid on $\mathcal{Q}$.)¹

(ii) $\Rightarrow$ (i): When the starting point is statement (ii), we must first show that the function $\# : \mathcal{Q} \rightarrow \mathcal{E}_3$ defined by

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \mathcal{Q}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \mathcal{Q} \in \text{Orth } \mathcal{V}_3$$

is a deformation, and we turn to this task next.

By Theorems A6.9.2, A5.5.2, and A5.5.3, $\#$ is differentiable on the interior $\mathring{\mathcal{Q}}$, and

$$D\# = \mathcal{Q} \text{ on } \mathring{\mathcal{Q}}.$$

Of course, the function $\underline{F}_e : \mathcal{Q} \rightarrow \text{Lin } \mathcal{V}_3$ defined by

$$\underline{F}_e(\mathbb{X}) = \mathcal{Q}$$

is continuous on $\mathcal{Q}$ and agrees with $D\#$ on $\mathring{\mathcal{Q}}$. $\therefore \# \in C^1(\mathcal{Q}, \mathcal{E}_3)$, and

$$D\# = \mathcal{Q} \text{ on } \mathcal{Q}.$$

Since $D\# = \mathcal{Q} \in \text{Orth } \mathcal{V}_3$,

¹ See Theorem B1.2.2 for elucidation of this point.

Let $D\# \neq 0$ on $\mathcal{Q}$

by Theorem A3.10.6 (iii).

To see that $\#$ is one-to-one, we note that for any $X, Y \in \mathcal{Q}$ the statement

$$\#(X) = \#(Y)$$

$$\Rightarrow \#(\overset{\circ}{X}) + \underline{Q}(X - \overset{\circ}{X}) = \#(\overset{\circ}{Y}) + \underline{Q}(Y - \overset{\circ}{Y}) \quad (\text{substitution})$$

$$\Rightarrow \underline{Q}(X - \overset{\circ}{X}) = \underline{Q}(Y - \overset{\circ}{Y}) \quad (\text{Theorem A6.1.1 (xviii)})$$

$$\Rightarrow X - \overset{\circ}{X} = Y - \overset{\circ}{Y} \quad (\text{multiplication on left by } \underline{Q}^T)$$

$$\Rightarrow X = Y \quad (\text{Theorem A.6.1.1 (xviii)}).$$

Thus, $\#$ is, indeed, a deformation. Then to show that (ii) $\Rightarrow$ (i) is a straightforward calculation, which is left as Exercise B1.2.1.

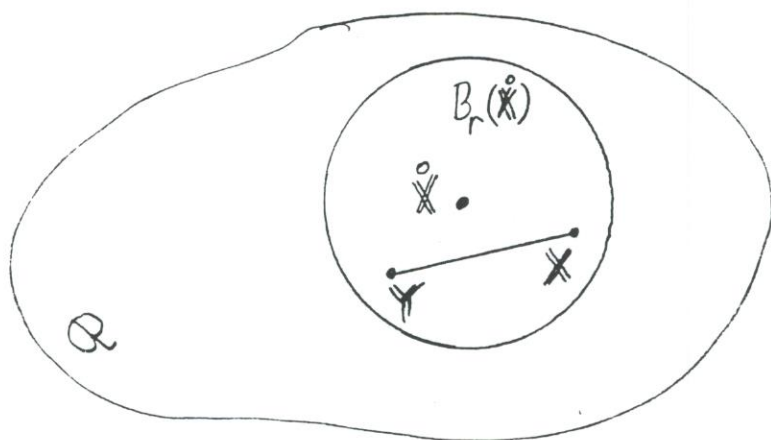
(ii) $\Rightarrow$ (iii): From the analysis above, $\underline{F} = D\# = \underline{Q} \in \text{Orth } \mathcal{V}_3$; and $\therefore$

$$\underline{C} = \underline{F}^T \underline{F} = \underline{Q}^T \underline{Q} = \underline{I} \text{ on } \mathcal{Q}.$$

(iii) $\Rightarrow$ (ii):¹ Let $\mathring{x} \in \mathring{Q}$. Since $\mathring{Q}$ is open, $\exists r > 0$ B1.2.6

$\exists$ the open ball $B_r(\mathring{x}) \subset \mathring{Q}$. Let $\mathring{x}, \mathring{y} \in B_r(\mathring{x})$. Then the line segment which connects $\mathring{x}$ and $\mathring{y}$ lies in $B_r(\mathring{x})$; i.e., for $\underline{e} = \mathring{x} - \mathring{y}$,

$$\{\mathring{y} + \tau \underline{e} : \tau \in [0, 1]\} \subset B_r(\mathring{x}). \quad)^{2,3}$$



Since $\mathbb{f}$ is differentiable, the mean-value theorem for derivatives (Theorem AS.9.1) $\Rightarrow$

$$\mathbb{f}(\mathring{y} + \underline{e}) - \mathbb{f}(\mathring{y}) = \int_0^1 D\mathbb{f}(\mathring{y} + \tau \underline{e})[\underline{e}] d\tau$$

or

$$\mathbb{f}(\mathring{x}) - \mathbb{f}(\mathring{y}) = \int_0^1 \underline{E}(\mathring{y} + \tau \underline{e}) \underline{e} d\tau.$$

$\therefore$

$$|\mathbb{f}(\mathring{x}) - \mathbb{f}(\mathring{y})| = \left| \int_0^1 \underline{E}(\mathring{y} + \tau \underline{e}) \underline{e} d\tau \right| \leq \int_0^1 |\underline{E}(\mathring{y} + \tau \underline{e}) \underline{e}| d\tau.$$

¹For a more elementary proof under the additional hypothesis that $\mathbb{f} \in C^2(Q, \mathbb{E}_3)$ see T.Y. THOMAS's Plastic Flow and Fracture in Solids.

²Cf. Definition B1.3.1 and the surrounding discussion.

³Draw this as a line in Exercise B1.7.2.

Now our starting hypothesis for this part of the proof, i.e. (iii), says that $\underline{F}$ is orthogonal at each point of $\mathcal{Q}$, $\therefore$ by Theorem A3.10.1, $\underline{F}$ preserves magnitudes; and $\forall \tau \in [0, 1]$

$$|\underline{F}(\underline{Y} + \tau \underline{e}) \underline{e}| = |\underline{e}| = |\underline{X} - \underline{Y}|.$$

Hence,

$$|\underline{f}(\underline{X}) - \underline{f}(\underline{Y})| \leq \int_0^1 |\underline{X} - \underline{Y}| d\tau = |\underline{X} - \underline{Y}|.$$

Of course, what we are looking for is that

$$|\underline{f}(\underline{X}) - \underline{f}(\underline{Y})| = |\underline{X} - \underline{Y}|. \quad (\#)$$

rather than

$$|\underline{f}(\underline{X}) - \underline{f}(\underline{Y})| \leq |\underline{X} - \underline{Y}|. \quad (\#\#)$$

We will reach the desired conclusion by applying the above argument to the inverse deformation to get the reversed inequality

$$|\underline{X} - \underline{Y}| \leq |\underline{f}(\underline{X}) - \underline{f}(\underline{Y})|. \quad (\#\#\#)$$

Then, of course, $(\#\#)$ and $(\#\#\#) \Rightarrow (\#)$. The details follow.

First, note that since

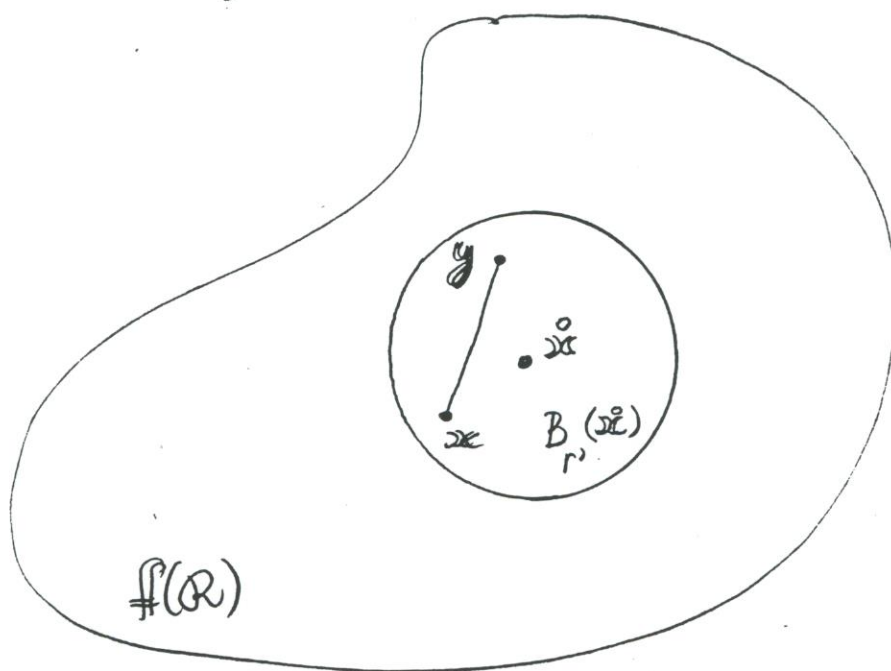
$$D \underline{f}^{-1}(\underline{f}(\underline{X})) = (D \underline{f}(\underline{X}))^{-1}, \quad)^1$$

¹ See Theorem B1.1.1 (vi).

$$\begin{aligned}
 (D\mathbb{f}^{-1})(D\mathbb{f}^{-1})^T &= (D\mathbb{f})^{-1}((D\mathbb{f})^{-1})^T \\
 &= (D\mathbb{f})^{-1}((D\mathbb{f})^T)^{-1} \quad (\text{Theorem A3.9.12}) \\
 &= ((D\mathbb{f})^T D\mathbb{f})^{-1} \quad (\text{Theorem A3.9.7 (iii)}) \\
 &= (\mathbb{I})^{-1} \quad (\text{iii}) \\
 &= \mathbb{I} \quad (\text{Theorem A3.9.7 (iv)}) ;
 \end{aligned}$$

The point fed
 $D\mathbb{f}^{-1}$ evaluated
 at $\mathbb{f}(\mathbb{X})$

i.e., $D\mathbb{f}^{-1}$ also is orthogonal. By Theorem B1.1.1 (iv), since $\mathbb{X}$ is an interior point of $\mathcal{Q}$, its image $\mathbb{x} = \mathbb{f}(\mathbb{X})$ is an interior point of $\mathbb{f}(\mathcal{Q})$. Consequently, $\exists r' > 0 \ni$ the open ball $B(\mathbb{x}) \subset \mathbb{f}(\mathcal{Q})$. By the continuity of $\mathbb{f}$, we can take $B_{r'}(\mathbb{X}) \subset \mathcal{Q}$ small enough that the image points $\mathbb{x} = \mathbb{f}(\mathbb{X})$ and $\mathbb{y} = \mathbb{f}(\mathbb{Y})$ lie in $B_{r'}(\mathbb{x}) \subset \mathbb{f}(\mathcal{Q})$.



Now, apply the mean-value theorem for derivatives to the function f^{-1} and the line segment connecting x and y , and conclude as before that

$$|f^{-1}(x) - f^{-1}(y)| \leq |x - y|.$$

But $x = f(X)$, $f^{-1}(x) = X$, etc., so that

$$|X - Y| \leq |f(X) - f(Y)|,$$

which is (###). Then as discussed previously, we have

$$|f(X) - f(Y)| = |X - Y| \quad \forall X, Y \in B_r(\dot{X})$$

Thus, f is rigid on $B_r(\dot{X})$. Since we have already shown that (i) $\Rightarrow$ (ii), f admits the representation

$$f(X) = f(\dot{X}) + \underline{Q}(X - \dot{X}), \quad \underline{Q} \in \text{Orth } V_3$$

on $B_r(\dot{X})$. Differentiation of this expression yields

$$\underline{F} = \nabla f = \underline{Q} \quad \text{on } B_r(\dot{X}).$$

Then since $\underline{Q}$ is uniform, $\underline{F}$ is differentiable on $B_r(\dot{X})$ and

$$D\underline{F} = \underline{Q} \quad \text{on } B_r(\dot{X}). \quad)^1$$

¹ Cf. Theorem A5.5.2.

In particular, $DF(\dot{x}) = \underline{Q}$; but $\dot{x}$ is an arbitrary point in the interior $\dot{Q}$, and $\therefore$

$$DF = \underline{Q} \text{ on } \dot{Q}.$$

Since Q is pathwise connected, $\dot{Q} \subset Q$ also is pathwise connected.¹ In fact being open, it is easy to see that it is polygonally connected. Then we conclude from Theorem A5.9.3 that F is constant on $\dot{Q}$, and since $F(\dot{x}) = \underline{Q}$,

$$F = DF = \underline{Q} \text{ on } \dot{Q}.$$

Then by the point field variant of Theorem A6.6.1,

$$f(x) = f(\dot{x}) + \underline{Q}(x - \dot{x}).$$

Thus, we have the desired representation on the interior $\dot{Q}$ and for $\dot{x} \in \dot{Q}$. For the moment leave $\dot{x} \in \dot{Q}$, then by continuity,

$$f(x) = f(\dot{x}) + \underline{Q}(x - \dot{x})$$

holds $\forall x \in Q$.² Now let $\dot{y}$ be any point in Q . Then

$$f(\dot{y}) = f(\dot{x}) + \underline{Q}(\dot{y} - \dot{x}),$$

¹ Prove this assertion as Exercise B1.2.3.

² Again, look ahead to Theorem B1.2.2.

and subtraction of this equation from the preceding one yields

$$f(x) = f(y) + Q(x-y) \quad (1)$$

Here, both x and y are arbitrary points of $\mathbb{R}$, and (ii) is established. $\square$

'Work out the details of this calculation as Exercise
B1.2.4.

Twice in the above proof, we had two continuous functions on $\mathcal{Q}$ which agreed on the interior $\mathring{\mathcal{Q}}$. We then claimed by continuity that they agreed on $\mathcal{Q}$. This situation occurs so frequently in continuum mechanics that we formalize it in

Theorem B1.2.2. (The Extension Theorem) Let
 $\underline{h} \in C^0(\mathcal{Q}, \mathcal{W})$, where $\mathcal{Q}$ is a compact region in E_3 and $\mathcal{W}$ is
a finite-dimensional linear space. Then

$$\underline{h} = \underline{0} \text{ on } \mathring{\mathcal{Q}} \Rightarrow \underline{h} = \underline{0} \text{ on } \mathcal{Q}. \quad)^1$$

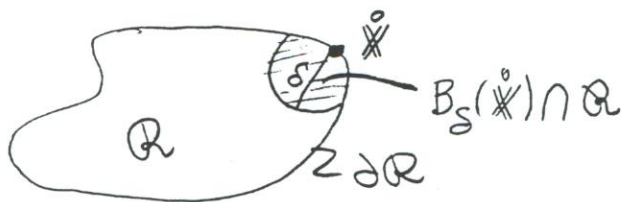
Proof. Since $\mathcal{Q}$ is compact $\bar{\mathcal{Q}} = \mathcal{Q}$ (Theorem A6.3.5 (i)),
 and then by Theorem A6.3.4

$$\mathcal{Q} = \mathring{\mathcal{Q}} \cup \partial\mathcal{Q}.$$

Consider any boundary point $\mathbb{X} \in \partial\mathcal{Q}$. Since $\underline{h}$ is continuous at $\mathbb{X}$, given any $\varepsilon > 0 \exists \delta > 0 \exists$

$$\mathbb{X} \in B_\delta(\mathbb{X}) \cap \mathcal{Q} \Rightarrow |\underline{h}(\mathbb{X}) - \underline{h}(\mathbb{X})| < \varepsilon.$$

Since $\mathcal{Q}$ is connected, it has no isolated points; and the situation pictured here is typical.



¹In practice, $\underline{h}$ is often the difference of two continuous functions.

Now since $\overset{\circ}{Q} \subset Q$, choose

$$x \in B_\delta(\overset{\circ}{x}) \cap \overset{\circ}{Q} \Rightarrow x \in B_\delta(\overset{\circ}{x}) \cap Q \Rightarrow |h(x)| < \varepsilon,$$

where we have noted that $h(x) = 0$ because $x \in \overset{\circ}{Q}$. By the arbitrariness of ε , we conclude that

$$|h(\overset{\circ}{x})| = 0.$$

For suppose not. Then say $|h(\overset{\circ}{x})| = h^0 > 0$, and

$$0 < h^0 < \varepsilon.$$

The choice $\varepsilon = h^0/2$ gives an obvious contradiction.

Finally, $|h(\overset{\circ}{x})| = 0 \Rightarrow h(\overset{\circ}{x}) = 0$. But $\overset{\circ}{x}$ was an arbitrary boundary point, so $h = 0$ on ∂Q . Consequently, $h = 0$ on $Q = \overset{\circ}{Q} \cup \partial Q$. $\square$

It is worth noting that the finite dimensionality of E_3 and W was not used in the above proof. The argument goes through unchanged if they are abstract metric spaces.

Let $\#$ be a rigid deformation on a region $\mathcal{Q}$. Then by Theorem B1.2.1, $\#$ admits the representation

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \underline{Q} \in \text{Orth } \mathcal{V}_3.$$

Differentiation at any interior point of $\mathcal{Q}$ yields

$$D\#(\mathbb{X}) = \underline{Q};$$

consequently, we have

Theorem B1.2.3. The orthogonal tensor $\underline{Q}$ in the representation

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}})$$

(given)

of a rigid deformation $\#$ is uniquely determined by the deformation.

Writing $\#(\mathring{\mathbb{X}}) = \underline{\mathcal{C}}$, the above representation becomes

$$\#(\mathbb{X}) = \underline{\mathcal{C}} + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}}).$$

Since the r.h.s. is well-defined $\forall \mathbb{X} \in \mathcal{E}_3$, it is customary to take rigid deformations as extended to all of $\mathcal{E}_3$. This will be our view in all that follows. Then we can dispense with explicit reference to any particular domain $\mathcal{Q}$.

The representation

$$\#(\mathbb{X}) = \#(\overset{\circ}{\mathbb{X}}) + \underline{Q}(\mathbb{X} - \overset{\circ}{\mathbb{X}})$$

provides us with a convenient vehicle for the investigation of rigid deformations. There are two classes of rigid deformations that are of particular importance. To obtain the first class, we consider the case where $\underline{Q} = \underline{I}$. Then

$$\#(\mathbb{X}) = \#(\overset{\circ}{\mathbb{X}}) + (\mathbb{X} - \overset{\circ}{\mathbb{X}})$$

or

$$\#(\mathbb{X}) - \mathbb{X} = \#(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}} \quad (\text{Theorem A6.1.1 (xv)}).$$

But by Definition Definition B1.1.2, $\#(\mathbb{X}) - \mathbb{X} = \underline{u}(\mathbb{X})$ is the displacement of the point $\mathbb{X}$. Thus, in the present case,

$$\underline{u}(\mathbb{X}) = \#(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}.$$

Since the displacement field is uniform, $\#$ is a translation of amount $\#(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}$.¹ Hence, we are lead to

Theorem B1.2.4. A rigid deformation of the form

$$\#(\mathbb{X}) = \#(\overset{\circ}{\mathbb{X}}) + (\mathbb{X} - \overset{\circ}{\mathbb{X}}) \quad (\underline{Q} = \underline{I})$$

is a translation of amount $\underline{u} = \#(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}$.

¹ Cf. Definition B1.1.3.

Conversely, a translation of amount $\underline{u}$ is a rigid deformation of the form

$$\#(\underline{x}) = \#(\underline{x}^0) + (\underline{x} - \underline{x}^0),$$

where $\#(\underline{x}^0) = \underline{x}^0 + \underline{u}$.

Proof. Only the converse assertion remains to be proved. This is left as Exercise B1.2.5. $\square$

Next suppose that $\#$ is a rigid deformation that leaves the point $\underline{x}^0$ fixed; i.e., $\#(\underline{x}^0) = \underline{x}^0$. Then the representation reduces to

$$\#(\underline{x}) = \underline{x}^0 + \underline{Q}(\underline{x} - \underline{x}^0),$$

and it is natural to make

Definition B1.2.2. A rigid deformation of the form

$$\#(\underline{x}) = \underline{x}^0 + \underline{Q}(\underline{x} - \underline{x}^0), \quad \underline{Q} \in \text{Orth } V_3$$

is a rotation about point $\underline{x}^0$.

Exercise B1.2.6. Show that the set of all rotations about a given point is a group w.r.t. composition.

The next result shows that as far as rigid deformations are concerned,

we can limit our study to translations and rotations. B1.2.17

Theorem B1.2.5. Any rigid deformation can be regarded as the composition of a translation and a rotation, in either order.

Proof. Let $\#$ be a rigid deformation. Then by Theorem B1.2.1, $\#$ admits the representation

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \underline{Q} \in \text{Orth } \mathcal{V}_3.$$

Then we see that $\#$ can be expressed as the composition

$$\# = \#_2 \circ \#_1,$$

where

$$\#_1: \mathbb{X} \mapsto \mathring{\mathbb{X}} + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}}) =: \mathbb{X}_1,$$

and

$$\#_2: \mathbb{X}_1 \mapsto \#(\mathring{\mathbb{X}}) + (\mathbb{X}_1 - \mathring{\mathbb{X}})$$

$$= \#(\mathring{\mathbb{X}}) + [(\mathring{\mathbb{X}} + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}})) - \mathring{\mathbb{X}}] \quad (\text{substitution})$$

$$= \#(\mathring{\mathbb{X}}) + \underline{Q}(\mathbb{X} - \mathring{\mathbb{X}}) \quad (\text{Theorem A6.1.1 (xiii)}).$$

By Definition B1.2.2, the deformation $\#_1$ is a rotation about point $\mathring{\mathbb{X}}$. Since

$$\#_2(\mathbb{X}_1) = \#(\mathring{\mathbb{X}}) + (\mathbb{X}_1 - \mathring{\mathbb{X}}),$$

$$\#_2(\mathring{\mathbb{X}}) = \#(\mathring{\mathbb{X}}) + \underline{Q} \quad (\text{substitution})$$

$$= \#(\mathring{\mathbb{X}}) \quad (\text{Theorem A6.1.1 (iii)}).$$

Thus,

$$\#_2(x_1) = \#_2(\overset{\circ}{x}) + (x_1 - \overset{\circ}{x}) ;$$

and by Theorem B1.2.4, $\#_2$ is seen to be a translation of amount $\#(\overset{\circ}{x}) - \overset{\circ}{x}$.

That we can do the translation first is not quite so transparent. Let us try

$$\# = \hat{\#}_2 \circ \hat{\#}_1 ,$$

where

$$\hat{\#}_1 : \overset{\circ}{x} \mapsto \hat{\#}_1(\overset{\circ}{x}) + (\overset{\circ}{x} - \overset{\circ}{x}) =: x_1,$$

and

$$\hat{\#}_2 : x_1 \mapsto \overset{\circ}{p} + \hat{Q}(x_1 - \overset{\circ}{p})$$

$$= \overset{\circ}{p} + \hat{Q}[(\hat{\#}_1(\overset{\circ}{x}) + (\overset{\circ}{x} - \overset{\circ}{x})) - \overset{\circ}{p}] \quad (\text{substitution})$$

$$= \overset{\circ}{p} + \hat{Q}[(\hat{\#}_1(\overset{\circ}{x}) - \overset{\circ}{p}) + (\overset{\circ}{x} - \overset{\circ}{x})] \quad (\text{Theorem Ab.1.1 (xi)})$$

$$= \overset{\circ}{p} + [\hat{Q}(\hat{\#}_1(\overset{\circ}{x}) - \overset{\circ}{p}) + \hat{Q}(\overset{\circ}{x} - \overset{\circ}{x})] \quad (\hat{Q} \in \text{Lin } \mathcal{V}_3)$$

$$= [\overset{\circ}{p} + \hat{Q}(\hat{\#}_1(\overset{\circ}{x}) - \overset{\circ}{p})] + \hat{Q}(\overset{\circ}{x} - \overset{\circ}{x}) \quad (\text{Theorem Ab.1.1 (xiv)})$$

Of course, what we want here is that

$$[\hat{\mathbf{p}} + \hat{\mathbf{Q}}(\hat{\mathbf{f}}_1(\check{\mathbf{x}}) - \hat{\mathbf{p}})] + \hat{\mathbf{Q}}(\check{\mathbf{x}} - \check{\mathbf{x}}) = \mathbf{f}(\check{\mathbf{x}}) + \mathbf{Q}(\check{\mathbf{x}} - \check{\mathbf{x}}).$$

We can make this happen by choosing $\hat{\mathbf{Q}} = \mathbf{Q}$ and then $\hat{\mathbf{p}}$ and $\hat{\mathbf{f}}_1(\check{\mathbf{x}}) \ni$

$$\hat{\mathbf{p}} + \mathbf{Q}(\hat{\mathbf{f}}_1(\check{\mathbf{x}}) - \hat{\mathbf{p}}) = \mathbf{f}(\check{\mathbf{x}})$$

$\Rightarrow$

$$\mathbf{Q}(\hat{\mathbf{f}}_1(\check{\mathbf{x}}) - \hat{\mathbf{p}}) = \mathbf{f}(\check{\mathbf{x}}) - \hat{\mathbf{p}} \quad (\text{Theorem A6.1.1 (xvi)})$$

$\Rightarrow$

$$\hat{\mathbf{f}}_1(\check{\mathbf{x}}) - \hat{\mathbf{p}} = \mathbf{Q}^T(\mathbf{f}(\check{\mathbf{x}}) - \hat{\mathbf{p}}) \quad (\mathbf{Q}^T \mathbf{Q} = \mathbf{I})$$

$\Rightarrow$

$$\hat{\mathbf{f}}_1(\check{\mathbf{x}}) = \hat{\mathbf{p}} + \mathbf{Q}^T(\mathbf{f}(\check{\mathbf{x}}) - \hat{\mathbf{p}}) \quad (\text{notation}).$$

Point $\hat{\mathbf{p}}$ is arbitrary; in particular, it could be $\check{\mathbf{x}}$. $\square$

Note that when the order of the composition in Theorem B1.2.6 is reversed, the translation is altered except when $\mathbf{Q} = \mathbf{I}$, which means that the original rigid deformation being decomposed is a translation.

The following theorem is expected, and it could just as well have appeared earlier.

Theorem B1.2.6. The composition of any two rigid deformations is a rigid deformation.

Proof. Exercise B1.2.7. $\square$

Supplementary Reading

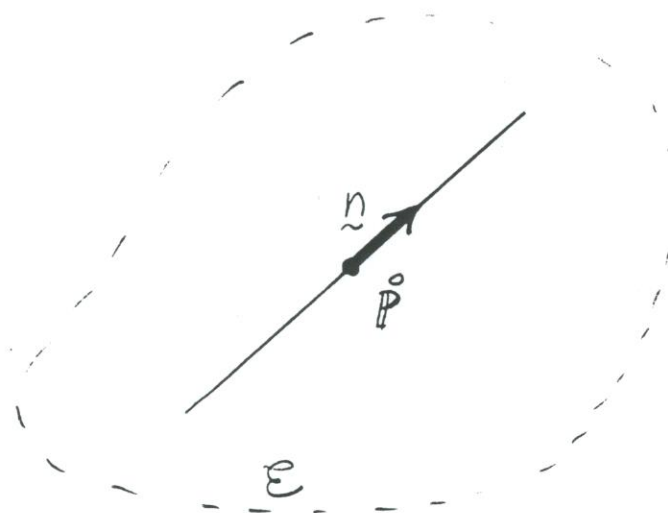
Same as for § B1.1 plus

GURTIN and WILLIAMS, "A note on rigid fields", J. Elasticity 6
(1976) 331-333.

A rotation about a point $\overset{\circ}{x}$ necessarily leaves $\overset{\circ}{x}$ fixed. The following famous result shows that $\overset{\circ}{x}$ leaves a whole line through $\overset{\circ}{x}$ fixed. Before stating the result it is appropriate to be precise about what we mean by a line in a point space.¹ with an important additional restriction

Definition B1.3.1. Let $\mathcal{E}$ be a Euclidean point space with translation space $\mathcal{V}$. Given a point $\overset{\circ}{p} \in \mathcal{E}$ and a unit vector $\underline{n} \in \mathcal{V}$, the line through point $\overset{\circ}{p}$ in the direction $\underline{n}$ is the point set

$$\{p \in \mathcal{E} : p = \overset{\circ}{p} + \alpha \underline{n}, \alpha \in \mathbb{R}\}$$



As the following exercise demonstrates, the "direction" of the unit vector $\underline{n}$ in the above definition is important but its "sense" is not.

Exercise B1.3.1. Show the line through point $\overset{\circ}{p}$ in the

¹For a line in a linear space, see Definition AS.9.1.

direction $-\underline{n}$ is the same point set as the line through point $\underline{p}$ in the direction $\underline{n}$.

Also it should be clear that we did not gain anything by characterizing the direction of the line by a unit vector. Formulate and prove a theorem to that effect as Exercise B1.3.2. Actually, in an important sense we lost because the translation space needs to be a normed linear space in order to speak of $\underline{n}$ being a unit vector.

For the restriction alluded to above, we need

Definition B1.3.2. A rotation $\mathbb{f}$ about a point $\underline{x}^0$; i.e. a deformation of the form

$$\mathbb{f}(\underline{x}) = \underline{x}^0 + \underline{Q}(\underline{x} - \underline{x}^0), \quad \underline{Q} \in \text{Orth } \mathcal{V}_3,$$

is proper if the orthogonal tensor $\underline{Q}$ is proper.

Note that if we had required deformations to meet

$$\det D\mathbb{f}(\underline{x}) > 0 \quad \forall \underline{x} \in \mathcal{R},$$

as was discussed on p. B1.1.3, then all rotations would be proper.

Exercise B1.3.3. Show that the set of all proper rotations about a given point is a subgroup of the set of all rotations about the same point (cf. Exercise B1.2.6).

Finally, we are prepared for

B1.3.3

Theorem B1.3.1 (EULER's Theorem) Let $\#$ be a proper rotation about a point $\overset{\circ}{X}$. If $\#$ is not the identity transformation, then $\exists$ a unique line through point $\overset{\circ}{X}$ that $\#$ leaves fixed.¹

Proof - Consider the line through $\overset{\circ}{X}$ in the direction $\underline{n}$:

$$\{ \overset{\circ}{X} : \overset{\circ}{X} = \overset{\circ}{X} + \alpha \underline{n}, \alpha \in \mathbb{R} \}.$$

The image of the typical point $\overset{\circ}{X}$ on this line under the proper rotation defined by

$$\#(\overset{\circ}{X}) = \overset{\circ}{X} + Q(\overset{\circ}{X} - \overset{\circ}{X})$$

is

$$\begin{aligned} \#(\overset{\circ}{X} + \alpha \underline{n}) &= \overset{\circ}{X} + Q((\overset{\circ}{X} + \alpha \underline{n}) - \overset{\circ}{X}) \\ &= \overset{\circ}{X} + \alpha Q \underline{n} \quad (\text{Theorem A6.1.1 (xiii)}). \end{aligned}$$

The typical point $\overset{\circ}{X} = \overset{\circ}{X} + \alpha \underline{n}$ will be fixed under $\#$ iff

$$\overset{\circ}{X} + \alpha \underline{n} = \overset{\circ}{X} + \alpha Q \underline{n}$$

or

$$\alpha \underline{n} = \alpha Q \underline{n} \quad (\text{Theorem A6.1.1 (vii)}).$$

For the whole line to be fixed this must hold $\forall \alpha \in \mathbb{R}$, which is equivalent to

$$Q \underline{n} = \underline{n}.$$

¹Of course, the identity transformation is a proper rotation about any point; and it leaves all lines fixed.

Thus, we need the existence of a unit vector $\underline{n} \neq \underline{0}$

$$(\underline{Q} - \underline{I})\underline{n} = \underline{0}.$$

By Theorem A3.8.7, a necessary and sufficient condition for the existence of such an $\underline{n}$ is

$$\det(\underline{Q} - \underline{I}) = 0.$$

Now

$$\begin{aligned} \det(\underline{Q} - \underline{I}) &= \det(\underline{Q} - \underline{I})^T && (\text{Theorem A3.8.6}) \\ &= \det(\underline{Q}^T - \underline{I}^T) && (\text{Theorem A3.6.3 (ii)}) \\ &= \det(\underline{Q}^T - \underline{I}) && (\text{Theorem A3.6.3 (v)}) \\ &= \det(\underline{Q}^T - \underline{I}) \det \underline{Q} && (\# \text{ a proper rotation}) \\ &= \det[(\underline{Q}^T - \underline{I})\underline{Q}] && (\text{Theorem A3.8.4}) \\ &= \det(\underline{Q}^T \underline{Q} - \underline{I} \underline{Q}) && (\text{Theorem A3.5.1 (ii) and (iii)}) \\ &= \det(\underline{I} - \underline{Q}) && (\text{Theorems A3.10.4 and A3.5.1 (iv)}) \\ &= \det[(-1)(\underline{Q} - \underline{I})] && (\text{Linear space properties of } \text{Lin } V_3) \\ &= (-1)^3 \det(\underline{Q} - \underline{I}) && (\text{Theorem A3.8.2 (iv)})^1 \\ &= -\det(\underline{Q} - \underline{I}) && (\text{arithmetic}). \end{aligned}$$

$\therefore$, in the present case, $\det(\underline{Q} - \underline{I}) = 0$; and $\exists \underline{n} \neq \underline{0} \ni \underline{Q}\underline{n} = \underline{n}$.

Thus, a proper rotation about $\dot{\ast}$ does leave a line through $\dot{\ast}$ fixed.

Suppose the unit vector $\underline{n}' \neq \pm \underline{n}$ defines a different line through $\dot{\ast}$ which is left fixed by the proper rotation under consideration. Then according to the above analysis, we have

¹ Notice the critical role played here by the dimension of the space.

both

$$\underline{Q}\underline{n} = \underline{n} \quad \text{and} \quad \underline{Q}\underline{n}' = \underline{n}'.$$

Then by the linearity, $\forall \alpha, \beta \in \mathbb{R}$

$$\underline{Q}(\alpha \underline{n} + \beta \underline{n}') = \alpha \underline{n} + \beta \underline{n}';$$

i.e., $\underline{Q}$ leaves the plane $Lsp\{\underline{n}, \underline{n}'\}$ fixed.

Exercise B1.3.4. Here, of course, we mean plane in the linear space context (see Definition A2.B.1). In the context of a three-dimensional point space, formulate a suitable definition of a plane through a point (cf. Definition B1.3.1), and show that in the present case the proper rotation $\mathbb{F}$ about $\mathbb{X}$ leaves the plane through $\mathbb{X}$ spanned by $\underline{n}$ and $\underline{n}'$ fixed.

Let $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}\}$ be any orthonormal basis for the two-dimensional linear subspace $Lsp\{\underline{n}, \underline{n}'\}$ of V_3 , and choose $\underline{e}_{\langle 3 \rangle}$ to be one of the two unit normals to the plane of $\underline{n}$ and $\underline{n}'$ (cf. Theorem A2.B.3). Then $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ is an orthonormal basis for V_3 . (Confirm this as Exercise B1.3.5.) Since

$$Q_{\langle ij \rangle} = \underline{e}_{\langle i \rangle} \cdot \underline{Q} \underline{e}_{\langle j \rangle},$$

$$[Q_{\langle ij \rangle}] = \begin{bmatrix} 1 & 0 & Q_{\langle 13 \rangle} \\ 0 & 1 & Q_{\langle 23 \rangle} \\ 0 & 0 & Q_{\langle 33 \rangle} \end{bmatrix},$$

Then by Theorem A3.8.5,

$$1 = \det Q = \det [Q^{<i>j>}] \Rightarrow Q^{<3>3} = 1.$$

Then by Theorems A3.2.7, A3.5.4, and A3.6.4, the orthogonality condition $\underline{I} = \underline{Q}^T \underline{Q}$ is equivalent to

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Q^{<1>3} & Q^{<2>3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & Q^{<1>3} \\ 0 & 1 & Q^{<2>3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & Q^{<1>3} \\ 0 & 1 & Q^{<2>3} \\ Q^{<1>3} & Q^{<2>3} & \underbrace{(Q^{<1>3})^2 + (Q^{<2>3})^2}_{1} \end{bmatrix}$$

$$\Rightarrow Q^{<1>3} = Q^{<2>3} = 0.$$

Thus, $[Q^{<i>j>}] = [\delta_{ij}]$ or $\underline{Q} = \underline{I}$; and we conclude that if $\#$ leaves more than one line fixed it must be the identity transformation on E_3 . $\square$

The algebraic content of the above theorem is that $\underline{Q}\eta = \eta$ has a unique (to within sense) nontrivial solution; or 1 is an eigenvalue of a any proper orthogonal tensor on a three-dimensional vector space, and the corresponding eigenspace is one-dimensional.

Supplementary Reading

Same as for $\S B1.1$

B1.4. Homogeneous Deformations

As promised at the end of § B1.1, we now turn to the important class of deformations characterized by the property that their derivatives, i.e., their deformation gradients, are uniform.

Definition B1.4.1. A deformation $\mathbb{F}$ on a region $\mathcal{Q}$ is homogeneous if its deformation gradient $\underline{F} = D\mathbb{F}$ is uniform on $\mathcal{Q}$; i.e.,

$$\underline{F}(\mathbb{X}) = \underline{F}(\mathbb{Y}) \quad \forall \mathbb{X}, \mathbb{Y} \in \mathcal{Q}.$$

The following representation theorem is the major tool in the study of homogeneous deformations.

Theorem B1.4.1. A deformation $\mathbb{F}$ on a region $\mathcal{Q}$ is homogeneous iff it has the form

$$\mathbb{F}(\mathbb{X}) = \mathbb{F}(\mathring{\mathbb{X}}) + \underline{\overset{\circ}{F}}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \mathbb{X} \in \mathcal{Q}, \quad)^1$$

where $\mathring{\mathbb{X}}$ is an arbitrary point of $\mathcal{Q}$ and $\underline{\overset{\circ}{F}}$ is a uniform invertible tensor on V_3 .

Proof. Suppose that $\mathbb{F}$ has the above form. Then by essentially the same argument used in the proof of Theorem B1.2.1 ((ii) $\Rightarrow$ (i)), we see that $\mathbb{F}$ is, indeed,

¹Because homogeneous deformations admit this representation, they are sometimes called affine deformations.

a deformation on $\mathcal{Q}$, and

$$D\# = \tilde{F}^{\circ} \text{ on } \mathcal{Q}.$$

$\therefore \#$ is homogeneous.

Conversely, suppose that $\#$ is a homogeneous deformation with

$$D\# = \tilde{F}^{\circ} \text{ on } \mathcal{Q}.$$

As was noted on p. B1.2.10, $\mathcal{Q}$ is polygonally connected because $\mathcal{Q}$ is pathwise connected. Then by Theorem A6.6.1, $\#$ admits the representation

the point space variant of

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \mathring{\underline{F}}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \mathbb{X} \in \mathring{\mathcal{Q}};$$

and by the extension theorem (Theorem B1.2.2),

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \mathring{\underline{F}}(\mathbb{X} - \mathring{\mathbb{X}}) \quad \forall \mathbb{X} \in \mathcal{Q}.$$

To remove the restriction that $\mathring{\mathbb{X}} \in \mathring{\mathcal{Q}}$, let $\mathring{\mathbb{Y}} \in \mathcal{Q}$ to get

$$\#(\mathring{\mathbb{Y}}) = \#(\mathring{\mathbb{X}}) + \mathring{\underline{F}}(\mathring{\mathbb{Y}} - \mathring{\mathbb{X}}),$$

Then subtraction yields

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{Y}}) + \mathring{\underline{F}}(\mathbb{X} - \mathring{\mathbb{Y}}) \quad \forall \mathbb{X}, \mathring{\mathbb{Y}} \in \mathcal{Q}. \quad \square$$

The next theorem mimics the corresponding result for a rigid deformation.

Theorem B1.4.2. The invertible tensor $\mathring{\underline{F}}$ in the
representation

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \mathring{\underline{F}}(\mathbb{X} - \mathring{\mathbb{X}})$$

of a given homogeneous deformation $\#$ is uniquely determined by
the deformation.

Proof. Exercise B1.4.1. $\square$

Again, as with rigid deformations, the representation

$$f(X) = f(\overset{\circ}{X}) + \overset{\circ}{F}(X - \overset{\circ}{X})$$

of a homogeneous deformation is well-defined on all of E_3 , and it is customary to take homogeneous deformations as extended to all of E_3 ; this will be our point of view in the sequel.

Next, we utilize the above representation to undertake a general investigation of the geometry of homogeneous deformations. Such a study is of interest in its own right, but it has the added importance that, in a sense to be made precise in the next section, general deformations are locally homogeneous. Furthermore, in view of the following result, the geometrical aspects of homogeneous deformations are quite transparent.

Theorem B1.4.3. A homogeneous deformation carries a straight line into a straight line.)¹

Proof. By Definition B1.3.1, a line L through point $\overset{\circ}{P}$ in the direction $\underline{m}$ ($|\underline{m}|=1$) is the point set

$$L = \{ X : X = \overset{\circ}{P} + \alpha \underline{m}, \alpha \in \mathbb{R} \}.$$

¹The adjective straight is just for emphasis; a straight line is simply a line.

By Theorem B1.4.1, the homogeneous deformation $\mathbb{F}$ under consideration admits the representation

$$\mathbb{F}(\mathbb{X}) = \mathbb{F}(\mathbb{X}^{\circ}) + \underline{\underline{\mathbb{F}}}(\mathbb{X} - \mathbb{X}^{\circ}), \quad \underline{\underline{\mathbb{F}}} \in \text{Inv } \mathcal{V}_3.$$

The choice of the base point $\mathbb{X}^{\circ}$ is arbitrary, and it is convenient here to choose $\mathbb{X}^{\circ} = \underline{\underline{\mathbb{P}}}$. Then the image of $\mathbb{X} \in \mathcal{L}$ under $\mathbb{F}$ is

$$\alpha = \mathbb{F}(\mathbb{X}) = \mathbb{F}(\underline{\underline{\mathbb{P}}} + \alpha \underline{\underline{m}}) \quad (\text{substitution})$$

$$= \mathbb{F}(\underline{\underline{\mathbb{P}}}) + \underline{\underline{\mathbb{F}}}[(\underline{\underline{\mathbb{P}}} + \alpha \underline{\underline{m}}) - \underline{\underline{\mathbb{P}}}] \quad (\text{representation of } \mathbb{F})$$

$$= \mathbb{F}(\underline{\underline{\mathbb{P}}}) + \underline{\underline{\mathbb{F}}}(\alpha \underline{\underline{m}}) \quad (\text{Theorem A6.1.1 (xiii)})$$

$$= \mathbb{F}(\underline{\underline{\mathbb{P}}}) + \alpha \underline{\underline{\mathbb{F}}} \underline{\underline{m}} \quad (\text{linearity of } \underline{\underline{\mathbb{F}}})$$

$$= \mathbb{F}(\underline{\underline{\mathbb{P}}}) + \alpha \frac{|\underline{\underline{\mathbb{F}}} \underline{\underline{m}}|}{|\underline{\underline{\mathbb{F}}} \underline{\underline{m}}|} \underline{\underline{\mathbb{F}}} \underline{\underline{m}} \quad)^1$$

Thus, setting

$$\underline{\underline{n}} = \frac{\underline{\underline{\mathbb{F}}} \underline{\underline{m}}}{|\underline{\underline{\mathbb{F}}} \underline{\underline{m}}|} \quad (|\underline{\underline{n}}| = 1),$$

we see that the image of the line $\mathcal{L}$ under $\mathbb{F}$ is the point set

¹ Prove that $\underline{\underline{\mathbb{F}}} \underline{\underline{m}} \neq \underline{\underline{0}}$ as Exercise B1.4.2.

$$\ell = \{x : x = f(\tilde{p}) + \alpha |\tilde{F}m| \tilde{n}, \alpha \in \mathbb{R}\}$$

$$= \{x : x = f(\tilde{p}) + \beta \tilde{n}, \beta \in \mathbb{R}\} \quad (\text{Exercise B1.4.3})$$

— a line through point $f(\tilde{p})$ in the direction $\tilde{n}$.

In the same way, a segment of a line, i.e., a point set

$$\{x : x = \tilde{p} + \alpha \tilde{m}, |\tilde{m}| = 1, \alpha \in [a, b]\},$$

gets carried into a segment of another line by a homogeneous deformation; and the following result makes sense.

Theorem B1.4.4. Let $x_1, x_2 \in E_3$. Then the image of the vector $\tilde{m} = \overrightarrow{x_1 x_2}$ under the homogeneous deformation defined by

$$x = f(x) = f(x^\circ) + \tilde{F}^\circ (x - x^\circ), \quad \tilde{F}^\circ \in \text{Im } \mathcal{V}_3$$

is

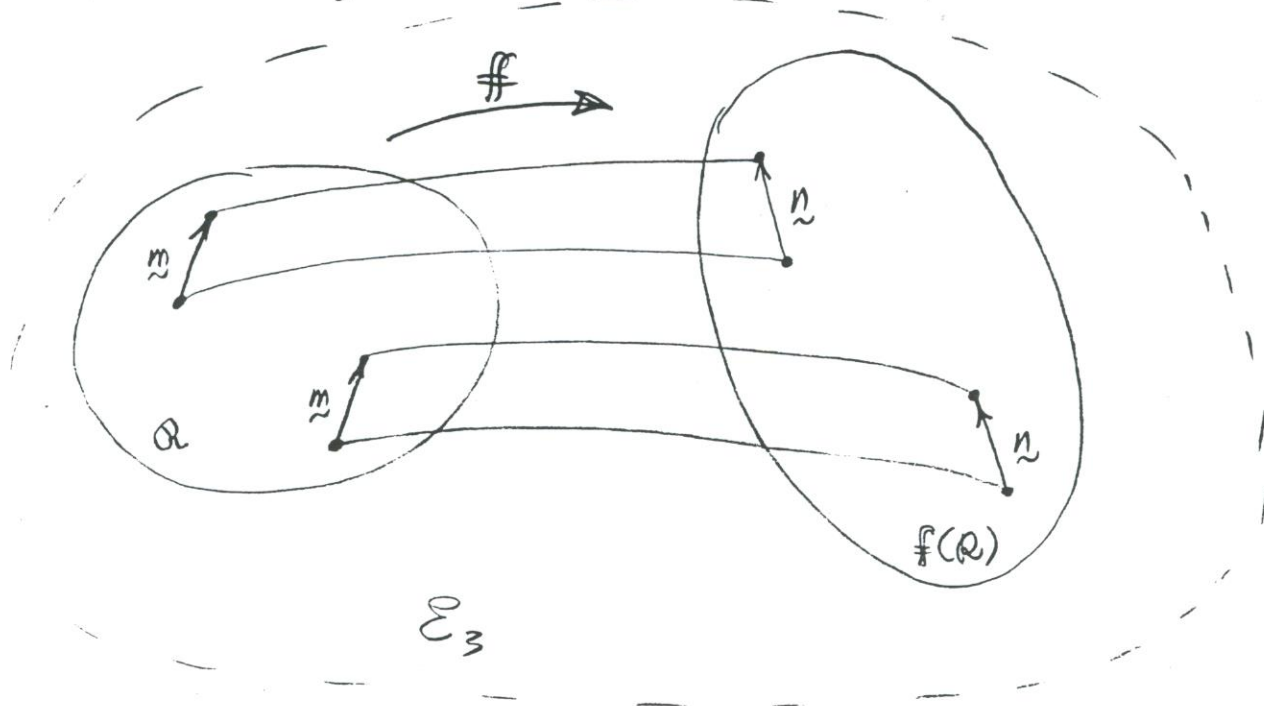
$$\tilde{n} = \overrightarrow{x_1 x_2} = \tilde{F}^\circ \tilde{m}. \quad)^1$$

Proof. Exercise B1.4.4. $\square$

Note that the image vector $\tilde{n}$ is independent of the location of the preimage vector $\tilde{m}$, i.e., $\tilde{n}$ depends on x_1 and x_2 through only their difference $\overrightarrow{x_1 x_2}$. In this sense, all

¹Here, of course, $x_1 = f(x_1)$ and $x_2 = f(x_2)$. Also the vectors $\tilde{m}$ and $\tilde{n}$ need not be of unit length.

parts of E_3 experience the same change of shape (see the figure below) under a homogeneous deformation — another reason for the terminology homogeneous.



Since the length of a vector and the angle between two vectors are defined through the inner product, the following result will be of great value.

Theorem B1.4.5. Let the vectors $\underline{n}_1$ and $\underline{n}_2$ be the images of the vectors $\underline{m}_1$ and $\underline{m}_2$ under the homogeneous deformation defined by

$$\underline{x} = f(\underline{X}) = \underline{F}(\underline{X}) + \underline{\tilde{F}}(\underline{X} - \underline{X}^0), \quad \underline{\tilde{F}} \in \text{Inv } \mathcal{V}_3.$$

Then

$$\underline{n}_1 \cdot \underline{n}_2 = \underline{m}_1 \cdot \underline{\tilde{C}} \underline{m}_2,$$

where

$$\underline{\tilde{C}} = \underline{\tilde{F}}^T \underline{\tilde{F}}.$$

Proof. Exercise B1.4.5. $\square$

For study of length changes caused by a homogeneous deformation, we introduce

Definition B1.4.2. Let the vector $\underline{m} \neq \underline{0}$ have the image $\underline{n}$ under a homogeneous deformation. Then the stretch in the direction of $\underline{m}$ is

$$\Delta(\underline{m}) = \frac{|\underline{n}|}{|\underline{m}|},$$

and the extension ratio in the direction of $\underline{m}$ is

$$\Delta(\underline{m}) = \frac{|\underline{n}| - |\underline{m}|}{|\underline{m}|} = \Delta(\underline{m}) - 1.$$

In other words, the stretch is the final length divided by the initial length, and the extension ratio is the change in length divided by the initial length. Thus, the extension ratio is at least close to the notion of normal strain in elementary solid mechanics. Note that these definitions do not make sense (at least without considerable adjustment) for general deformations because they are in terms of vectors being mapped into vectors.

The next theorem shows that the length of the preimage vector $\underline{m}$ in Definition B1.4.2 is of no importance, except for $|\underline{m}| \neq 0$,

Theorem B1.4.6. For $\alpha \neq 0$ and $\underline{m} \neq 0$,

$$\Delta(\alpha \underline{m}) = \Delta(\underline{m}).$$

Proof. Exercise B1.4.6. $\square$

Thus, there is no loss of generality in Definition B1.4.2 if we take $|\underline{m}| = 1$. This is understood in the sequel.

Application of Theorem B1.4.5 leads easily to

Theorem B1.4.7. For the homogeneous deformation defined by

$$\underline{x} = \underline{f}(\underline{X}) = \underline{f}(\underline{X}^\circ) + \underline{\underline{F}}(\underline{X} - \underline{X}^\circ), \quad \underline{\underline{F}} \in \text{Inv } \mathcal{V}_3,$$

$$\Delta(\underline{m}) = \sqrt{\underline{m} \cdot \underline{\underline{C}} \underline{m}}, \quad)^1$$

where $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$.

Proof. Exercise B1.4.7. $\square$

Thus, the tensor $\underline{\underline{C}}$ determines the stretches and hence the extension ratios.

¹As we saw in the proof of the polar decomposition theorem (Theorem A3.15.4), for $\underline{\underline{F}} \in \text{Inv } \mathcal{V}_3$, $\underline{\underline{F}}^T \underline{\underline{F}} \in \text{Psym } \mathcal{V}_3 \Rightarrow \underline{m} \cdot \underline{\underline{C}} \underline{m} \geq 0$.

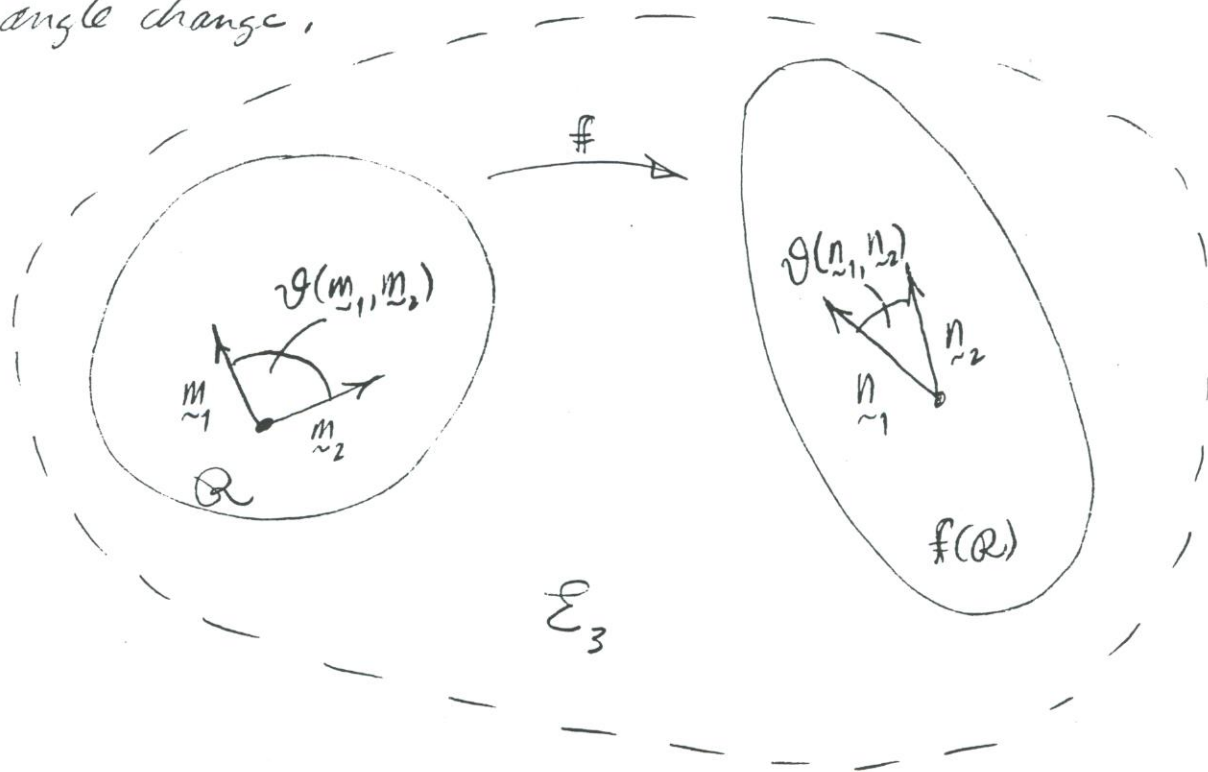
Consequently, the square root here is never a problem.

For the investigation of angle changes caused by a homogeneous deformation, we introduce

Definition B1.4.3. Let the unit vectors $\underline{m}_1$ and $\underline{m}_2$ have the images $\underline{n}_1$ and $\underline{n}_2$, respectively, under a homogeneous deformation. Denote the angles between $\underline{m}_1$ and $\underline{m}_2$ and between $\underline{n}_1$ and $\underline{n}_2$ by $\mathcal{V}(\underline{m}_1, \underline{m}_2)$ and $\mathcal{V}(\underline{n}_1, \underline{n}_2)$, respectively. Then the shear of the directions $\underline{m}_1$ and $\underline{m}_2$ is

$$\Gamma(\underline{m}_1, \underline{m}_2) = \mathcal{V}(\underline{m}_1, \underline{m}_2) - \mathcal{V}(\underline{n}_1, \underline{n}_2).$$

Thus, the shear is the angle decrease or the negative angle change.



In view of the homogeneity, cf. the remark following Theorem B1.4.4, it is not necessary that $\underline{m}_1$ and $\underline{m}_2$ issue from the same point of E_3 in Definition B1.4.3.

Since

$$\underline{n}_1 \cdot \underline{n}_2 = |\underline{n}_1| |\underline{n}_2| \cos \vartheta(\underline{n}_1, \underline{n}_2),$$

Theorem B1.4.5 leads to

(Let $\underline{n}_1$ and $\underline{n}_2$ be the images of the unit vectors $\underline{m}_1$ and $\underline{m}_2$ under Theorem B1.4.8, the homogeneous deformation defined by

$$\underline{x} = \underline{f}(\underline{X}) = \underline{f}(\underline{X}^0) + \underline{\tilde{F}}(\underline{X} - \underline{X}^0), \quad \underline{\tilde{F}} \in \text{Inv } \mathcal{V}_3.$$

Then

$$\cos \vartheta(\underline{n}_1, \underline{n}_2) = \cos [\vartheta(\underline{m}_1, \underline{m}_2) - \Pi(\underline{m}_1, \underline{m}_2)] = \frac{\underline{m}_1 \cdot \underline{\tilde{C}} \underline{m}_2}{\Lambda(\underline{m}_1) \Lambda(\underline{m}_2)}, \quad)^1$$

where $\underline{\tilde{C}} = \underline{\tilde{F}}^T \underline{\tilde{F}}$.

Proof. Exercise B1.4.8. $\square$

Thus, the tensor $\underline{\tilde{C}}$ also determines the shears.

It is useful to see what happens to an orthonormal basis under a homogeneous deformation.

Theorem B1.4.9. Let $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ be an orthonormal basis for $\mathcal{V}_3$. Then for the homogeneous deformation defined by

$$\underline{x} = \underline{f}(\underline{X}) = \underline{f}(\underline{X}^0) + \underline{\tilde{F}}(\underline{X} - \underline{X}^0), \quad \underline{\tilde{F}} \in \text{Inv } \mathcal{V}_3,$$

$$\Lambda(\underline{e}_{\langle i \rangle}) = \sqrt{\underline{\tilde{C}}_{\langle ii \rangle}}, \quad i \in \Pi_3$$

¹ In view of Theorem B1.4.7 and the positive definiteness of $\underline{\tilde{C}}$, the presence of $\Lambda(\underline{m})$ in the denominator here is never a problem.

and

$$\sin \Gamma(\underline{e}_{\langle i \rangle}, \underline{e}_{\langle j \rangle}) = \frac{\dot{\underline{C}}^{\langle ij \rangle}}{\sqrt{\dot{\underline{C}}^{\langle ii \rangle}} \sqrt{\dot{\underline{C}}^{\langle jj \rangle}}} , \quad i \neq j \in \Pi_3 \quad (\text{no sum}).$$

Proof. Exercise B1.4.9. $\square$

Hence, the diagonal components of $\dot{\underline{C}}$ determine the stretches in the coordinate directions¹; while the off-diagonal components are associated closely with the shears of the coordinate directions. In fact, the following result can be read off from Theorem B1.4.9.

Theorem B1.4.10. Let the hypotheses of Theorem B1.4.9 hold.

Then

$$\Delta(\underline{e}_{\langle i \rangle}) = 0 \iff \dot{\underline{C}}^{\langle ii \rangle} = 1 , \quad i \in \Pi_3$$

and

$$\Gamma(\underline{e}_{\langle i \rangle}, \underline{e}_{\langle j \rangle}) = 0 \iff \dot{\underline{C}}^{\langle ij \rangle} = 0 , \quad i \neq j \in \Pi_3 .$$

Certainly, this is consistent with our prior knowledge that
a deformation is rigid iff $\dot{\underline{C}} = \underline{\underline{I}} \quad (\dot{\underline{C}}^{\langle ij \rangle} = \delta_{ij})$.²

Since $\dot{\underline{C}} = \dot{\underline{F}}^T \dot{\underline{F}}$ is symmetric, the following result can be read off from the spectral theorem (Theorem A3.13.1).

Theorem B1.4.11. Let the hypotheses of Theorem B1.4.9 hold.
Then $\exists$ at least one orthonormal basis $\{\underline{\hat{e}}_{\langle 1 \rangle}^*, \underline{\hat{e}}_{\langle 2 \rangle}^*, \underline{\hat{e}}_{\langle 3 \rangle}^*\}$ for $\mathcal{V}_3$,

¹ Cf. Definition A6.2.3.

² See Theorem B1.2.1.

called principal axes of strain of the given homogeneous deformation,
 $\ni$

$$[\hat{\underline{\underline{C}}}^{*ij}] = \begin{bmatrix} \hat{\underline{\underline{C}}}_1 & 0 & 0 \\ 0 & \hat{\underline{\underline{C}}}_2 & 0 \\ 0 & 0 & \hat{\underline{\underline{C}}}_3 \end{bmatrix}.$$

The eigenvalues, $\hat{\underline{\underline{C}}}_1, \hat{\underline{\underline{C}}}_2, \hat{\underline{\underline{C}}}_3$, of $\hat{\underline{\underline{C}}}$ are called the principal strains.

Similarly, from Theorems A3.13.2 and B1.4.7, we get

Theorem B1.4.12. Let the hypotheses of Theorem B1.4.11 hold.
Order the principal strains according to $\hat{\underline{\underline{C}}}_1 \leq \hat{\underline{\underline{C}}}_2 \leq \hat{\underline{\underline{C}}}_3$. Then

$$\min_{|\underline{m}|=1} \Lambda(\underline{m}) = \sqrt{\hat{\underline{\underline{C}}}_1} \quad \text{and} \quad \max_{|\underline{m}|=1} \Lambda(\underline{m}) = \sqrt{\hat{\underline{\underline{C}}}_3}. \quad)^1$$

It is natural to make

Definition B1.4.4. Refer to Theorem B1.4.11. (For $i \in \Pi_3$),
the principal stretch Δ_i and the principal extension ratio λ_i are
the stretch and the extension ratio in a principal direction of $\hat{\underline{\underline{C}}}$
associated with the principal strain $\hat{\underline{\underline{C}}}_i$.

Then it follows from Theorem B1.4.7 that

¹ In view of the positive definiteness of $\hat{\underline{\underline{C}}}$ and Theorem A3.15.1, the square roots here are never a problem.

Theorem B1.4.13. $\Delta_i = \sqrt{\tilde{C}_i}$ and $\Delta_i = \sqrt{\tilde{C}_i} - 1$, $i \in \mathbb{I}_3$.

An interesting corollary of Theorem B1.4.11 is

Theorem B1.4.14. Every homogeneous deformation carries at least one triplet of mutually orthogonal vectors into a triplet of mutually orthogonal vectors.

Proof. Exercise B1.4.10. $\square$

INSERT material on pp. B1.14.14 a-d

Next, we decompose a given homogeneous deformation into its simplest ingredients. We are interested particularly in separating the part of a deformation associated with change of shape from the part that is rigid. To get started, we need

Theorem B1.4.15. The composition of any two homogeneous deformations is a homogeneous deformation.

Proof. Call the two homogeneous deformations under consideration $\mathbb{F}_1$ and $\mathbb{F}_2$. By Theorem B1.4.1, they admit the representations

$$\mathbb{F}_1(\mathbb{X}) = \mathbb{F}_1(\mathring{\mathbb{X}}) + \tilde{\mathbb{F}}_1(\mathbb{X} - \mathring{\mathbb{X}}), \quad \tilde{\mathbb{F}}_1 \in \text{Inv } \mathcal{V}_3$$

and

$$\mathbb{F}_2(\mathbb{X}) = \mathbb{F}_2(\mathring{\mathbb{X}}) + \tilde{\mathbb{F}}_2(\mathbb{X} - \mathring{\mathbb{X}}), \quad \tilde{\mathbb{F}}_2 \in \text{Inv } \mathcal{V}_3.$$

Of course, it is not necessary to choose the same base point $\mathring{\mathbb{X}}$ for both representations; but there is no loss of generality in doing

Up to this point our geometrical analysis of the typical homogeneous deformation $\#$ defined by

$$\#(\mathbb{X}) = \#(\mathring{\mathbb{X}}) + \underline{\underline{\mathring{F}}}(\mathbb{X} - \mathring{\mathbb{X}}), \quad \underline{\underline{\mathring{F}}} \in \text{Inv } \mathcal{V}_3$$

has come out naturally in terms of the tensor

$$\underline{\underline{\mathring{C}}} = \underline{\underline{\mathring{F}}}^T \underline{\underline{\mathring{F}}}.$$

This is fine because obviously $\underline{\underline{\mathring{C}}}$ is easy to calculate. However, there is another tensor measure of the deformation that has a more direct geometrical interpretation; unfortunately, it is more difficult to calculate. We can get at it in several ways.

First we recall from the proof of the polar decomposition theorem¹ that

$$\underline{\underline{\mathring{C}}} = \underline{\underline{\mathring{F}}}^T \underline{\underline{\mathring{F}}}$$

is positive definite symmetric. Consequently, by Theorem A3.15.3 it has a unique positive definite symmetric square root; i.e., $\exists$

$$\underline{\underline{\mathring{U}}} \in \text{Psym } \mathcal{V}_3 \quad \ni \quad \underline{\underline{\mathring{U}}}^2 = \underline{\underline{\mathring{C}}}.$$

$\therefore$ Theorem B1.4.5 becomes

¹Theorem A3.15.4.

$$\begin{aligned}
n_{\sim 1} \cdot n_{\sim 2} &= m_{\sim 1} \cdot \overset{\circ}{\sim} m_{\sim 2} \\
&= m_{\sim 1} \cdot \overset{\circ}{\sim}^2 m_{\sim 2} \quad (\text{substitution}) \\
&= m_{\sim 1} \cdot \overset{\circ}{\sim} (\overset{\circ}{\sim} m_{\sim 2}) \quad (\text{definition of tensor multiplication}) \\
&= (\overset{\circ}{\sim} m_{\sim 2}) \cdot \overset{\circ}{\sim}^T m_{\sim 1} \quad (\text{definition of transpose}) \\
&= (\overset{\circ}{\sim} m_{\sim 2}) \cdot (\overset{\circ}{\sim} m_{\sim 1}) \quad (\text{symmetry of } \overset{\circ}{\sim}) \\
&= (\overset{\circ}{\sim} m_{\sim 1}) \cdot (\overset{\circ}{\sim} m_{\sim 2}) \quad (\text{commutativity of inner product}).
\end{aligned}$$

We can obtain the same result in another way by noting that in terms of this same $\overset{\circ}{\sim}$, $\overset{\circ}{F}$ has the polar decomposition

$$\overset{\circ}{F} = \overset{\circ}{R} \overset{\circ}{\sim}, \quad \overset{\circ}{R} \in \text{Orth } V_3.$$

Then by Theorem B1.4.4,

$$n_{\sim 1} \cdot n_{\sim 2} = (\overset{\circ}{F} m_{\sim 1}) \cdot (\overset{\circ}{F} m_{\sim 2}) = (\overset{\circ}{\sim} m_{\sim 1}) \cdot (\overset{\circ}{\sim} m_{\sim 2})$$

The details of the calculation are left as Exercise B1.4.11.

In the same way, Theorems B1.4.7 and 8 also come out simpler in terms of $\overset{\circ}{\sim}$, we gather these results as

Theorem B1.4.15. Let the vectors $\underline{n}_1$ and $\underline{n}_2$ be the images of the vectors $\underline{m}_1$ and $\underline{m}_2$ under the homogeneous deformation defined by

$$\underline{x} = \underline{f}(\underline{X}) = \underline{f}(\underline{X}^0) + \underline{\overset{\circ}{F}}(\underline{X} - \underline{X}^0), \quad \underline{\overset{\circ}{F}} \in \text{Inv } \mathcal{V}_3.$$

Let $\underline{\overset{\circ}{F}}$ have the polar decomposition

$$\underline{\overset{\circ}{F}} = \underline{\overset{\circ}{R}} \underline{\overset{\circ}{U}}, \quad \underline{\overset{\circ}{R}} \in \text{Orth } \mathcal{V}_3, \quad \underline{\overset{\circ}{U}} \in \text{Psym } \mathcal{V}_3.$$

Then

$$(i) \quad \underline{n}_1 \cdot \underline{n}_2 = (\underline{\overset{\circ}{U}} \underline{m}_1) \cdot (\underline{\overset{\circ}{U}} \underline{m}_2),$$

$$(ii) \quad \Delta(\underline{m}_1) = |\underline{\overset{\circ}{U}} \underline{m}_1|,$$

and

$$(iii) \quad \cos \mathcal{V}(\underline{n}_1, \underline{n}_2) = \frac{(\underline{\overset{\circ}{U}} \underline{m}_1) \cdot (\underline{\overset{\circ}{U}} \underline{m}_2)}{|\underline{\overset{\circ}{U}} \underline{m}_1| |\underline{\overset{\circ}{U}} \underline{m}_2|}.$$

The next two exercises contain additional additional results in this vein.

Exercise B1.4.12-16. Work out the $\underline{\overset{\circ}{U}}$ -analogs of Theorems B1.4.9-13.

Exercise B1.4.17. There also is the polar decomposition

$$\underline{\overset{\circ}{F}} = \underline{\overset{\circ}{V}} \underline{\overset{\circ}{R}}, \quad \underline{\overset{\circ}{R}} \in \text{Orth } \mathcal{V}_3, \quad \underline{\overset{\circ}{V}} \in \text{Psym } \mathcal{V}_3,$$

where

$$\underset{\sim}{V}^{\circ} = \underset{\sim}{F} \underset{\sim}{F}^{\circ T} =: \underset{\sim}{B}.$$

Show that

$$m_{\underset{\sim}{1}} \cdot m_{\underset{\sim}{2}} = (\underset{\sim}{V}^{\circ-1}_{\underset{\sim}{n}_1}) \cdot (\underset{\sim}{V}^{\circ-1}_{\underset{\sim}{n}_2}) = \underset{\sim}{n}_1 \cdot \underset{\sim}{B}^{-1} \underset{\sim}{n}_2.$$

Return to p. B1.4.14

so, and it does simplify the following calculation.

$$\begin{aligned}
 \#_2 \circ \#_1(\mathbb{X}) &= \#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2 [\#_1(\mathbb{X}) - \overset{\circ}{\mathbb{X}}] \quad (\text{substitution}) \\
 &= \#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2 [(\#_1(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_1(\mathbb{X} - \overset{\circ}{\mathbb{X}})) - \overset{\circ}{\mathbb{X}}] \quad (\text{substitution}) \\
 &= \#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2 [(\#_1(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_1(\mathbb{X} - \overset{\circ}{\mathbb{X}})] \quad (\text{Theorem A6.1.1 (xi)}) \\
 &= \#_2(\overset{\circ}{\mathbb{X}}) + [\overset{\circ}{F}_2(\#_1(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2(\overset{\circ}{F}_1(\mathbb{X} - \overset{\circ}{\mathbb{X}}))] \quad (\text{linearity of } \overset{\circ}{F}_2) \\
 &= [\#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2(\#_1(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}})] + \overset{\circ}{F}_2(\overset{\circ}{F}_1(\mathbb{X} - \overset{\circ}{\mathbb{X}})) \quad (\text{Theorem A6.1.1 (xiv)}) \\
 &= [\#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2(\#_1(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}})] + (\overset{\circ}{F}_2 \overset{\circ}{F}_1)(\mathbb{X} - \overset{\circ}{\mathbb{X}}) \quad (\text{tensor multiplication})
 \end{aligned}$$

Evaluation at $\overset{\circ}{\mathbb{X}} \Rightarrow$

$$\#_2 \circ \#_1(\overset{\circ}{\mathbb{X}}) = [\#_2(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{F}_2(\#_1(\overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}})],$$

and $\therefore$

$$\#_2 \circ \#_1(\mathbb{X}) = \#_2 \circ \#_1(\overset{\circ}{\mathbb{X}}) + (\overset{\circ}{F}_2 \overset{\circ}{F}_1)(\mathbb{X} - \overset{\circ}{\mathbb{X}}).$$

By Theorem B1.4.1 in the other direction, this is seen to be a homogeneous deformation provided $\overset{\circ}{F}_2 \overset{\circ}{F}_1 \in \text{Inv } \mathcal{V}_3$. But

$$\det(\overset{\circ}{F}_2 \overset{\circ}{F}_1) = (\det \overset{\circ}{F}_2)(\det \overset{\circ}{F}_1) \neq 0 \Rightarrow \overset{\circ}{F}_2 \overset{\circ}{F}_1 \in \text{Inv } \mathcal{V}_3. \quad \square$$

Now consider the typical homogeneous deformation $\#$. By Theorem B1.4.1, $\#$ has the form

$$\#(\underline{X}) = \#(\underline{X}^\circ) + \underline{\tilde{F}}(\underline{X} - \underline{X}^\circ), \quad \underline{\tilde{F}} \in \text{Inv } \mathcal{V}_3,$$

Since $\underline{\tilde{F}} \in \text{Inv } \mathcal{V}_3$, we can apply the polar decomposition theorem (Theorem A3.15.4) to write

$$\underline{\tilde{F}} = \underline{\tilde{R}} \underline{\tilde{U}} = \underline{\tilde{V}} \underline{\tilde{R}},$$

where $\underline{\tilde{R}} \in \text{Orth } \mathcal{V}_3$ and $\underline{\tilde{U}}, \underline{\tilde{V}} \in \text{Psym } \mathcal{V}_3$. Then we see that $\#$ can be expressed as the composition

$$\# = \#_3 \circ \#_2 \circ \#_1,$$

where

$$\#_1: \underline{X} \mapsto \underline{X}^\circ + \underline{\tilde{U}}(\underline{X} - \underline{X}^\circ) =: \underline{X}_1,$$

$$\#_2: \underline{X}_1 \mapsto \underline{X}^\circ + \underline{\tilde{R}}(\underline{X}_1 - \underline{X}^\circ) =: \underline{X}_2$$

$$= \underline{X}^\circ + \underline{\tilde{R}}[(\underline{X}^\circ + \underline{\tilde{U}}(\underline{X} - \underline{X}^\circ)) - \underline{X}^\circ] \quad (\text{substitution})$$

$$= \underline{X}^\circ + \underline{\tilde{R}}[\underline{\tilde{U}}(\underline{X} - \underline{X}^\circ)] \quad (\text{Theorem A6.1.1 (xiii)})$$

$$= \underline{X}^\circ + \underline{\tilde{R}} \underline{\tilde{U}}(\underline{X} - \underline{X}^\circ) \quad (\text{tensor multiplication})$$

$$= \underline{X}^\circ + \underline{\tilde{F}}(\underline{X} - \underline{X}^\circ) \quad (\text{substitution}),$$

and

$$\begin{aligned}
\mathbb{F}_3: \mathbb{X}_2 &\mapsto \mathbb{F}(\overset{\circ}{\mathbb{X}}) + (\mathbb{X}_2 - \overset{\circ}{\mathbb{X}}) \\
&= \mathbb{F}(\overset{\circ}{\mathbb{X}}) + [\overset{\circ}{\mathbb{X}} + \overset{\circ}{\mathbb{F}}(\mathbb{X} - \overset{\circ}{\mathbb{X}}) - \overset{\circ}{\mathbb{X}}] \quad (\text{substitution}) \\
&= \mathbb{F}(\overset{\circ}{\mathbb{X}}) + \overset{\circ}{\mathbb{F}}(\mathbb{X} - \overset{\circ}{\mathbb{X}}) \quad (\text{Theorem A6.1.1 (xiii)}) \\
&= \mathbb{F}(\mathbb{X}).
\end{aligned}$$

By Theorem B1.2.4, $\mathbb{F}_3$ is a translation; and by Definition B1.2.2, $\mathbb{F}_2$ is a rotation about $\overset{\circ}{\mathbb{X}}$. Since translations and rotations are rigid deformations, any change of shape to be associated with $\mathbb{F}$ must be in $\mathbb{F}_1$. This leads us to

Definition B1.4.5. A homogeneous deformation of the form

$$\mathbb{F}(\mathbb{X}) = \overset{\circ}{\mathbb{X}} + \overset{\circ}{\mathbb{U}}(\mathbb{X} - \overset{\circ}{\mathbb{X}}), \quad \overset{\circ}{\mathbb{U}} \in \text{Psym } \mathcal{V}_3$$

is a pure stretch¹ from point $\overset{\circ}{\mathbb{X}}$.

With this terminology, we see that the calculation above has established part of the following famous theorem due to CAUCHY.

Theorem B1.4.17. Any homogeneous deformation can be regarded as a composition of the following three basic homogeneous deformations:

(1) pure stretch; (2) rotation; (3) translation,

¹Often, pure strain, pure deformation.

in any order.

Proof. We have already proven the theorem for the order listed above. We leave it as Exercise B1.4.18 to show that any other order is possible. Of course, we cannot expect the actual pure stretch, rotation, and translation needed to be independent of the ordering. In this regard, recall the other polar decomposition $\underline{\underline{\tilde{F}}} = \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{R}}}$. $\square$

While it is clear that in the decomposition of a homogeneous deformation into a pure stretch, a rotation, and a translation any change of shape involved must be associated with the pure stretch, there is the possibility that a pure stretch might include some rigid contributions. In order to see that this is not the case, we consider the typical pure stretch defined by

$$\underline{\underline{F}}(\underline{\underline{X}}) = \underline{\underline{X}} + \underline{\underline{U}}(\underline{\underline{X}} - \underline{\underline{X}}), \quad \underline{\underline{U}} \in \text{Psym } \mathcal{V}_3$$

in some detail.

First we note that in the present case,

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2.$$

$\therefore$ by Theorems B1.4.11 and B1.4.13 and Definition B1.4.4, if $\{\underline{\underline{e}}_{\langle 1 \rangle}^*, \underline{\underline{e}}_{\langle 2 \rangle}^*, \underline{\underline{e}}_{\langle 3 \rangle}^*\}$ are principal axes of strain and $\Lambda_1, \Lambda_2, \Lambda_3$ are the corresponding principal stretches, then

$$\underline{\underline{U}}^{\circ} = \Lambda_1 \underline{\underline{e}}_{\sim\langle 1 \rangle}^* \otimes \underline{\underline{e}}_{\sim\langle 1 \rangle}^* + \Lambda_2 \underline{\underline{e}}_{\sim\langle 2 \rangle}^* \otimes \underline{\underline{e}}_{\sim\langle 2 \rangle}^* + \Lambda_3 \underline{\underline{e}}_{\sim\langle 3 \rangle}^* \otimes \underline{\underline{e}}_{\sim\langle 3 \rangle}^* .$$

Secondly, we observe that in the degenerate special case of $\underline{\underline{U}}^{\circ} = \underline{\underline{I}}$, $\#(\mathbb{X}) = \mathbb{X}$ — the identity map. These observations lead us to

Definition B1.4.6. A pure stretch from $\mathbb{X}^{\circ}$ of the form

$$\begin{aligned} \#(\mathbb{X}) &= \mathbb{X} + [\underline{\underline{I}} + (\Lambda - 1) \underline{\underline{e}} \otimes \underline{\underline{e}}] (\mathbb{X} - \mathbb{X}^{\circ}) \\ &= \mathbb{X} + (\Lambda - 1) \underline{\underline{e}} \otimes \underline{\underline{e}} (\mathbb{X} - \mathbb{X}^{\circ}), \end{aligned}$$

where $\Lambda > 0$ and $|\underline{\underline{e}}| = 1$ is an extension from point $\mathbb{X}^{\circ}$ of amount Λ in the direction $\underline{\underline{e}}$.

As an easy consequence of Theorem B1.4.7, we have

Theorem B1.4.18. For the extension defined by

$$\#(\mathbb{X}) = \mathbb{X} + (\Lambda - 1) \underline{\underline{e}} \otimes \underline{\underline{e}} (\mathbb{X} - \mathbb{X}^{\circ}),$$

the stretch in the direction $\underline{\underline{e}}$ is

$$\Delta(\underline{\underline{e}}) = \Lambda.$$

To get a better feel for the above extension, we introduce

¹ We leave it as Exercise B1.4.19 to show that $\underline{\underline{U}}^{\circ} = \underline{\underline{I}} + (\Lambda - 1) \underline{\underline{e}} \otimes \underline{\underline{e}}$ with $\Lambda > 0$ and $|\underline{\underline{e}}| = 1$ is positive definite symmetric.

the reference frame $\{\overset{\circ}{X}; \underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ for $\mathcal{E}_3$, where $\underline{e}_{\langle 1 \rangle} = \underline{e}$. The displacement field associated with $\#$ is given by

$$\underline{u}(\overset{\circ}{X}) = \#(\overset{\circ}{X}) - \overset{\circ}{X} = (\Delta - 1) \underline{e} \otimes \underline{e} (\overset{\circ}{X} - \overset{\circ}{X}),$$

Letting

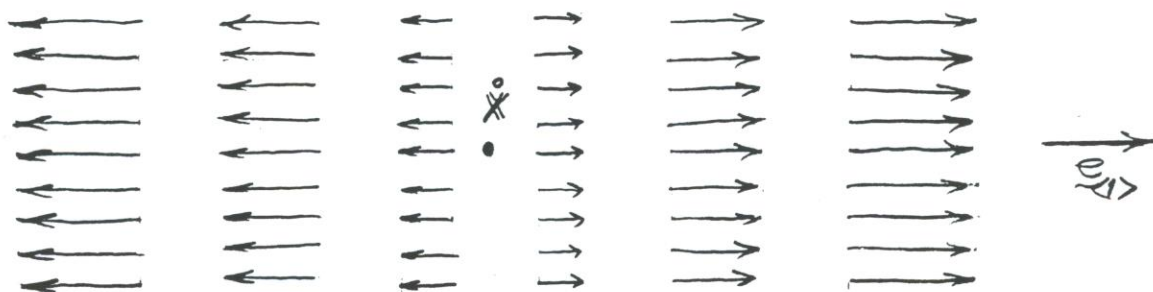
$$\overset{\circ}{X} - \overset{\circ}{X} = \underline{\hat{x}} = x^i \underline{e}_{\langle i \rangle},$$

we see that the displacement component fields¹ are given by²

$$\hat{u}^{\langle 1 \rangle}(\underline{\hat{x}}) = (\Delta - 1)x^1,$$

$$\hat{u}^{\langle 2 \rangle}(\underline{\hat{x}}) = \hat{u}^{\langle 3 \rangle}(\underline{\hat{x}}) = 0.$$

Thus, a point $\overset{\circ}{X}$ is displaced from $\overset{\circ}{X}$ in the $\underline{e}_{\langle 1 \rangle}$ direction by an amount proportional to its x^1 coordinate.



This makes it clear that there is no extraneous rigid portion contained in an extension. It involves nothing but a change of shape, except in the degenerate case of $\Delta = 1$ when it

¹ cf. the notation of § A6.6.

² Work this out as Exercise B1.4.20.

collapses to the identity transformation.

Returning to a general pure stretch, we get

Theorem B1.4.19. Any pure stretch from $\mathbb{X}$ can be regarded as the composition of three extensions from $\mathbb{X}$ in mutually orthogonal directions. Specifically, consider the pure stretch from $\mathbb{X}$ defined by

$$\mathbb{F}(\mathbb{X}) = \mathbb{X} + \underline{\underline{\mathbb{U}}} (\mathbb{X} - \mathbb{X}^{\circ}), \quad \underline{\underline{\mathbb{U}}} \in \text{Psym } \mathcal{V}_3,$$

where $\underline{\underline{\mathbb{U}}}$ has the spectral resolution

$$\underline{\underline{\mathbb{U}}} = \Lambda_1 \underline{\underline{\mathbb{e}}}_{\langle 1 \rangle}^* \otimes \underline{\underline{\mathbb{e}}}_{\langle 1 \rangle}^* + \Lambda_2 \underline{\underline{\mathbb{e}}}_{\langle 2 \rangle}^* \otimes \underline{\underline{\mathbb{e}}}_{\langle 2 \rangle}^* + \Lambda_3 \underline{\underline{\mathbb{e}}}_{\langle 3 \rangle}^* \otimes \underline{\underline{\mathbb{e}}}_{\langle 3 \rangle}^* .)^1$$

Then

$$\mathbb{F} = \mathbb{F}_3 \circ \mathbb{F}_2 \circ \mathbb{F}_1,$$

where

$$\mathbb{F}_i(\mathbb{X}) = \mathbb{X} + [\underline{\underline{\mathbb{I}}} + (\Lambda_i - 1) \underline{\underline{\mathbb{e}}}_{\langle i \rangle}^* \otimes \underline{\underline{\mathbb{e}}}_{\langle i \rangle}^*] (\mathbb{X} - \mathbb{X}^{\circ}).$$

Moreover, the ordering of the successive extensions is arbitrary.

Proof. Exercise B1.4.21. $\square$

¹We saw just before Definition B1.4.6 that $\{\underline{\underline{\mathbb{e}}}_{\langle 1 \rangle}^*, \underline{\underline{\mathbb{e}}}_{\langle 2 \rangle}^*, \underline{\underline{\mathbb{e}}}_{\langle 3 \rangle}^*\}$ are principal axes of strain and $\Lambda_1, \Lambda_2, \Lambda_3$ are the corresponding principal stretches for $\mathbb{F}$.

~~Supplementary~~ Reading

Same as for $\S$ B1.1

B1.5. The Geometry of General Deformations

Throughout this section, $\#$ is a deformation of a compact regular region $\mathcal{Q} \subset \mathbb{E}_3$. As usual

$$\underline{F} = D\#$$

denotes the deformation gradient.

The differentiability¹ of $\# \Rightarrow$

$$\#(\mathbb{X}) = \#(\dot{\mathbb{X}}) + \underline{F}(\dot{\mathbb{X}})(\mathbb{X} - \dot{\mathbb{X}}) + o(\mathbb{X} - \dot{\mathbb{X}}) \text{ as } \mathbb{X} \rightarrow \dot{\mathbb{X}},$$

Comparing this with the representation Theorem B1.4.1, we see that the deformation of a small neighborhood of point $\dot{\mathbb{X}}$ is approximated by a homogeneous deformation whose gradient is $\underline{F}(\dot{\mathbb{X}})$; and the smaller the neighborhood the better the approximation. In this sense, deformations are locally homogeneous.

The geometrical properties of homogeneous deformations were considered in the previous section. That study motivates

Definition B1.5.1: Let the deformation gradient $\underline{F} = D\#$ of the deformation $\#$ on $\mathcal{Q}$ have the polar decompositions

¹G. Definition A6.B.1.

$$\underline{F}(\mathbb{X}) = \underline{R}(\mathbb{X}) \underline{U}(\mathbb{X}) = \underline{V}(\mathbb{X}) \underline{R}(\mathbb{X})$$

at $\mathbb{X} \in \mathcal{Q}$, where $\underline{R}(\mathbb{X}) \in \text{Orth } \mathcal{V}_3$ and $\underline{U}(\mathbb{X}), \underline{V}(\mathbb{X}) \in \text{Psym } \mathcal{V}_3$.
Then we introduce the following terminology:

$\underline{R} : \mathcal{Q} \rightarrow \text{Orth } \mathcal{V}_3$ --- the rotation tensor field;

$\underline{U} : \mathcal{Q} \rightarrow \text{Psym } \mathcal{V}_3$ --- the right stretch tensor field;

$\underline{V} : \mathcal{Q} \rightarrow \text{Psym } \mathcal{V}_3$ --- the left stretch tensor field;

$\underline{C} : \mathcal{Q} \rightarrow \text{Psym } \mathcal{V}_3$ defined by $\underline{C}(\mathbb{X}) = [\underline{F}(\mathbb{X})]^T \underline{F}(\mathbb{X})$

--- the right deformation tensor¹ field;

and

$\underline{B} : \mathcal{Q} \rightarrow \text{Psym } \mathcal{V}_3$ defined by $\underline{B}(\mathbb{X}) = \underline{F}(\mathbb{X}) [\underline{F}(\mathbb{X})]^T$

--- the left deformation tensor² field.

It is important to understand that the above relations are definitions, and as such they are exact. The only approximation involved is that the geometrical interpretation of the deformation in terms of the approximating homogeneous deformation is valid only locally in general.

¹ Usually, right CAUCHY-GREEN tensor.

² Usually, left CAUCHY-GREEN tensor.

The first pair of the following identities explains why $\underline{C}$ and $\underline{B}$ are labeled the right and left deformation tensors, respectively.

Theorem B1.5.1. Refer to Definition B1.5.1. Then

$$(i) \quad \underline{C} = \underline{U}^2, \quad \underline{B} = \underline{V}^2;$$

$$(ii) \quad \underline{V} = \underline{R} \underline{U} \underline{R}^T, \quad \underline{U} = \underline{R}^T \underline{V} \underline{R};$$

and

$$(iii) \quad \underline{B} = \underline{R} \underline{C} \underline{R}^T, \quad \underline{C} = \underline{R}^T \underline{B} \underline{R}.$$

Proof. Exercise B1.5.1. $\square$

In order to gain additional insight into the geometrical significance of the above fields, we calculate the change in length of a "material curve" caused by a deformation. Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^n$; i.e.,

$$\mathcal{C} = \{ \mathbb{X} : \mathbb{X} = \underline{C}(\tau), \tau \in [0, 1] \},$$

where

$$\underline{C} \in C^1([0, 1], \mathbb{R}^n).$$

Now

$$\text{length } \mathcal{C} = \int_0^1 |\underline{C}'(\tau)| d\tau, \quad)^1$$

¹See pp. 276-280 of Vol. I of COURANT's Differential and Integral Calculus.

where c' denotes the ordinary derivative of c . The image of C under the deformation f is the curve

$$c = \{x : x = f(c(\tau)), \tau \in [0, 1]\},$$

and

$$\text{length } c = \int_0^1 |d'(\tau)| d\tau,$$

where

$$d = f \circ c.$$

By the chain rule,

$$Dd(\tau)[\alpha] = Df(c(\tau))[Dc(\tau)[\alpha]],$$

but

$$Dd(\tau)[\alpha] = \alpha d'(\tau) \text{ and } Dc(\tau)[\alpha] = \alpha c'(\tau),$$

so

$$d'(\tau) = F(c(\tau))c'(\tau).$$

$\therefore$

$$|d'(\tau)|^2 = [F(c(\tau))c'(\tau)] \cdot [F(c(\tau))c'(\tau)]$$

$$= c'(\tau) \cdot [F(c(\tau))]^T [F(c(\tau))c'(\tau)] \quad \left(\begin{array}{l} \text{definition} \\ \text{of transpose} \end{array} \right)$$

$$= c'(\tau) \cdot \{([F(c(\tau))]^T [F(c(\tau))])c'(\tau)\} \quad \left(\begin{array}{l} \text{definition of} \\ \text{tensor mult.} \end{array} \right)$$

$$= c'(\tau) \cdot C(c(\tau))c'(\tau) \quad \left(\text{definition of } C \right).$$

$$= c'(\tau) \cdot [U(c(\tau))U(c(\tau))]c'(\tau) \quad (\text{Theorem B1.5.1})$$

$$\begin{aligned}
&= \mathbf{C}'(\tau) \cdot \underline{\underline{U}}(\mathbf{C}(\tau)) [\underline{\underline{U}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)] \quad (\text{tensor multiplication}) \\
&= [\underline{\underline{U}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)] \cdot [\underline{\underline{U}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)] \quad (\text{definition of transpose and symmetry of } \underline{\underline{U}}) \\
&= |\underline{\underline{U}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)|^2 \quad (\text{definition of } |\cdot|).
\end{aligned}$$

Thus, we have

Theorem B1.5.2. Let

$$\mathcal{C} = \{ \mathbb{X} : \mathbb{X} = \mathbf{C}(\tau), \tau \in [0, 1] \}$$

and

$$\mathcal{C} = \{ \mathbb{X} : \mathbb{X} = \#(\mathbf{C}(\tau)), \tau \in [0, 1] \}$$

be a smooth curve in $\mathcal{Q}$ and its image¹ under the deformation $\#$. Then

$$\text{length } \mathcal{C} = \int_0^1 |\mathbf{C}'(\tau)| d\tau$$

and

$$\begin{aligned}
\text{length } \mathcal{C} &= \int_0^1 \sqrt{\mathbf{C}'(\tau) \cdot \underline{\underline{C}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)} d\tau \\
&= \int_0^1 |\underline{\underline{U}}(\mathbf{C}(\tau)) \mathbf{C}'(\tau)| d\tau.
\end{aligned}$$

¹Sometimes, we refer to the curve as a material curve since it is carried along with the body as it deforms; i.e., the curve always consists of the same "material points".

Thus, both the right stretch tensor field $\underline{U}$ and the right deformation tensor field $\underline{C}$ determine length changes exactly.

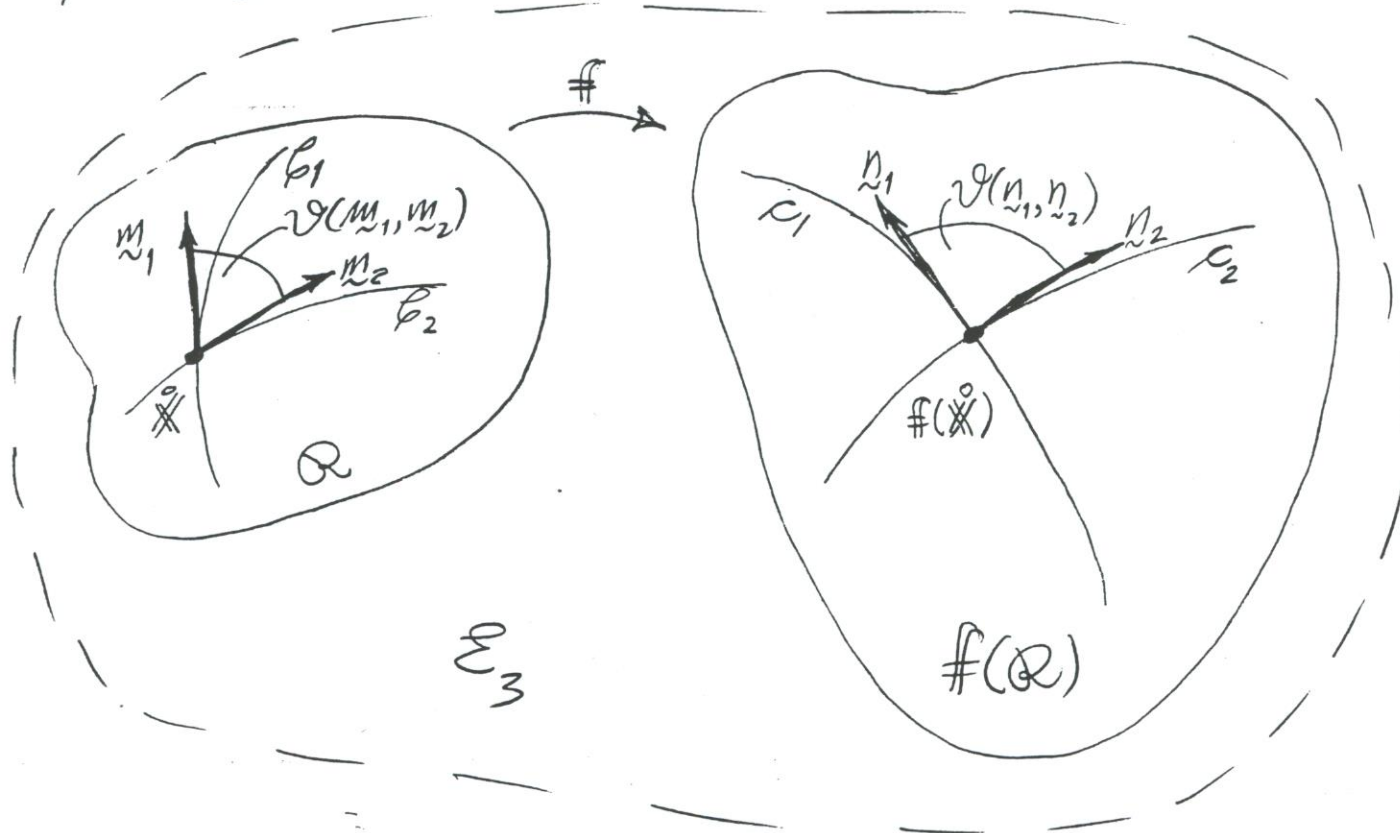
To investigate angle changes in the general case, consider two smooth curves in $\mathcal{Q}$

$$\mathcal{C}_i = \{ \mathbb{X} : \mathbb{X} = \mathcal{C}_i(\tau), \tau \in [0, 1] \}, \quad i \in \mathbb{I}_2$$

whose images under $\#$ are

$$\mathcal{C}_i = \{ \mathcal{X} : \mathcal{X} = \#(\mathcal{C}_i(\tau)), \tau \in [0, 1] \} \quad i \in \mathbb{I}_2.$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at $\mathbb{X}^\circ = \mathcal{C}_1(\bar{\tau}) = \mathcal{C}_2(\bar{\tau})$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at $\mathcal{X}^\circ = \#(\mathbb{X}^\circ)$.



A tangent¹ to C_i at $\overset{\circ}{X}$ is provided by $C_i'(\bar{c})$.
 Similarly, a tangent to C_i at $\#(\overset{\circ}{X})$ is given by

$$(\# \circ C_i)'(\bar{c}) = \underline{F}(\overset{\circ}{X}) C_i'(\bar{c});$$

$(\# \circ C_i)'$ was calculated on p. B1.5.4. Then the unit tangent to C_i at $\#(\overset{\circ}{X})$ in the direction defined by the parameterization is

$$\underline{n}_i = \frac{\underline{F}(\overset{\circ}{X}) C_i'(\bar{c})}{|\underline{F}(\overset{\circ}{X}) C_i'(\bar{c})|} \quad (\text{no sum})$$

By the calculations on pp. B1.5.4-5,

$$|\underline{F}(\overset{\circ}{X}) C_i'(\bar{c})| = \sqrt{C_i'(\bar{c}) \cdot \underline{C}(\overset{\circ}{X}) C_i'(\bar{c})} = |\underline{U}(\overset{\circ}{X}) C_i'(\bar{c})|.$$

Then

$$\begin{aligned} \underline{n}_1 \cdot \underline{n}_2 &= \frac{\underline{F}(\overset{\circ}{X}) C_1'(\bar{c})}{|\underline{U}(\overset{\circ}{X}) C_1'(\bar{c})|} \cdot \frac{\underline{F}(\overset{\circ}{X}) C_2'(\bar{c})}{|\underline{U}(\overset{\circ}{X}) C_2'(\bar{c})|} \\ &= \frac{C_1'(\bar{c}) \cdot \underline{C}(\overset{\circ}{X}) C_2'(\bar{c})}{|\underline{U}(\overset{\circ}{X}) C_1'(\bar{c})| |\underline{U}(\overset{\circ}{X}) C_2'(\bar{c})|} \\ &= \frac{\underline{U}(\overset{\circ}{X}) C_1'(\bar{c}) \cdot \underline{U}(\overset{\circ}{X}) C_2'(\bar{c})}{|\underline{U}(\overset{\circ}{X}) C_1'(\bar{c})| |\underline{U}(\overset{\circ}{X}) C_2'(\bar{c})|} \end{aligned}$$

¹See p. 263 of Vol. I of COURANT's Differential and Integral Calculus.

by computations similar to those on pp. B1.5.4-5 (see also p. B1.4.14b). Now

$$\mathcal{C}_i'(\vec{c}) = |\mathcal{C}_i'(\vec{c})| \underline{m}_i,$$

where $\underline{m}_i$ is the unit tangent to $\mathcal{C}_i$ at $\mathbb{X}$ in the direction defined by the parameterization, $\therefore$

$$\begin{aligned} \cos \mathcal{V}(\underline{n}_1, \underline{n}_2) &= \frac{\underline{U}(\mathbb{X}) \underline{m}_1}{|\underline{U}(\mathbb{X}) \underline{m}_1|} \cdot \frac{\underline{U}(\mathbb{X}) \underline{m}_2}{|\underline{U}(\mathbb{X}) \underline{m}_2|} \\ &= \frac{\underline{m}_1 \cdot \underline{C}(\mathbb{X}) \underline{m}_2}{\sqrt{\underline{m}_1 \cdot \underline{C}(\mathbb{X}) \underline{m}_1} \sqrt{\underline{m}_2 \cdot \underline{C}(\mathbb{X}) \underline{m}_2}} \end{aligned}$$

Hence, we have

Theorem B1.5.3. Let $\underline{m}_1$ and $\underline{m}_2$ be unit tangents at $\mathbb{X}$ to smooth curves in $\mathcal{Q}$ which intersect at $\mathbb{X}$. Let $\underline{n}_1$ and $\underline{n}_2$ be the unit tangents at $\#(\mathbb{X})$ to the image curves under the deformation $\#$. Let $\mathcal{V}(\underline{n}_1, \underline{n}_2)$ be the angle between $\underline{n}_1$ and $\underline{n}_2$. Then $\cos \mathcal{V}(\underline{n}_1, \underline{n}_2)$ is given by the formula above.

Thus, $\underline{U}$ and $\underline{C}$ also determine angle changes exactly:

Supplementary Reading
Same as for § B1.1

B1.6. Isochoric Deformations. The Localization Theorem

For the study of materials that are idealized as being "incompressible" we shall need

Definition B1.6.1. A deformation $\#$ on a region $\mathcal{Q}$ is isochoric if it preserves volumes; i.e., if

$$\int_{\#(R)} dV_x = \int_R dV_x \quad \forall \text{ regular regions } R \subset \mathcal{Q}.$$

It is crucial to realize that an isochoric deformation preserves the volumes of arbitrary subregions of $\mathcal{Q}$ as well as the volume of the full region $\mathcal{Q}$.

Isochoric deformations are easily characterized.

Theorem B1.6.1. A deformation $\#$ is isochoric iff

$$J(\mathbb{X}) := |\det F(\mathbb{X})| = 1 \quad \forall \mathbb{X} \in \mathcal{Q}. \quad)^1$$

Proof. By the change of variable theorem,²

¹As usual, $F = D\#$ is the deformation gradient.

²Theorem AG.10.2.

$$\int_{\#(R)} dV_x = \int_R J(\mathbb{X}) dV_x, \quad (*)$$

where R is any regular subregion of $\mathcal{Q}$;

Suppose that $J(\mathbb{X}) = 1 \quad \forall \mathbb{X} \in \mathcal{Q}$. Then $(*) \Rightarrow$

$$\int_{\#(R)} dV_x = \int_R dV_x,$$

and since this holds $\forall$ regular subregions $R \subset \mathcal{Q}$, $\#$ is isochoric.

Conversely, suppose that $\#$ is isochoric. Then

$$\int_{\#(R)} dV_x = \int_R dV_x \quad \forall \text{ regular } R \subset \mathcal{Q},$$

and $(*) \Rightarrow$

$$\int_R [J(\mathbb{X}) - 1] dV_x = 0 \quad \forall \text{ regular } R \subset \mathcal{Q}. \quad (**)$$

Since this holds $\forall$ regular subregions $R \subset \mathcal{Q}$, we conclude that

$$[J(\mathbb{X}) - 1] = 0 \quad \forall \mathbb{X} \in \mathcal{Q}.$$

For if not, then $\exists$ some point $\mathbb{X}^* \in \mathcal{Q} \ni$

$$[J(\mathbb{X}^*) - 1] \neq 0.$$

For definiteness, suppose that $[J(\mathbb{X}^*) - 1] > 0$. Then by continuity, $\exists \delta > 0 \ni$

$$[J(\mathbb{X}) - 1] > 0 \quad \forall \mathbb{X} \in B_\delta(\mathbb{X}^*) \cap \mathcal{R}. \quad)^{1,2}$$

Then

$$\int_{B_\delta(\mathbb{X}^*) \cap \mathcal{R}} [J(\mathbb{X}) - 1] dV_{\mathbb{X}} > 0,$$

$$B_\delta(\mathbb{X}^*) \cap \mathcal{R}$$

which contradicts $(**) \therefore$

$$J(\mathbb{X}) = 1 \quad \forall \mathbb{X} \in \mathcal{R}. \quad \square$$

The above argument is needed so often in continuum mechanics that we formalize it next for later use.

Theorem B1.6.2. (The Localization Theorem) Let $\underline{\mathbb{I}}^{(m)}$ be a continuous m^{th} -order tensor field on a regular region $\mathcal{R}$. If

$$\int_{\mathcal{R}} \underline{\mathbb{I}}^{(m)}(\mathbb{X}) dV_{\mathbb{X}} = \underline{\underline{0}} \quad \forall \text{ regular regions } \mathcal{R} \subset \mathcal{R},$$

¹ The kind of continuity argument needed here is worked out in detail in the proof of Theorem A5.8.2.

² Is the field defined by $[J(\mathbb{X}) - 1]$ continuous? — Exercise B1.6.1.

then

$$\int_{\sim}^{(m)} (\mathbb{X}) = 0 \quad \forall \mathbb{X} \in \mathcal{Q}.$$

Proof. For a zero-order tensor or scalar field, the result follows from the proof of the previous theorem. For higher-order tensor fields, it follows from the scalar case through the use of components,

E.g., suppose that $\underline{T} \in C^0(\mathcal{Q}, \text{Lin } \mathcal{V}_3)$ and

$$\int_R \underline{T}(\mathbb{X}) dV_x = 0 \quad \forall \text{ regular } R \subset \mathcal{Q}.$$

Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ be a basis for $\mathcal{V}_3$. Then

$$\begin{aligned} \left(\int_R \underline{T}(\mathbb{X}) dV_x \right) \underline{e}_i &= 0 \underline{e}_i \quad (\text{substitution}) \\ &= 0 \quad (\text{definition of } 0) \end{aligned}$$

$$\Rightarrow \int_R \underline{T}(\mathbb{X}) \underline{e}_i dV_x = 0 \quad (\text{linearity of } \int)$$

$$\begin{aligned} \Rightarrow \underline{e}_j \cdot \left(\int_R \underline{T}(\mathbb{X}) \underline{e}_i dV_x \right) &= \underline{e}_j \cdot 0 \quad (\text{substitution}) \\ &= 0 \quad (\text{linearity of inner product}) \end{aligned}$$

$$\Rightarrow \int_R \underline{e}_j \cdot \underline{T}(\mathbb{X}) \underline{e}_i dV_x = 0 \quad (\text{linearity of } \int).$$

$$\Rightarrow \int_R T_{ji}(\mathbb{X}) dV_x = 0 \quad (\text{definition of components}).$$

And since this must hold $\forall$ regular $R \subset \mathcal{R}$, we conclude that

$$T_{ji}(\mathbb{X}) = 0 \quad \forall \mathbb{X} \in \mathcal{R}.$$

$\therefore$

$$\underline{T}(\mathbb{X}) = T_{ji}(\mathbb{X}) \underline{e}^j \otimes \underline{e}^i = \underline{0} \quad \forall \mathbb{X} \in \mathcal{R}, \quad \square$$

Supplementary Reading

Same as for $\S$ B1.1

B1.7. Bodies and Configurations

As was noted in $\S B1.1$, the most evident property of bodies is that they occupy regions of E_3 . We formalize this in the following

Definition B1.7.1. A body B is a set $\{X, Y, \dots\}$, whose elements are called particles¹, together with a set of configurations² $\{\mathcal{X}, \mathcal{Y}, \dots\}$. A configuration $\mathcal{X}$ of a body B is a one-to-one correspondence between B and a compact regular region $\mathcal{X}(B) \subset E_3$; we write

$$x = \mathcal{X}(X), \quad X = \mathcal{X}^{-1}(x),$$

and we say that x is the place occupied by the particle X in the configuration $\mathcal{X}$ of the body B .

A body and its configurations are required to satisfy the following axioms:

(B1) If $\mathcal{X}$ and $\mathcal{Y}$ are configurations of a body B , then the point field

$$f = \mathcal{Y} \circ \mathcal{X}^{-1} : \mathcal{X}(B) \rightarrow \mathcal{Y}(B)$$

$$[f(x) = \mathcal{Y}(\mathcal{X}^{-1}(x)), \quad x \in \mathcal{X}(B)],$$

¹ Often, material points or body points.

² Sometimes, placements.

which maps $\mathcal{K}(\mathcal{B})$ onto $\Psi(\mathcal{B})$, is a deformation¹.

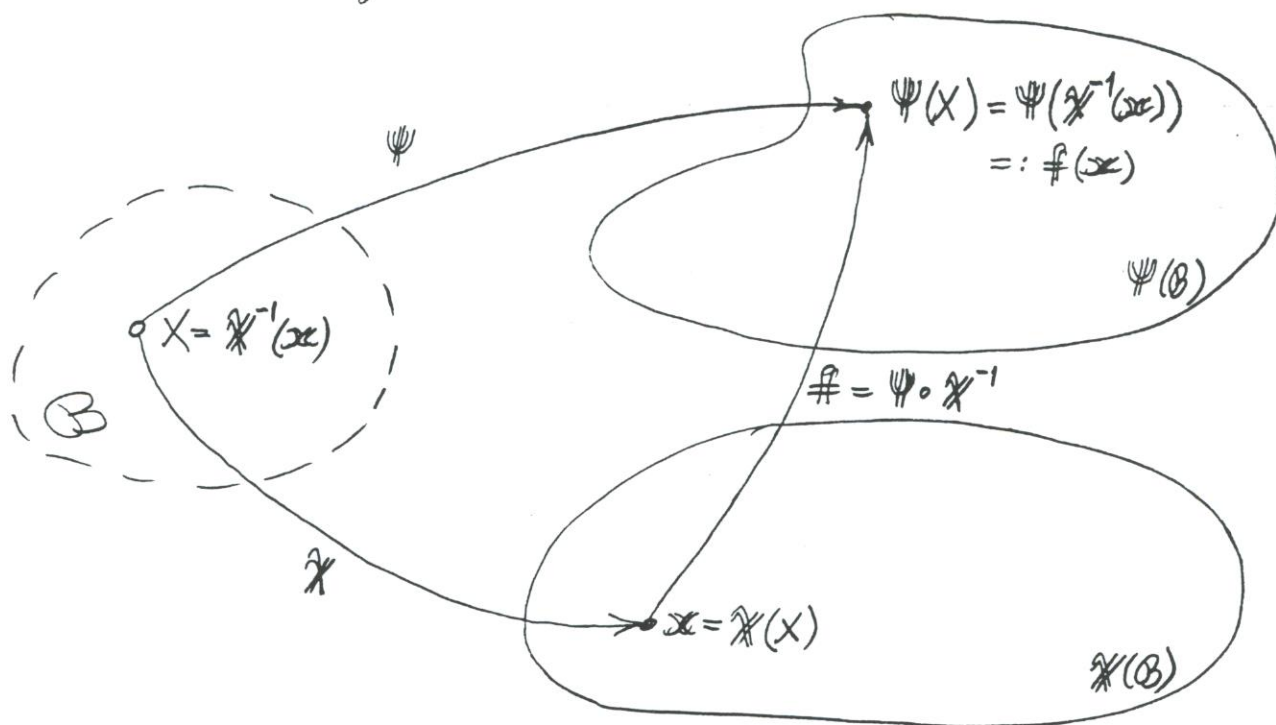
(B2) If $\mathcal{K}$ is a configuration of a body $\mathcal{B}$ and $\#$ is a deformation of the region $\mathcal{K}(\mathcal{B})$,² then the function

$$\Psi = \# \circ \mathcal{K} : \mathcal{B} \rightarrow \#(\mathcal{K}(\mathcal{B}))$$

$$[\Psi(X) = \#(\mathcal{K}(X)), X \in \mathcal{B}],$$

which associates particles in $\mathcal{B}$ with points in $\mathcal{E}_3$, is a configuration of $\mathcal{B}$.

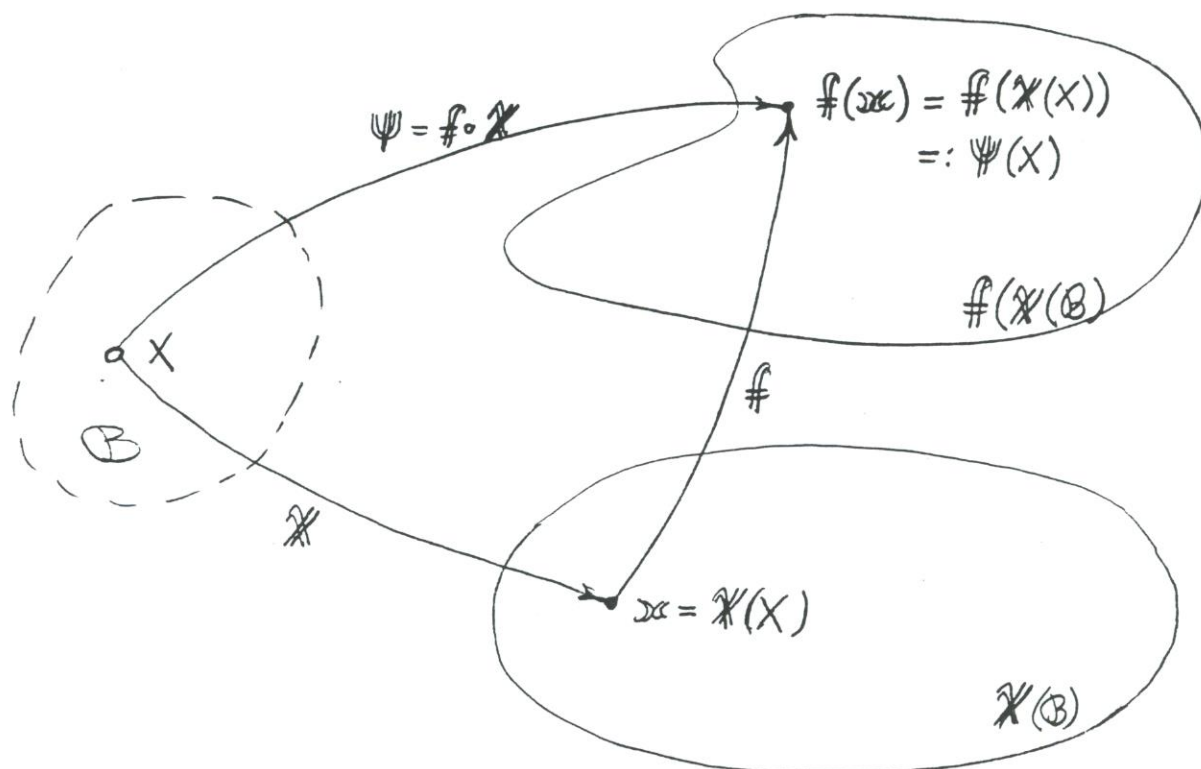
Axiom (B1) simply states that any two configurations of a body are connected by a deformation. The following figure makes this natural requirement even more transparent.



¹See Definition B1.1.1.

² At this stage in our development of continuum mechanics, we envision any deformation of $\mathcal{K}(\mathcal{B})$ as theoretically possible. Sometimes the internal constitution of a body is such that it is appropriate to restrict the allowed deformations to some special class such as rigid deformations, isochoric deformations, etc. The topic of internally constrained bodies will be taken up in Chapter B5

Axiom (B2) requires that if a body is deformed from one configuration it winds up in another configuration. The figure depicting this follows.



Of course, the concept of configuration is just our formal device for associating elements of a body (particles) with the points in space that they occupy. Often, particularly in the special branches of continuum mechanics, this association is left implicit, and particles are identified with the places that they occupy in some special configuration of the body.

The reader should not miss the fact that in Definition B1.7.1 the term deformation is used in the technical sense of Definition B1.1.1. Consequently, there

are certain, perhaps unexpected, implications. E.g., consider a particle of a body which has a place on the boundary (interior) of the region that the body occupies in some configuration; by Theorem B1.1.1, this particle cannot be "deformed" into the interior (boundary) of the region that the body would occupy in any other configuration.

Supplementary Reading
~~nothing more~~

Same as for § B1.1

B1.8. Motions. Material and Spatial Descriptions. Functions of Position and Time

In the previous section, we introduced a scheme to associate a body and the region of space that it occupies. However, in general, a body ~~does~~ not just sit in one region of space, but rather the body moves as time progresses. For this we need

Definition B1.8.1. A motion of a body B is a one-parameter family

$$\{\mathbb{X}_t : t \in \mathcal{Q}\}$$

(perhaps unbounded)

of its configurations.¹ The index set $\mathcal{Q}$ is an interval in $\mathbb{R}$, and the parameter t is called the time.

Thus, the point

$$x = \mathbb{X}_t(X)$$

is the place occupied at time t by the particle X in the motion $\{\mathbb{X}_t : t \in \mathcal{Q}\}$ of the body B . For given t , $\mathbb{X}_t$ is a configuration and its inverse is $\mathbb{X}_t^{-1}$. Hence,

$$X = \mathbb{X}_t^{-1}(x)$$

¹Subsequently, we shall have to make smoothness hypotheses beyond those implied here.

is the particle of body B which occupies the place x at time t in the motion $\{\mathbb{X}_t : t \in \mathcal{Q}\}$ of the body B ,

We will find it convenient to adopt the less precise but more common notation introduced next.

Definition B1.8.2. Given a motion $\{\mathbb{X}_t : t \in \mathcal{Q}\}$ of a body B , we write

$$x = \mathbb{X}_t(X) =: \mathbb{X}(X, t), \quad X = \mathbb{X}_t^{-1}(x) =: \mathbb{X}^{-1}(x, t);$$

and we denote the motion by $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$.

It must be emphasized that $\mathbb{X}^{-1}(x, t)$ denotes the inverse of the particular configuration $\mathbb{X}(\cdot, t) = \mathbb{X}_t$ evaluated at the point x . When $\mathbb{X}$ is viewed as a function of the pair (X, t) , i.e. as a function on $B \times \mathcal{Q}$, it generally does not have an inverse because it generally is not one-to-one. This is easily seen by the following reasoning. Suppose the function

$$\mathbb{X} : B \times \mathcal{Q} \rightarrow E_3$$

were one-to-one. Then¹

$$\mathbb{X}(X_1, t_1) = \mathbb{X}(X_2, t_2) \Rightarrow (X_1, t_1) = (X_2, t_2);$$

¹cf. Definition A1.2.3.

and, in particular,

$$\mathbb{X}(X, t_1) = \mathbb{X}(X, t_2) \Rightarrow t_1 = t_2.$$

Thus, a given particle would not be allowed to occupy the same place at two different times. Certainly, we must admit this possibility into our theory.

In order to define velocity and acceleration in the usual way as the first and second derivatives, respectively, of position w.r.t. time, we make the following assumption.

(M1) Given a motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{D}\}$ of a body $\mathcal{B}$, for each particle $X \in \mathcal{B}$ the function

$$\mathbb{X}(X, \cdot) \in C^2(\mathcal{D}, \mathcal{E}_3).$$

We will view (M1) as an axiom adjoined to Definition B1.B.1,

Now we are in position for

Definition B1.B.3. Given a motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{D}\}$ of a body $\mathcal{B}$, the velocity and acceleration of the particle X at time t are, respectively,

$$\underline{v}(X, t) = \mathbb{X}_{,2}(X, t)$$

and

$$\underline{a}(X, t) = \underline{v}_{,2}(X, t) = \mathbb{X}_{,22}(X, t).$$

Of course, the ordinary partial derivative $\mathbb{X}_{,2}$ is defined by

$$\mathbb{X}_{,2}(X,t) = \lim_{h \rightarrow 0} \frac{1}{h} [\mathbb{X}(X, t+h) - \mathbb{X}(X, t)]; \quad)^1$$

and because of the point difference in the numerator, it is indeed vector-valued as our notation for the velocity suggests.

It is sometimes, especially in fluid mechanics, more convenient to work with places and times rather than with particles and times. Accordingly, we introduce

$$\underline{v}_s(\mathbb{x}, t) = \underline{v}(\mathbb{X}^{-1}(\mathbb{x}, t), t) \quad [\underline{v}_s(\cdot, t) = \underline{v}(\cdot, t) \circ \mathbb{X}^{-1}(\cdot, t)]$$

and

$$\underline{a}_s(\mathbb{x}, t) = \underline{a}(\mathbb{X}^{-1}(\mathbb{x}, t), t) \quad [\underline{a}_s(\cdot, t) = \underline{a}(\cdot, t) \circ \mathbb{X}^{-1}(\cdot, t)].$$

Thus, $\underline{v}_s(\mathbb{x}, t)$ and $\underline{a}_s(\mathbb{x}, t)$ are, respectively, the velocity and the acceleration at time t of that particle which is at the place $\mathbb{x}$ at the time t .

We have not been precise about the domain of the functions $\underline{v}_s$ and $\underline{a}_s$, at least when they are viewed as functions of the pair $(\mathbb{x}, t)$. To this end, we make

¹cf. Theorem A5.12.8 and the discussion that precedes it.

Definition B1.8.4. Given a motion $\{X(\cdot, t): t \in \mathcal{Q}\}$ of a body $\mathcal{B}$, we write

$$\mathcal{B}_t = X(\mathcal{B}, t) \quad)^1$$

for the compact regular region of E_3 occupied by the body $\mathcal{B}$ at time t . The trajectory $\mathcal{I}$ of a body $\mathcal{B}$ under a motion $\{X(\cdot, t): t \in \mathcal{Q}\}$ is the set of pairs

$$\mathcal{I} = \{(x, t): x \in \mathcal{B}_t, t \in \mathcal{Q}\}.$$

Thus, the trajectory is the set of all pairs of places and times that correspond to the motion of the body.

Exercise B1.8.1. What is the distinction, if any, between the trajectory $\mathcal{I}$ and the set product $\mathcal{B}_t \times \mathcal{Q}$?

Now we formalize the scheme by which we introduced V_a and $\mathcal{Q}_a$ in

Definition B1.8.5. Let $\{X(\cdot, t): t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Given a function Θ on $\mathcal{B} \times \mathcal{Q}$, the function Θ_a is defined on the trajectory $\mathcal{I}$ by

$$\Theta_a(x, t) = \Theta(X^{-1}(x, t), t), \quad (x, t) \in \mathcal{I}.$$

¹ Cf. Definition A1.2.2.

Given a function Φ on $\mathcal{Y}$, the function Φ_m is defined on $\mathcal{B} \times \mathcal{Q}$ by

$$\Phi_m(X, t) = \Phi(\chi(X, t), t), \quad (X, t) \in \mathcal{B} \times \mathcal{Q}.$$

The pairs (X, t) and (χ, t) are called material and spatial¹ variables, respectively. Their respective uses are referred to as the material and the spatial descriptions. Accordingly, Φ_m is called a material field, while Φ_s is a spatial field.²

As expected, the material and spatial descriptions are related by

Theorem B1.8.1. Refer to Definition B1.8.5. Then

$$(\Phi_s)_m = \Phi \quad \text{and} \quad (\Phi_m)_s = \Phi.$$

Proof. Exercise B1.8.2, $\square$

Quite obviously then, we can always convert between the material and the spatial descriptions. However, there is a problem in general continuum mechanics due to the large

¹Often, EULERIAN variables even though they were introduced by DANIEL BERNOULLI and D'ALEMBERT.

²In the strict sense of SA6.4, Φ_m and Φ_s are not fields because of the presence of the time variable; and, of course, (cont.)

number of quantities which arise. It is cumbersome to have one symbol for a quantity when the material description is used and another symbol when the spatial description is used.¹ Yet mathematical precision often dictates, especially when differentiation is involved, that different symbols be utilized. We will compromise with the following convention.

First Notational Agreement, When the meaning is unambiguous, a given quantity will be denoted by a plain symbol, say Ψ , and it may be regarded as a function of either the material or the spatial variables. When more precision is demanded, we will denote the function by Ψ_m when the independent variables are the material variables (X, t) and by Ψ_s when the independent variables are the spatial variables (x, t) .

For example, we use the suggestive symbol $\underline{v}$ for the velocity. In the material description,

$$\underline{v} = \underline{v}_m(X, t);$$

(continuation of footnote from previous page) the elements of $\mathcal{B}$ are not points. This traditional usage (for continuum mechanics) should cause no confusion.

Two additional descriptions shall be introduced in subsequent sections.

and in the spatial description,

$$\underline{v} = \underline{v}_s(\underline{x}, t).$$

For corresponding X and $\underline{x}$, i.e. X and $\underline{x}$ related through the motion by

$$\underline{x} = \underline{\chi}(X, t),$$

the function values $\underline{v}_m(X, t)$ and $\underline{v}_s(\underline{x}, t)$ are equal; but, of course, the functions $\underline{v}_m$ and $\underline{v}_s$ are not equal — they do not even have the same domains.

As a rule, general studies in continuum mechanics are most easily formulated and carried out in the material description since in this description it is always clear what particles are under consideration. The spatial description is often better for the kinds of problems that arise in fluid mechanics.

An instructive and nontrivial exercise, which also will produce a useful result, is to relate the spatial acceleration field to the spatial velocity field. In order to do this, we assume

(M2) Any motion $\{\underline{\chi}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$ is $\exists$ the corresponding spatial velocity field $\underline{v}_s \in C^1(\mathcal{I}, \mathcal{V}_3)$.

As with (M1), we view (M2) as an axiom adjoined to Definition B1.B.1. Occasionally, (M2) is relaxed on various singular sets so as to permit the modeling of phenomena such as shock waves in terms of fields with jump discontinuities. Such instances will be made clear.

Since the trajectory

$$\gamma = \{(x, t); x \in B_t, t \in \mathcal{I}\}$$

is not an open subset of a linear space, Axiom (M2) needs some clarification. First, γ is a subset of $E_3 \times \mathbb{R}$ for which

$$d: (E_3 \times \mathbb{R}) \times (E_3 \times \mathbb{R}) \rightarrow \mathbb{R}$$

defined by

$$d((x, t), (y, s)) = |y - x| + |s - t|$$

is a suitable distance function¹. Hence, $E_3 \times \mathbb{R}$ is a metric space, and we can do limits in the usual way.

Viewing,

$$\gamma_\alpha: \gamma \subset E_3 \times \mathbb{R} \rightarrow \gamma_3$$

as a function of two variables, differentiation can be treated as in §A5.12. The fact that the first variable

¹Work this out as Exercise B1.B.3. See §A5.1 for background on metric spaces.

lives in a point space rather than a linear space can be handled as in §§ A6.4 and A6.6. By Theorem A5.12.6, $\underline{v}_s \in C^1(\mathcal{Y}, \mathcal{V}_3)$ is equivalent to the existence of continuous first partial derivatives $D_1 \underline{v}_s$ and $D_2 \underline{v}_s$ on $\mathcal{Y}$, and then

$$D_{\underline{v}_s}(\underline{x}, t)[(\underline{u}, \tau)] = D_1 \underline{v}_s(\underline{x}, t)[\underline{u}] + D_2 \underline{v}_s(\underline{x}, t)[\tau].$$

Of course, $D_{\underline{v}_s}(\underline{x}, t)$ is the linear operator on $\mathcal{V}_3 \times \mathbb{R}$ to $\mathcal{V}_3$ with the property that

$$\underline{v}_s(\underline{x} + \underline{u}, t + \tau) = \underline{v}_s(\underline{x}, t) + D_{\underline{v}_s}(\underline{x}, t)[(\underline{u}, \tau)] + o((\underline{u}, \tau)) \text{ as } (\underline{u}, \tau) \rightarrow (0, 0).$$

Since the second variable is real, it follows from Theorem A5.3.1¹ that

$$D_2 \underline{v}_s(\underline{x}, t)[\tau] = \underline{v}_{s,2}(\underline{x}, t) \tau,$$

where $\underline{v}_{s,2}$ is the ordinary partial derivative defined by

$$\underline{v}_{s,2}(\underline{x}, t) = \lim_{h \rightarrow 0} \frac{1}{h} [\underline{v}_s(\underline{x}, t+h) - \underline{v}_s(\underline{x}, t)].$$

Traditionally, the notations

$$D_1 \underline{v}_s = \nabla \underline{v}_s \quad \text{and} \quad \underline{v}_{s,2} = \frac{\partial \underline{v}_s}{\partial t}$$

are used in the continuum mechanics literature.

¹See also Theorem A5.11.8.

Finally, there is the technical detail that since the region B is closed, the trajectory $\mathcal{J}$ is not open in $E_3 \times \mathbb{R}$.¹ Note also that we have not been specific about the interval $\mathcal{J}$; it might be open or closed or neither one. As usual,² to say $\underline{v}_2 \in C'(\mathcal{J}, \mathcal{V}_3)$ means $\underline{v}_2 \in C^0(\mathcal{J}, \mathcal{V}_3)$, $\underline{v}_2 \in C'(\mathring{\mathcal{J}}, \mathcal{V}_3)$, and the derivative $D\underline{v}_2$ has a continuous extension from $\mathring{\mathcal{J}}$ to $\mathcal{J}$.

It is important that the student understand the above remarks concerning the differentiation of functions defined on a subset of $E_3 \times \mathbb{R}$, because most of the objects that arise in continuum mechanics are naturally functions of position and time.

We now return to the problem of relating $\underline{a}_s$ to $\underline{v}_s$. We start in the material description where it is easy to see what is happening. By Definition B1.8.3 and Theorem A5.3.1,

$$\underline{a}_{\underline{m}}(X, t) = \underline{v}_{\underline{m}, 2}(X, t) = D_2 \underline{v}_{\underline{m}}(X, t) [1].$$

By Definition B1.8.5,

$$\underline{v}_{\underline{m}}(X, t) = \underline{v}_s(\mathcal{X}(X, t), t).$$

¹ Show this as Exercise B1.8.4.

² Cf. the footnote on p. A6.10.2.

By Axioms (M1) and (M2), $\underline{v}_s$ and $\mathbb{X}(X, \cdot)$ are differentiable, as is the identity map $t \mapsto t$, on $\mathbb{R}$; $\therefore$ the chain rule in the form Theorem A5.13.3 may be applied to the composite function above. To sort this out, let us fix X and write

$$\text{where } \underline{v}_m(X, t) = \underline{v}_s(g_1(t), g_2(t)),$$

$$g_1(t) := \mathbb{X}(X, t), \quad g_2(t) := t.$$

Then by Theorem A5.13.3,

$$D_{\underline{v}_m}(X, t)[\alpha] = D_{\underline{v}_s}(g_1(t), g_2(t))[Dg_1(t)[\alpha]] + D_{\underline{v}_s}(g_1(t), g_2(t))[Dg_2(t)[\alpha]]$$

But by theorem A5.3.1,

$$Dg_1(t)[\alpha] = g_1'(t)\alpha = \mathbb{X}_{,2}(X, t)\alpha = \underline{v}_m(X, t)\alpha,$$

$$Dg_2(t)[\alpha] = g_2'(t)\alpha = \alpha,$$

$$\text{and } D_{\underline{v}_m}(X, t)[\alpha] = \underline{v}_{m,2}(X, t)\alpha = \underline{a}_m(X, t)\alpha,$$

$$D_{\underline{v}_s}(g_1(t), g_2(t))[\beta] = \underline{v}_{s,2}(g_1(t), g_2(t))\beta;$$

whence

$$\underline{a}_m(X, t)\alpha = D_{\underline{v}_s}(\mathbb{X}(X, t), t)[\underline{v}_m(X, t)\alpha] + \underline{v}_{s,2}(\mathbb{X}(X, t), t)\alpha.$$

Converting to the spatial variables, i.e. replacing X by $X = \mathcal{X}^{-1}(\mathcal{x}, t)$, we get EULER's famous

Theorem B1.B.2. The spatial acceleration field is given in terms of the spatial velocity field by

$$\underline{a}_s(\mathcal{x}, t) = D_1 \underline{v}_s(\mathcal{x}, t) [\underline{v}_s(\mathcal{x}, t)] + \underline{v}_{s,2}(\mathcal{x}, t), \quad (\mathcal{x}, t) \in \mathcal{I}.$$

This theorem makes it clear that in general

$$\underline{a}_s \neq \underline{v}_{s,2}.$$

Actually, this is obvious on somewhat simpler grounds. For by definition,

$$\underline{v}_{s,2}(\mathcal{x}, t) = \lim_{h \rightarrow 0} \frac{1}{h} [\underline{v}_s(\mathcal{x}, t+h) - \underline{v}_s(\mathcal{x}, t)];$$

and $\underline{v}_s(\mathcal{x}, t+h)$ is the velocity at time $t+h$ of that particle which is at the place $\mathcal{x}$ at time $t+h$, while $\underline{v}_s(\mathcal{x}, t)$ is the velocity at time t of that particle which is at the place $\mathcal{x}$ at time t . Thus, the difference quotient in the above definition generally will be comparing the velocities of two different particles. Consequently, $\underline{v}_{s,2}(\mathcal{x}, t)$ generally will not be the acceleration at time t of that particle which is at the place $\mathcal{x}$ at time t .

The partial derivative $D_1 \underline{v}_s$ occurs sufficiently often

to warrant

Definition B1.8.6. Given a motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$, the following fields are defined on the trajectory $\mathcal{J} = \{(\mathbf{x}, t) : \mathbf{x} \in \mathcal{B}_t, t \in \mathcal{I}\}$:

$\underline{\underline{L}} = D_1 \underline{\underline{v}}_s$ --- the spatial velocity gradient field¹;

$\underline{\underline{D}} = \text{sym } \underline{\underline{L}}$ --- the stretching field²;

$\underline{\underline{W}} = \text{skw } \underline{\underline{L}}$ --- the spin tensor field³;

$\underline{\underline{w}} = 2 \text{ax } \underline{\underline{W}}$ ⁴ --- the spin vector field⁵.

In terms of the spatial velocity gradient field, Theorem B1.8.2 appears as

¹ Sometimes, denoted by $\underline{\underline{G}}$.

² Often, the rate of deformation tensor field.

³ Often, the vorticity tensor field.

⁴ Recall from Theorem A3.6.9 that the axial vector of $\underline{\underline{W}}$, $\text{ax } \underline{\underline{W}}$, is defined by $(\text{ax } \underline{\underline{W}}) \times \underline{\underline{u}} = \underline{\underline{W}} \underline{\underline{u}} \quad \forall \underline{\underline{u}} \in \mathcal{V}_3$. Of course, the use of the cross product requires that $\mathcal{V}_3$ be oriented.

⁵ Often, the vorticity vector field.

$$\underline{a}_s = \underline{L} \underline{v}_s + \underline{v}_{s,2}.$$

The factor of 2 is inserted in the definition of the spin vector so that in accordance with Definition A6.6.1.

$$\underline{w} = \text{curl } \underline{v}_s.$$

Since, given a material field Θ on $\mathcal{B} \times \mathcal{Q}$, $\Theta_{,2}$ is the ordinary partial derivative w.r.t. the time variable holding the particle or material point fixed, $\Theta_{,2}$ often is referred to as the material time derivative of the material field Θ . The notation

$$\dot{\Theta} := \Theta_{,2}$$

is common as are the more hazardous notations

$$\frac{D\Theta}{Dt} := \Theta_{,2} \quad \text{and} \quad \frac{\partial \Theta}{\partial t} := \Theta_{,2}.$$

In the same way, given a spatial field Φ on $\mathcal{Y}$,

$$\dot{\Phi} := ((\Phi_m)')_s$$

is called the material time derivative of the spatial field Φ because the material point is held fixed in the time differentiation. Again, the notation

$$\frac{D\Phi}{Dt} := \dot{\Phi}$$

is common, but here the usual interpretation of $\frac{\partial \Phi}{\partial t}$ means something else.

Exercise B1.8.5. Show that material time differentiation commutes with both the material and the spatial "transformations" in the sense that, given a material field Θ and a spatial field Φ ,

$$(\dot{\Theta})_s = (\Theta_s)^\cdot =: \dot{\Theta}_s$$

and

$$(\dot{\Phi})_m = (\Phi_m)^\cdot =: \dot{\Phi}_m.$$

The calculation $((\Phi_m)^\cdot)_s$ is exactly the way that we got to $\dot{a}_s$ from $\dot{v}_s$ in the proof of Theorem B1.8.2; thus,

$$\dot{a}_s = \dot{v}_s = (D_1 v_s) v_s + v_{s,2}.$$

More generally, we have the results stated in the following exercise.

Exercise B1.8.6. Consider the smooth spatial fields $f \in C'(\mathcal{I}, \mathbb{R})$ and $\underline{f} \in C'(\mathcal{I}, \mathbb{V}_2)$. Show that

$$\dot{f} = D_1 f[v_s] + f_{,2} = (\nabla_1 f) \cdot v_s + f_{,2}$$

and

$$\dot{\underline{f}} = D_1 \underline{f}[v_s] + \underline{f}_{,2},$$

where the field ∇, f is defined by

$$D_t f(x, t)[a] = (\nabla_t f(x, t)) \cdot a$$

(cf. § A6.5).

Supplementary Reading
~~university~~

Same as for § B1.1.

B1.10. Rigid Motions

In § B1.2, rigid deformations were defined as distance preserving. Rigid motions are defined in the same way.

Definition B1.10.1. A motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$ is rigid if it preserves distance; i.e.,

$$|\mathbb{X}(X, t) - \mathbb{X}(Y, t)| = |\mathbb{X}(X, s) - \mathbb{X}(Y, s)| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t, s \in \mathcal{I}.$$

In order to use what we already know about rigid deformations, we recall that a motion is just a family of configurations and any two configurations are connected by a deformation. This leads us to

Theorem B1.10.1. For a given motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$, let $t_0 \in \mathcal{I}$ be an arbitrarily fixed reference time and write

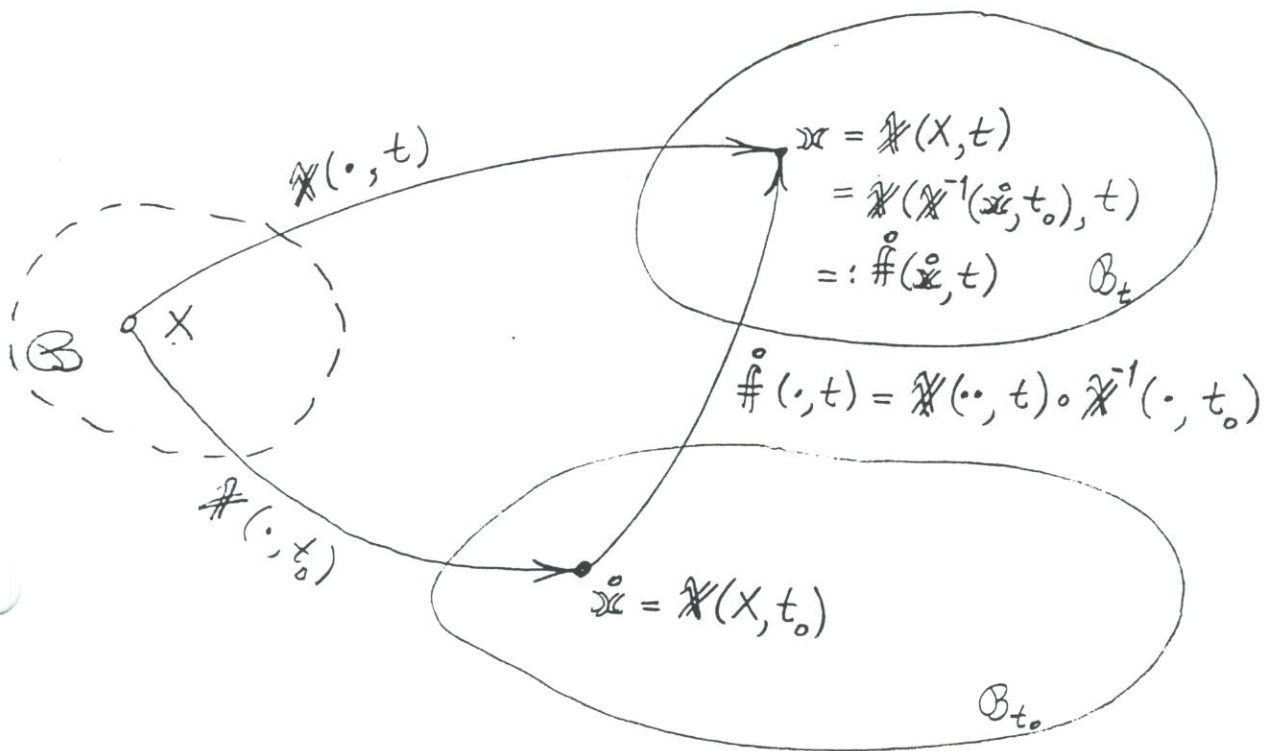
$$\tilde{x} = \mathbb{X}(X, t_0)$$

for the position of the particle $X \in \mathcal{B}$ at time t_0 . Then the configurations $\mathbb{X}(\cdot, t_0)$ and $\mathbb{X}(\cdot, t)$ are connected by the deformation

$$\mathring{\mathbb{F}}(\cdot, t) = \mathbb{X}(\cdot, t) \circ \mathbb{X}^{-1}(\cdot, t_0) : \mathcal{B}_{t_0} \rightarrow \mathcal{B}_t$$

$$[\mathring{\mathbb{F}}(\tilde{x}, t) = \mathbb{X}(\mathbb{X}^{-1}(\tilde{x}, t_0), t), \tilde{x} \in \mathcal{B}_{t_0}].$$

Proof. The theorem is simply Axiom (B1) of Definition B1.7.1 applied to the case at hand. The following diagram will help keep things straight. $\square$



The following result will come as no surprise. It could just as well have served as the definition of a rigid motion.

Theorem B1.10.2. A motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body B is rigid iff each member of the one-parameter family of deformations $\{\mathring{\mathbb{X}}(\cdot, t) : t \in \mathcal{I}\}$ is a rigid deformation.

Proof. For $Y \in B$, write

$$\overset{\circ}{y} = \mathcal{X}(Y, t_0) \quad (*)$$

and

$$y = \mathcal{X}(Y, t) = \overset{\circ}{F}(\overset{\circ}{y}, t) \quad (**)$$

Then

$$|\overset{\circ}{F}(\overset{\circ}{x}, t) - \overset{\circ}{F}(\overset{\circ}{y}, t)| = |\mathcal{X}(X, t) - \mathcal{X}(Y, t)| \quad (\#)$$

Suppose that the motion $\{\mathcal{X}(\cdot; t) : t \in \mathcal{Q}\}$ is rigid.
Then by Definition B1.10.1,

$$|\mathcal{X}(X, t) - \mathcal{X}(Y, t)| = |\mathcal{X}(X, t_0) - \mathcal{X}(Y, t_0)| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t \in \mathcal{Q}.$$

With (*) this becomes

$$|\mathcal{X}(X, t) - \mathcal{X}(Y, t)| = |\overset{\circ}{x} - \overset{\circ}{y}| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t \in \mathcal{Q},$$

and then (#) yields

$$|\overset{\circ}{F}(\overset{\circ}{x}, t) - \overset{\circ}{F}(\overset{\circ}{y}, t)| = |\overset{\circ}{x} - \overset{\circ}{y}| \quad \forall \overset{\circ}{x}, \overset{\circ}{y} \in \mathcal{B}_{t_0} \text{ and } \forall t \in \mathcal{Q}.$$

Thus, for each $t \in \mathcal{Q}$ the deformation $\overset{\circ}{F}(\cdot, t)$ is rigid.

Conversely, suppose that for each $t \in \mathcal{Q}$ the deformation $\overset{\circ}{F}(\cdot, t)$ is rigid. Then

$$|\overset{\circ}{x} - \overset{\circ}{y}| = |\overset{\circ}{F}(\overset{\circ}{x}, t) - \overset{\circ}{F}(\overset{\circ}{y}, t)| \quad \forall \overset{\circ}{x}, \overset{\circ}{y} \in \mathcal{B}_{t_0} \text{ and } \forall t \in \mathcal{Q},$$

which by (**) becomes

$$|\dot{x} - \dot{y}| = |\dot{x}(X, t) - \dot{x}(Y, t)| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t \in \mathcal{I}.$$

Since this holds $\forall t \in \mathcal{I}$, we can write

$$|\dot{x} - \dot{y}| = |\dot{x}(X, s) - \dot{x}(Y, s)| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall s \in \mathcal{I}.$$

These last two results combine to yield

$$|\dot{x}(X, t) - \dot{x}(Y, t)| = |\dot{x}(X, s) - \dot{x}(Y, s)| \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t, s \in \mathcal{I},$$

i.e., the motion $\{\dot{x}(\cdot, t) : t \in \mathcal{I}\}$ of $\mathcal{B}$ is rigid. $\square$

INSERT material on p. B1.10.4a

Theorem B1.10.2 provides us with the machinery to prove the following important result:

Theorem B1.10.4. Let $\{\dot{x}(\cdot, t) : t \in \mathcal{I}\}$ be a motion of a body $\mathcal{B}$. Then the following statements are equivalent:

- (i) the motion $\{\dot{x}(\cdot, t) : t \in \mathcal{I}\}$ is rigid;
- (ii) the velocity field of the motion $\{\dot{x}(\cdot, t) : t \in \mathcal{I}\}$ has the form

$$\dot{x}_m(X, t) = \dot{x}_m(Y, t) + \underline{\underline{\Omega}}(t) [\dot{x}(X, t) - \dot{x}(Y, t)] \quad \forall X, Y \in \mathcal{B} \text{ and } \forall t \in \mathcal{I},$$

or

$$\dot{x}_s(x, t) = \dot{x}_s(y, t) + \underline{\underline{\Omega}}(t) (x - y) \quad \forall (x, t), (y, t) \in \mathcal{I},$$

INSERT for p. B1.10.4

On noting that the reference time t_0 in Theorems B1.10.1 and 2 is arbitrary, we can get the following more symmetric result by simply setting $t_0 = s$.

Theorem B1.10.3. A motion $\{\mathcal{X}(\cdot, t) : t \in \mathcal{D}\}$ of a body $\mathcal{B}$ is rigid iff the configurations $\mathcal{X}(\cdot, t)$ and $\mathcal{X}(\cdot, s)$ are connected¹ by a rigid deformation $\forall t, s \in \mathcal{D}$.

¹G. Axiom (B1) of Definition B1.7.1.

Now return to p. B1.10.4.

where $\underline{\Omega}(t)$ is a uniform skew tensor on $\mathcal{V}_3$, moreover

$$\underline{\Omega} \in C^1(\mathcal{Q}, \text{Skw } \mathcal{V}_3);^1$$

(iii) the stretching field

$$\underline{D}_x = \text{sym } \underline{L}_x = \underline{Q} \text{ on } \mathcal{V}.$$

Proof. (i) $\Rightarrow$ (ii):² Suppose that the motion $\{\mathcal{X}(\cdot, t): t \in \mathcal{Q}\}$ is rigid. Then by the previous theorem for each $t \in \mathcal{Q}$ the associated deformation $\mathring{\mathcal{F}}(\cdot, t)$ is a rigid deformation. Consequently, by Theorem B1.2.1 $\exists$ a unique $\underline{Q}(t) \in \text{Orth } \mathcal{V}_3 \ni$

$$\mathring{\mathcal{F}}(\mathring{x}, t) = \mathring{\mathcal{F}}(\mathring{y}, t) + \underline{Q}(t)(\mathring{x} - \mathring{y}) \quad \forall \mathring{x}, \mathring{y} \in \mathcal{B}_{t_0} \text{ and } t \in \mathcal{Q}. (*)$$

But

$$\mathring{x} = \mathcal{X}(X, t_0) \text{ and } \mathring{y} = \mathcal{X}(Y, t_0),$$

so that

$$\mathring{\mathcal{F}}(\mathring{x}, t) = \mathcal{X}(X, t) \text{ and } \mathring{\mathcal{F}}(\mathring{y}, t) = \mathcal{X}(Y, t);$$

$\therefore (*)$ can be written as

¹We shall see in Theorem B1.10.5 that $\underline{\Omega}$ is uniquely determined by the rigid motion.

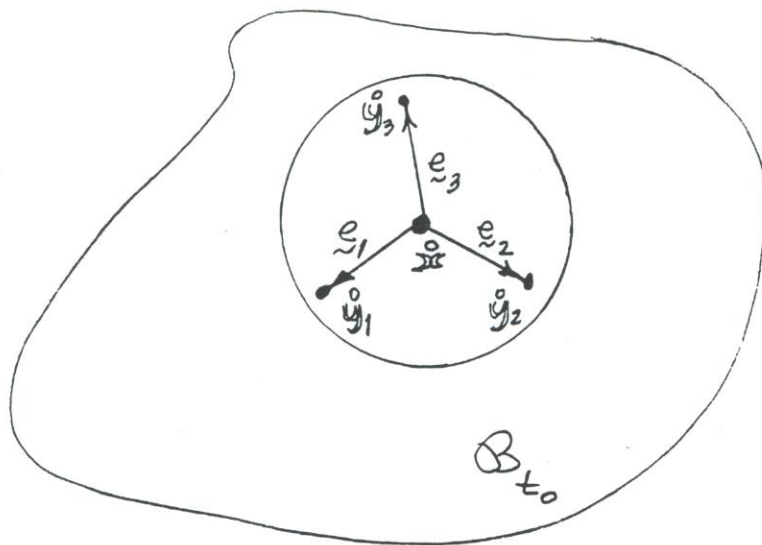
²Actually, Footnote 1 on p. B1.2.2 makes it clear that (i) $\Rightarrow$ (ii) without any smoothness assumptions on deformations.

$$\mathbb{X}(X, t) = \mathbb{X}(Y, t) + Q(t) (\tilde{x} - \tilde{y}) \quad (**)$$

Differentiation w.r.t. time will give the material velocity field, but we cannot carry this out until we establish that Q is differentiable. To this end, we momentarily choose the particle $X \ni$ the point $\tilde{x}$ is an interior point of B_{t_0} . Then we can always make three choices of the particle Y , say Y_1, Y_2, Y_3 , so that the corresponding points $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ are $\ni$ the three vectors

$$\tilde{x} - \tilde{y}_i =: \tilde{e}_i, \quad i \in \Pi_3$$

comprise a basis for V_3 .



Then $(**) \Rightarrow$

$$Q_{ij}(t) = \tilde{e}_i \cdot Q(t) \tilde{e}_j = \tilde{e}_i \cdot [\mathbb{X}(X, t) - \mathbb{X}(Y_j, t)] \quad)^1$$

¹ Cf. Theorem A3.2.2.

it follows from Axiom (M1) that the components

$$Q_{ij} \in C^2(\mathcal{Q}, \mathbb{R}).$$

$$Q(t) = Q_{ij}(t) \underline{e}^i \otimes \underline{e}^j,$$

$$Q \in C^2(\mathcal{Q}, \text{Orth} V_3).$$

(#)

Now we can return to (**), differentiate w.r.t. time, and use Definition B1.B.3 to get

(##)

$$\dot{V}_m(X, t) = \dot{V}_m(Y, t) + \dot{Q}(t) (\underline{x} - \underline{y}),$$

where $\dot{Q}$ denotes the ordinary derivative of $Q: \mathcal{Q} \rightarrow \text{Orth} V_3$.
Since $Q(t)$ is orthogonal, $Q^{-1} = Q^T$ and (**) $\Rightarrow$

$$(\underline{x} - \underline{y}) = [Q(t)]^T [\mathbb{X}(X, t) - \mathbb{X}(Y, t)],$$

which when combined with (##) yields

$$\dot{V}_m(X, t) = \dot{V}_m(Y, t) + \underline{\Omega}(t) [\mathbb{X}(X, t) - \mathbb{X}(Y, t)],$$

where

$$\underline{\Omega}(t) = \dot{Q}(t) [Q(t)]^T.$$

The smoothness claimed for $\underline{\Omega}$ follows from (#). To see that $\underline{\Omega}$ is skew, we differentiate the identity

$$\underline{Q}\underline{Q}^T = \underline{I}$$

to get

$$\dot{\underline{Q}}\underline{Q}^T + \underline{Q}\dot{\underline{Q}}^T = \underline{0}, \quad)^1$$

which by (i) and (iv) of Theorem A3.6.3 can be written as

$$\dot{\underline{Q}}\underline{Q}^T + (\dot{\underline{Q}}\underline{Q}^T)^T = \underline{0}$$

or

$$\underline{\Omega}^T = -\underline{\Omega}.$$

(ii) $\Rightarrow$ (i): Suppose that the velocity field of the motion $\{\mathcal{X}(\cdot; t): t \in \mathcal{D}\}$ of the body $\mathcal{B}$ has the form

$$\underline{v}_m(X, t) = \underline{v}_m(Y, t) + \underline{\Omega}(t) [\mathcal{X}(X, t) - \mathcal{X}(Y, t)],$$

where $\underline{\Omega} \in C^1(\mathcal{D}, \text{Skw } \mathcal{V}_3)$. Then for any $X, Y \in \mathcal{B}$

$$\begin{aligned} & \frac{d}{dt} \{ [\mathcal{X}(X, t) - \mathcal{X}(Y, t)] \cdot [\mathcal{X}(X, t) - \mathcal{X}(Y, t)] \} \\ &= 2 [\underline{v}_m(X, t) - \underline{v}_m(Y, t)] \cdot [\mathcal{X}(X, t) - \mathcal{X}(Y, t)] \quad)^2 \end{aligned}$$

¹ Here, we need the product rule $\underline{\dot{A}}\underline{B} = \underline{\dot{A}}\underline{B} + \underline{A}\underline{\dot{B}}$, which is covered by Theorem A5.6.7 (Show this as Exercise B1.10.1), and the commutation rule $(\underline{A}^T)^{\cdot} = (\underline{\dot{A}})^T$, the proof of which is left as Exercise B1.10.2 (A neat demonstration can be based on $\underline{u} \cdot \underline{A}^T \underline{v} = \underline{v} \cdot \underline{A} \underline{u}$; of course, straight forward use of the chain rule also works.)

² The product rule used here is covered by Theorem A5.6.5(i), and we are using Theorem A6.9.1 to differentiate the point difference.

$$\begin{aligned}
&= 2 \{ \underline{\Omega}(t) [\mathbb{X}(X,t) - \mathbb{X}(Y,t)] \} \cdot [\mathbb{X}(X,t) - \mathbb{X}(Y,t)] \quad (\text{substitution}) \\
&= 2(\underline{\Omega} \underline{d}) \cdot \underline{d} \quad (\text{substitution}) \\
&= 2(\underline{\Omega}^T \underline{d}) \cdot \underline{d} \quad (\text{definition of transpose}) \\
&= -2(\underline{\Omega} \underline{d}) \cdot \underline{d} \quad (\underline{\Omega} \in \text{Skw } \mathcal{V}_3) \\
&= 0 \quad (\alpha = -\alpha \Rightarrow \alpha = 0).
\end{aligned}$$

from Theorem A5.9.3

Since the interval $\mathcal{Q}$ is polygonally connected, we conclude that $|\mathbb{X}(X,t) - \mathbb{X}(Y,t)|$ is independent of t ; i.e.,

$$|\mathbb{X}(X,t) - \mathbb{X}(Y,t)| = |\mathbb{X}(X,s) - \mathbb{X}(Y,s)| \quad \forall t, s \in \mathcal{Q} \quad (\text{and } \forall X, Y \in \mathcal{B}).$$

$\therefore$ the motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ is rigid.

(ii) $\Rightarrow$ (iii) : Suppose that the velocity field of the motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ of the body $\mathcal{B}$ has the form

$$\underline{V}_x(x, t) = \underline{V}_x(y, t) + \underline{\Omega}(t)(x - y),$$

where $\underline{\Omega} \in C'(\mathcal{Q}, \text{Skw } \mathcal{V}_3)$. Then by Theorems A6.9.2, A5.5.2, and A5.5.3, for each $t \in \mathcal{Q}$, $\underline{V}_x$ is differentiable on $\hat{\mathcal{B}}_t$ and

$$D_{\underline{V}_x} = \underline{\Omega} \quad \text{on } \hat{\mathcal{B}}_t.$$

$\therefore$ since $\underline{Q}$ is skew,

$$\underline{D}_{\sim s} = \text{sym } \underline{D}_1 \underline{V}_s = \underline{Q} \quad \text{on } \hat{\mathcal{B}}_t;$$

and by the extension theorem (Theorem B1.2.2),

$$\underline{D}_{\sim s} = \underline{Q} \quad \text{on } \mathcal{B}_t;$$

and since this holds for each $t \in \mathcal{I}$,

$$\underline{D}_{\sim s} = \underline{Q} \quad \text{on } \mathcal{I}.$$

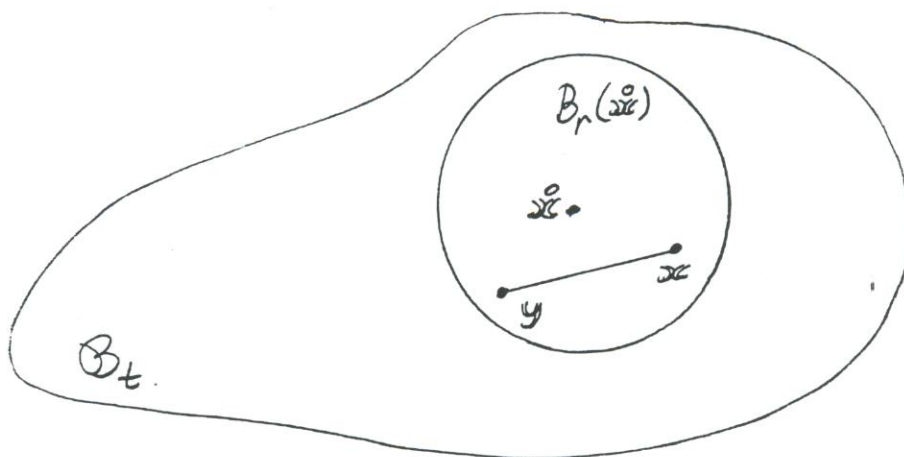
(iii) $\Rightarrow$ (ii):¹ Let $\bar{x}$ be the center of an open ball $B_r(\bar{x}) \subset \hat{\mathcal{B}}_t$. Let $x, y \in B_r(\bar{x})$. Then the line segment which connects x and y lies in $B_r(\bar{x})$; i.e.

$$\{y + \tau e : \tau \in [0, 1]\} \subset B_r(\bar{x}),$$

where

$$e = x - y.$$

¹This is the same kind of situation that we faced in proving (iii) $\Rightarrow$ (ii) of Theorem B1.2.1: We know something about the derivative of a field, and we wish to find a representation theorem for the field. The traditional and more elementary proof that (iii) $\Rightarrow$ (ii) (see e.g. §13 of GURTIN's "The linear theory of elasticity" in the Handbuch der Physik) requires that $\underline{V}_s \in C^2$. The proof given here is an expanded version of the demonstration in GURTIN and WILLIAMS, "A note on rigid fields", J. Elasticity 6 (1976) 331-333 (cf. also GURTIN's An Introduction to Continuum Mechanics).



Since $\underline{v}_s$ is differentiable, the mean-value theorem for derivatives (Theorem A5.9.1) $\Rightarrow$

$$\underline{v}_s(y+\underline{e}, t) - \underline{v}_s(y, t) = \int_0^1 D_1 \underline{v}_s(y + \tau \underline{e}, t) [\underline{e}] d\tau$$

or

$$\underline{v}_s(x, t) - \underline{v}_s(y, t) = \int_0^1 \underline{L}_s(y + \tau \underline{e}, t) \underline{e} d\tau. \quad (*)$$

Consequently,

$$[\underline{v}_s(x, t) - \underline{v}_s(y, t)] \cdot (x - y) = \int_0^1 \underline{e} \cdot \underline{L}_s(y + \tau \underline{e}) \underline{e} d\tau$$

$$= \int_0^1 \underline{e} \cdot \text{sym} \underline{L}_s(y + \tau \underline{e}) \underline{e} d\tau. \quad (\text{Exercise A3.15.1})$$

$$= 0 \quad (\text{sym} \underline{L}_s = \underline{0} \text{ by hypothesis}).$$

The result that here

$$[V_a(x, t) - V_a(y, t)] \cdot (x - y) = 0 \quad (**)$$

is sometimes expressed by saying that the field $V_a(\cdot, t)$ has the projection property (on $B_r(\bar{x})$). We capitalize on it as follows.

For each $y \in B_r(\bar{x})$ and $t \in \mathcal{Q}$, the scalar field α defined on $B_r(\bar{x})$ by

$$\alpha(x) = [V_a(x, t) - V_a(y, t)] \cdot (x - y)$$

is differentiable by Theorems A5.6.5 (i), A5.5.1, A5.5.2, A6.9.1, and A5.5.4 (ii), and with (**)

$$\nabla \alpha(x) = [D_1 V_a(x, t)]^T (x - y) + V_a(x, t) - V_a(y, t) = 0.$$

For each $x \in B_r(\bar{x})$ and $t \in \mathcal{Q}$, the vector field $\underline{a}$ defined on $B_r(\bar{x})$ by

$$\underline{a}(y) = [D_1 V_a(x, t)]^T (x - y) + V_a(x, t) - V_a(y, t)$$

is differentiable by Theorems A5.5.1, A5.6.6, A5.5.2, A6.9.1, A5.5.4 (ii), and

$$D \underline{a}(y) = -[D_1 V_a(x, t)]^T - D_1 V_a(y, t) = 0$$

or

$$-[L_a(x, t)]^T = L_a(y, t).$$

But under the present hypotheses, $\underline{L}_\alpha$ is skew; and $\therefore$

$$\underline{L}_\alpha(x, t) = \underline{L}_\alpha(y, t) \quad \forall x, y \in B_r(\bar{x}) \text{ and } \forall t \in \mathcal{Q}.$$

Thus, for each $t \in \mathcal{Q}$, $\underline{L}_\alpha(\cdot, t)$ is uniform on the ball $B_r(\bar{x})$, and we write

$$\underline{L}_\alpha(x, t) = \underline{L}_\alpha(y, t) =: \underline{\Omega}(t). \quad (\#)$$

Clearly,

$$\underline{\Omega} \in C^1(\mathcal{Q}, \text{Skw } V_3). \quad (\#\#)$$

(*) , (#), and (##) yield the desired representation

$$\underline{V}_\alpha(x, t) = \underline{V}_\alpha(y, t) + \underline{\Omega}(t)(x - y) \text{ on } B_r(\bar{x}). \quad (d)$$

The extension to B_t is not difficult.

First, for arbitrarily fixed $t \in \mathcal{Q}$, we differentiate (d) to get

$$\underline{L}_\alpha = D_1 \underline{V}_\alpha = \underline{\Omega} \text{ on } B_r(\bar{x}).$$

Then since $\underline{\Omega}$ is uniform, $\underline{L}_\alpha$ is differentiable on $B_r(\bar{x})$ and

$$D_1 \underline{L}_\alpha = \underline{\Omega} \text{ on } B_r(\bar{x}), \quad)^1$$

In particular, $D_1 \underline{L}_\alpha(\bar{x}, t) = \underline{\Omega}$; but $\bar{x}$ is an arbitrary

¹cf. Theorem A5.5.2.

point in $\mathring{B}_t$, and $\therefore$

$$D_1 \underline{L}_a = \underline{\Omega} \text{ on } \mathring{B}_t.$$

By the connectivity of $\mathring{B}_t$, it follows from Theorem A5.9.3 that $\underline{L}_a$ is uniform on $\mathring{B}_t$; and since $\underline{L}_a(\dot{x}, t) = \underline{\Omega}(t)$,

$$\underline{L}_a(x, t) = D_1 \underline{V}_a(x, t) = \underline{\Omega}(t) \text{ on } \mathring{B}_t.$$

Again by the connectivity, it follows from Theorem A6.6.1 that

$$\underline{V}_a(x, t) = \underline{V}_a(y, t) + \underline{\Omega}(t)(x - y) \quad \forall x, y \in \mathring{B}_t.$$

By the extension theorem (Theorem B1.2.2), $\mathring{B}_t$ can be replaced by B_t ; and then since this result holds for every $t \in \mathcal{I}$, we finally have

$$\underline{V}_a(x, t) = \underline{V}_a(y, t) + \underline{\Omega}(t)(x - y) \quad \forall (x, t), (y, t) \in \mathcal{I}. \quad \square$$

In the spirit of Theorems B1.2.3 and B1.4.2, we have

Theorem B1.10.5. The skew tensor function $\underline{\Omega} \in C^1(\mathcal{I}, \text{Skew } \mathcal{V}_3)$ in the representations

$$\underline{V}_m(X, t) = \underline{V}_m(Y, t) + \underline{\Omega}(t)[\mathcal{K}(X, t) - \mathcal{K}(Y, t)]$$

and

$$\underline{V}_a(x, t) = \underline{V}_a(y, t) + \underline{\Omega}(t)(x - y)$$

of the velocity field of a rigid motion $\{X(\cdot, t) : t \in \mathcal{Q}\}$ of a body $\mathcal{B}$ is uniquely determined by the motion.

Proof. Exercise B1.10.3. $\square$

As an immediate consequence of the definition of the axial vector¹ of a skew tensor, we have

Theorem B1.10.6. For a rigid motion, the representations

$$\underline{v}_m(X, t) = \underline{v}_m(Y, t) + \underline{\Omega}(t) [X(X, t) - X(Y, t)]$$

and

$$\underline{v}_s(x, t) = \underline{v}_s(y, t) + \underline{\Omega}(t) (x - y)$$

have the alternative forms

$$\underline{v}_m(X, t) = \underline{v}_m(Y, t) + \underline{\omega}(t) \times [X(X, t) - X(Y, t)]$$

and

$$\underline{v}_s(x, t) = \underline{v}_s(y, t) + \underline{\omega}(t) \times (x - y),$$

where $\underline{\omega} = \text{ax } \underline{\Omega}$. $\underline{\Omega}$ and $\underline{\omega}$ are called the angular velocity tensor and the angular velocity vector, respectively, of the rigid motion.

Differentiation w.r.t. time of the representation

¹Theorem A3.6.9. And for this, of course, we must assume that the translation space $\mathcal{V}_3$ of $\mathcal{E}_3$ has been oriented.

$$\underline{v}_m(X, t) = \underline{v}_m(Y, t) + \underline{\omega}(t) \times [\underline{x}(X, t) - \underline{x}(Y, t)]$$

leads to

Theorem B1.10.7. Let $\underline{\omega} \in C'(\mathcal{D}, \mathbb{R}^3)$ be the angular velocity vector of a rigid motion $\{\underline{x}(\cdot, t) : t \in \mathcal{D}\}$ of a body B . Then the acceleration field of the rigid motion is given by

$$\underline{a}_m(X, t) = \underline{a}_m(Y, t) + \underline{\alpha}(t) \times [\underline{x}(X, t) - \underline{x}(Y, t)] + \underline{\omega}(t) \times \{ \underline{\omega}(t) \times [\underline{x}(X, t) - \underline{x}(Y, t)] \}$$

$$\forall X, Y \in B \text{ and } \forall t \in \mathcal{D}$$

or

$$\underline{a}_a(\underline{x}, t) = \underline{a}_a(\underline{y}, t) + \underline{\alpha}(t) \times (\underline{x} - \underline{y}) + \underline{\omega}(t) \times [\underline{\omega}(t) \times (\underline{x} - \underline{y})] \quad \forall (\underline{x}, t), (\underline{y}, t) \in \mathcal{I},$$

where $\underline{\alpha} := \dot{\underline{\omega}}$ is the angular acceleration vector of the rigid motion.

Proof, Exercise B.1.10.4. $\square$

Of course, the above result must also follow from application of Theorem B1.8.2,

$$\underline{a}_a(\underline{x}, t) = D_1 \underline{v}_a(\underline{x}, t) [\underline{v}_a(\underline{x}, t)] + \underline{v}_{a,2}(\underline{x}, t), \quad (*)$$

to the representation

$$\underline{v}_a(\underline{x}, t) = \underline{v}_a(\underline{y}, t) + \underline{\omega}(t) \times (\underline{x} - \underline{y}). \quad (**)$$

This is not a very economical approach, but it does provide

B1.10.17

a good test of one's understanding of the spatial description.
Let us work it through.

By Theorems A5.5.1, A5.5.2, A5.6.5 (iii), A6.9.1, and A5.5.4 (ii), $(**) \Rightarrow$

$$D_1 \underline{v}_s(x, t) [\underline{z}] = \underline{\omega}(t) \times \underline{I} \underline{z} = \underline{\omega}(t) \times \underline{z}.$$

Whence,

$$D_1 \underline{v}_s(x, t) [\underline{v}_s(y, t)] = \underline{\omega}(t) \times [\underline{v}_s(y, t) + \underline{\omega}(t) \times (x - y)].$$

Here, of course, we have used the fact that $\underline{v}_s(y, t)$ is independent of x .

Similarly, $(**) \Rightarrow$

$$\underline{v}_{s,2}(x, t) = \underline{v}_{s,2}(y, t) + \underline{\omega}(t) \times (x - y).$$

And, of course, here x and y are held fixed in the $(\cdot)_{,2}$ computation.

Placing these results in $(*)$, we get

$$\underline{a}_s(x, t) = \underline{\omega}(t) \times \underline{v}_s(y, t) + \underline{\omega}(t) \times [\underline{\omega}(t) \times (x - y)] + \underline{v}_{s,2}(y, t) + \underline{\omega}(t) \times (x - y).$$

Evaluation at $x = y$ yields

$$\underline{a}_s(y, t) = \underline{\omega}(t) \times \underline{v}_s(y, t) + \underline{v}_{s,2}(y, t),$$

and $\therefore$

$$\underline{a}_s(x, t) = \underline{a}_s(y, t) + \dot{\underline{\omega}}(t) \times (x - y) + \underline{\omega}(t) \times [\underline{\omega}(t) \times (x - y)],$$

which is in agreement with Theorem B1.10.7.

Supplementary Reading

Same as for § B1.1

B1.12. Isochoric Motions . Parts of a Body

In § B1.6, isochoric deformations were defined as volume preserving. In order to extend this notion to the motion of bodies, it is convenient to introduce

Definition B1.12.1. A subset (not necessarily proper) $\mathcal{P}$ of a body $\mathcal{B}$ is a part of $\mathcal{B}$ if $\mathcal{P}$ itself satisfies all the conditions of being a body. If $\mathcal{P}$ is a part and $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ is a motion of a body $\mathcal{B}$, then

$$\mathcal{P}_t = \mathbb{X}(\mathcal{P}, t)$$

denotes the compact regular region of E_3 occupied by the part $\mathcal{P}$ at time t ,

Now Definition B1.6.1 suggests

Definition B1.12.2. A motion $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$ is isochoric if it preserves volume; i.e.,

$$\int_{\mathcal{P}_t} dV = \int_{\mathcal{P}_s} dV \quad \forall \text{ parts } \mathcal{P} \subset \mathcal{B} \text{ and } \forall t, s \in \mathcal{I}.$$

As in our investigation of rigid motions in § B1.10, we pick a reference time t_0 and use the deformation

$$\tilde{\mathbb{X}}(\cdot, t) = \mathbb{X}(\cdot, t) \circ \mathbb{X}^{-1}(\cdot, t_0)$$

as a vehicle for bringing to bear what we know about isochoric deformations.

Theorem B1.12.1. A motion $\{\mathbb{X}(\cdot, t): t \in \mathcal{Q}\}$ of a body $\mathcal{B}$ is isochoric iff each member of the one-parameter family of deformations $\{\mathbb{F}(\cdot, t): t \in \mathcal{Q}\}$ of Theorem B1.10.1 is isochoric.

Proof. Exercise B1.12.1. $\square$

Combining this result with Theorem B1.6.1, we obtain

Theorem B1.12.2. A motion $\{\mathbb{X}(\cdot, t): t \in \mathcal{Q}\}$ of a body $\mathcal{B}$ is isochoric iff

$$|\det D_{\mathbb{F}}(\mathbb{X}_0, t)| = 1 \quad \forall (\mathbb{X}_0, t) \in \mathcal{B}_{t_0} \times \mathcal{Q}.$$

We will use this theorem in §B1.14 to get at more useful characterizations of isochoric motions.

Of course, analogous to Theorem B1.10.3, we have the alternative more symmetric formulation provided by

Theorem B1.12.2. A motion $\{\mathbb{X}(\cdot, t): t \in \mathcal{Q}\}$ of a body $\mathcal{B}$ is isochoric iff the configurations $\mathbb{X}(\cdot, t)$ and $\mathbb{X}(\cdot, s)$ are connected by an isochoric deformation $(\forall t, s \in \mathcal{Q})$.

Supplementary Reading
Same as for §B1.1

B1.13. Reference Configurations. The Transport Theorem

While the treatment of motions and most other basic aspects of continuum mechanics in terms of abstract particles is conceptually very clear and while the spatial description is particularly useful in fluid mechanics, it is often convenient (especially in solid mechanics) to choose one configuration of a body as a reference configuration. In solid mechanics there is usually an obvious natural choice. Then we can view every other configuration as being obtained from the reference configuration by a deformation. This allows us to make use of the results of §§ B1.1-6. In fact, we used a special case of this procedure in §§ B1.10 and 12 in our studies of rigid and isochoric motions.

Let $\mathcal{U}$ be a configuration of a body $\mathcal{B}$. For $X \in \mathcal{B}$, we write

$$\mathbb{X} = \mathcal{U}(X), \quad X = \mathcal{U}^{-1}(\mathbb{X}).$$

Thus, $\mathbb{X}$ is the place occupied by the particle X in the configuration $\mathcal{U}$ of the body $\mathcal{B}$. If $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ is a motion of $\mathcal{B}$ so that

$$\mathcal{X} = \mathbb{X}(X, t)$$

is the place occupied by the particle X at time t , then

$$\mathcal{X} = \mathcal{X}(\mathcal{U}^{-1}(\mathcal{X}), t) =: \mathcal{X}_\kappa(\mathcal{X}, t)$$

is the position at time t of the particle that occupies the place $\mathcal{X}$ in the reference configuration $\mathcal{U}$.
By Axiom (B1) of Definition B1.7.1,

$$\mathcal{X}_\kappa(\cdot, t) = \mathcal{X}(\cdot, t) \circ \mathcal{U}^{-1} : \mathcal{U}(\mathcal{B}) \rightarrow \mathcal{E}_3$$

is a deformation on the region $\mathcal{U}(\mathcal{B})$ for each $t \in \mathcal{I}$.
Formally, we make

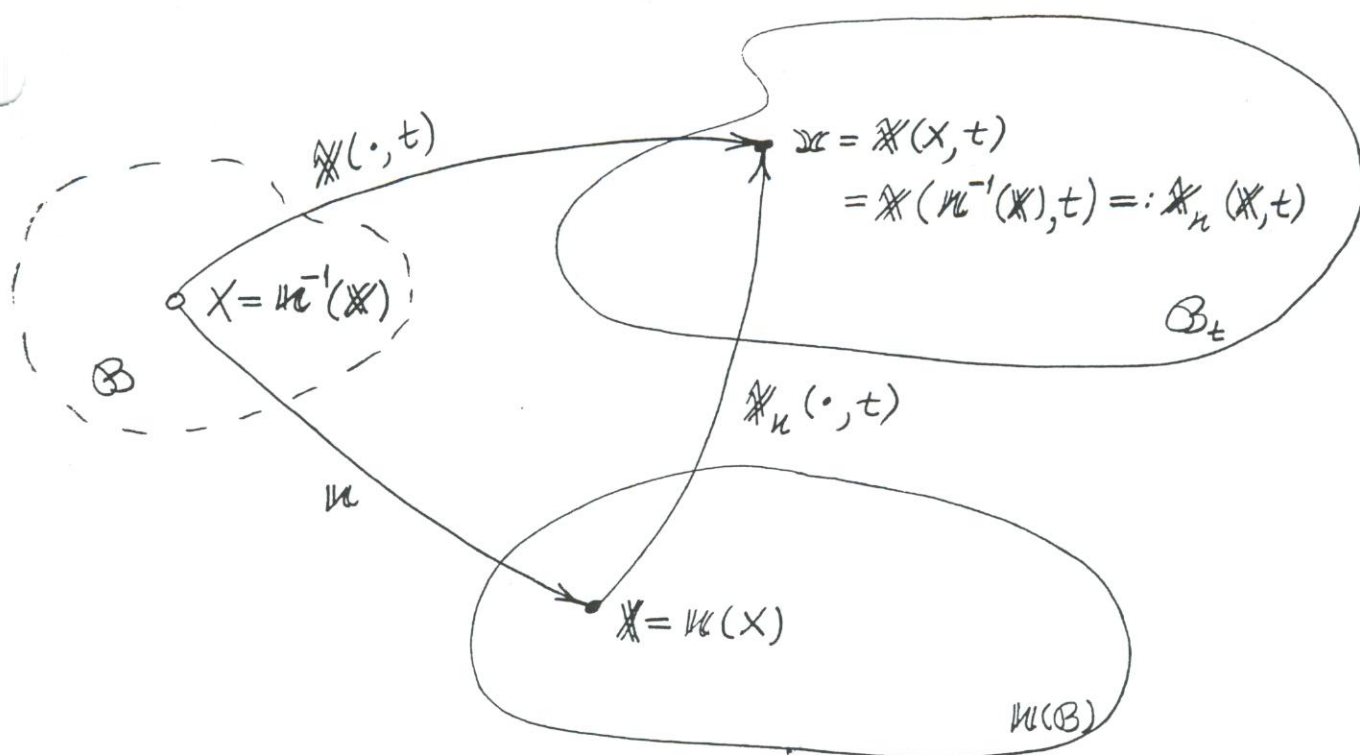
Definition B1.13.1. Given a configuration $\mathcal{U}$ and a motion $\{\mathcal{X}(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$, the one-parameter family of deformations $\{\mathcal{X}_\kappa(\cdot, t) : t \in \mathcal{I}\}$ on $\mathcal{U}(\mathcal{B})$.
defined by

$$\mathcal{X}_\kappa(\cdot, t) = \mathcal{X}(\cdot, t) \circ \mathcal{U}^{-1} \quad [\mathcal{X}_\kappa(\mathcal{X}, t) = \mathcal{X}(\mathcal{U}^{-1}(\mathcal{X}), t), \mathcal{X} \in \mathcal{U}(\mathcal{B})]$$

is the motion of $\mathcal{B}$ relative to the reference configuration $\mathcal{U}$.

The following figure makes it easy to keep the concept of a reference configuration straight.

¹ Sometimes, reference placement, and then the deformation $\mathcal{X}_\kappa$ is referred to as a transplacement.



Often, as in Theorem B1.10.1, it is assumed that $\exists$ a reference time $t_0 \ni \kappa = \mathbb{X}(\cdot, t_0)$. However, without explicit mention to the contrary, we will not assume that the body ever actually occupies the reference configuration being employed to study a given motion.

We have already noted that for each $t \in \mathcal{Q}$ $\mathbb{X}_\kappa(\cdot, t)$ is a deformation; $\therefore$, we can read off the following result from Definition B1.1.1 and Theorem B1.1.1.

Theorem B1.13.1. Let $\{\mathbb{X}_\kappa(\cdot, t): t \in \mathcal{Q}\}$ be a motion of a body B relative to a reference configuration κ . Then for each $t \in \mathcal{Q}$, $\mathbb{X}_\kappa(\cdot, t)$ is smooth on $\kappa(B)$ and its inverse $\mathbb{X}_\kappa^{-1}(\cdot, t)$ is smooth on B_t . Moreover,

$$J_\kappa(x, t) := |\det D_1 \mathbb{X}_\kappa(x, t)| \neq 0 \text{ on } \kappa(B) \times \mathcal{Q}.$$

Combining

$$\mathbb{X}_n(\mathbb{X}, t) = \mathbb{X}(\mathbb{U}^{-1}(\mathbb{X}), t)$$

with the smoothness Axiom (M1), we get

Theorem B1.13.2. Let $\{\mathbb{X}_n(\cdot, t): t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathbb{U}$. Then for each $\mathbb{X} \in \mathbb{U}(\mathcal{B})$, $\mathbb{X}_n(\mathbb{X}, \cdot) \in C^2(\mathcal{Q}, \mathcal{E}_3)$.

This last result allows us to calculate $\mathbb{X}_{n,2}(\mathbb{X}, t)$ and $\mathbb{X}_{n,2}(\mathbb{X}, t)$, which we expect to be the velocity and the acceleration at time t of the particle that is at $\mathbb{X}$ in the reference configuration $\mathbb{U}$. To see that this is indeed the case, we differentiate

$$\mathbb{X}_n(\mathbb{X}, t) = \mathbb{X}(\mathbb{U}^{-1}(\mathbb{X}), t)$$

to get

$$\mathbb{X}_{n,2}(\mathbb{X}, t) = \mathbb{X}_{,2}(\mathbb{U}^{-1}(\mathbb{X}), t) = \underline{v}_m(\mathbb{U}^{-1}(\mathbb{X}), t),$$

where in the last step we have used Definitions B1.8.3 and 5. Now $\underline{v}_m(\mathbb{U}^{-1}(\mathbb{X}), t)$ is the velocity at time t of the particle $X = \mathbb{U}^{-1}(\mathbb{X})$, but this particle is the one at place $\mathbb{X}$ in the reference configuration $\mathbb{U}$. Of course,

$$\underline{v}_m(\mathbb{U}^{-1}(\mathbb{X}), t) = \underline{v}_m(X, t) = \underline{v}_a(\mathbb{X}, t) = \underline{v}_a(\mathbb{X}_n(\mathbb{X}, t), t). \quad)^1$$

$\therefore$ it is natural to introduce the notations

¹G. Definition B1.8.5 and the surrounding discussion.

$$\underline{v}_n(\mathbb{X}, t) = \underline{v}_m(\mathbb{M}^{-1}(\mathbb{X}), t) = \underline{v}_s(\mathbb{X}_n(\mathbb{X}, t), t)$$

and

$$\underline{a}_n(\mathbb{X}, t) = \underline{a}_m(\mathbb{M}^{-1}(\mathbb{X}), t) = \underline{a}_s(\mathbb{X}_n(\mathbb{X}, t), t)$$

for the velocity and the acceleration, respectively, at time t of the particle that corresponds to the place $\mathbb{X}$ in the reference configuration $\mathbb{M}$. More generally, we make

Definition B1.13.2. Let $\mathbb{M}$ be a reference configuration and $\{\mathbb{X}(\cdot, t): t \in \mathcal{Q}\}$ a motion of a body $\mathcal{B}$. Given a function $\underline{\Psi}_m$ on $\mathcal{B} \times \mathcal{Q}$ or equivalently¹ a function $\underline{\Psi}_s$ on the trajectory $\mathcal{J}$, the function $\underline{\Psi}_n$ is defined on $\mathbb{M}(\mathcal{B}) \times \mathcal{Q}$ by

$$\underline{\Psi}_n(\mathbb{X}, t) = \underline{\Psi}_m(\mathbb{M}^{-1}(\mathbb{X}), t) = \underline{\Psi}_s(\mathbb{X}_n(\mathbb{X}, t), t).$$

Conversely, given a function $\underline{\Psi}_n$ on $\mathbb{M}(\mathcal{B}) \times \mathcal{Q}$, the functions $\underline{\Psi}_m$ on $\mathcal{B} \times \mathcal{Q}$ and $\underline{\Psi}_s$ on $\mathcal{J}$ are defined by

$$\underline{\Psi}_m(X, t) = \underline{\Psi}_n(\mathbb{M}(X), t)$$

and

$$\underline{\Psi}_s(\mathbb{X}, t) = \underline{\Psi}_n(\mathbb{X}_n^{-1}(\mathbb{X}, t), t).$$

The pair $(\mathbb{X}, t)$ are called the referential variables², and their use

¹G. Definition B1.8.5 and the First Notational Agreement.

²Often, LAGRANGEAN variables even though they were introduced by EULER. Sometimes, material variables since they do serve to label particles

is referred to as the referential description.

Second Notational Agreement. The First Notational Agreement (p. B1.13.7) is extended in the obvious way to include the referential description.

We have already established the following

Theorem B1.13.3. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{R}$. Then the referential velocity and acceleration fields are given by

$$\left. \begin{aligned} \underline{v}_n(\mathbb{X}, t) &= \mathbb{X}_{n,2}(\mathbb{X}, t) \\ \text{and} \\ \underline{a}_n(\mathbb{X}, t) &= \underline{v}_{n,2}(\mathbb{X}, t) = \mathbb{X}_{n,22}(\mathbb{X}, t) \end{aligned} \right\} (\mathbb{X}, t) \in \mathcal{R}(\mathcal{B}) \times \mathcal{Q}.$$

Theorem B1.13.2 gave the implications of the smoothness Axiom (M1) vis-a-vis the referential description. The implications of Axiom (M2) are only slightly more difficult to obtain. E.g.,

$$\underline{v}_n(\mathbb{X}, t) = \underline{v}_a(\mathbb{X}_n(\mathbb{X}, t), t)$$

and the chain rule lead to

$$D_1 \underline{v}_n(\mathbb{X}, t) = D_1 \underline{v}_a(\mathbb{X}_n(\mathbb{X}, t), t) D_1 \mathbb{X}_n(\mathbb{X}, t)$$

and

$$\dot{v}_{n,2}(\mathbb{X},t) = D_1 \dot{v}_n(\mathbb{X}_n(\mathbb{X},t),t) [\dot{\mathbb{X}}_{n,2}(\mathbb{X},t)] + \dot{v}_{n,2}(\mathbb{X}_n(\mathbb{X},t),t).$$

However, we do not appear to have a way to get continuity in $\mathbb{X}$ and t jointly. $\therefore$ we assume

(M3) Every motion $\{\mathbb{X}(\cdot,t): t \in \mathcal{D}\}$ and every reference configuration $\mathbb{B}$ of a body $\mathcal{B}$ are $\exists$

$$\mathbb{X}_n \in C^N(\mathcal{K}(\mathcal{B}) \times \mathcal{D}, \mathcal{E}_3), \quad N \geq 2.$$

As with (M1) and (M2), (M3) is viewed as an axiom adjoined to Definition B1.8.1. As with (M2), Axiom (M3) is relaxed occasionally on singular sets so that we can model shock waves, etc. We will always be explicit about such things. To simply specify that $\mathbb{X}_n$ be in class C^N allows us the flexibility to take however many derivatives we might need in our study of continuum mechanics. At this point, we need $N \geq 2$ for calculation of the acceleration. The reader should probably not go forward without a good understanding of the meaning of class C^N for functions of position and time; review the discussion on pp. B1.8.9-10 if necessary.

As was noted in Theorem B1.13.1, $\mathbb{X}_n^{-1}(\cdot,t)$ denotes the inverse¹ of the deformation $\mathbb{X}_n(\cdot,t)$. Thus,

¹cf. the discussion following Definition B1.8.2.

$$\mathbb{X} = \mathbb{X}_n^{-1}(\mathbb{x}, t)$$

is the position in the reference configuration $\mathbb{u}$ of the particle that is at the place $\mathbb{x}$ at time t . Of course, for this to make sense $\mathbb{x}$ and t must be related by the motion; i.e.,

$$(\mathbb{x}, t) \in \mathcal{J},$$

where $\mathcal{J} = \{(\mathbb{x}, t) : \mathbb{x} \in \mathcal{B}_t, t \in \mathcal{Q}\}$ is the trajectory!
In terms of $\mathbb{X}_n$, we can write

$$\mathcal{B}_t = \mathbb{X}(\mathcal{B}, t) = \mathbb{X}_n(\mathbb{u}(\mathcal{B}), t)$$

for the region of $\mathcal{E}_3$ occupied by the body at time t .
These remarks lead us to

Definition B1.13.3. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathbb{u}$. Then the function

$$\mathbb{X}_n^{-1} : \mathcal{J} \rightarrow \mathbb{u}(\mathcal{B})$$

is the reference map of the motion, and the function

$$\Gamma : \mathbb{u}(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathcal{J}$$

defined by

$$\Gamma(\mathbb{X}, t) = (\mathbb{X}_n(\mathbb{X}, t), t)$$

is the trajectory map of the motion.

¹(cf. Definition B1.8.4.)

A relatively recent result is

B1.13.9

Theorem B1.13.4. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{U}$ so that by Axiom (M3) $\mathbb{X}_n \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{E}_3)$, $N \geq 2$. Then the reference and trajectory maps are also of class C^N ; i.e.,

$$\mathbb{X}_n^{-1} \in C^N(\mathcal{Y}, \mathcal{U}(\mathcal{B})) \text{ and } \Gamma \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{Y}).$$

Moreover, the trajectory map Γ is one-to-one and $\Gamma^{-1} \in C^N(\mathcal{Y}, \mathcal{U}(\mathcal{B}) \times \mathcal{Q})$.

Proof. Consider the trajectory map

$$\Gamma : \mathcal{U}(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathcal{Y} \subset \mathcal{E}_3 \times \mathbb{R}$$

defined by

$$\Gamma(\mathbb{X}, t) = (\mathbb{X}_n(\mathbb{X}, t), t).$$

Focusing on the product structure of the codomain, we see from Theorem A5.13.1 that Γ is differentiable at each $(\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q}$ ² because of the differentiability of the "component" maps

$$\Gamma(\mathbb{X}, t) := \mathbb{X}_n(\mathbb{X}, t) \text{ and } \gamma(\mathbb{X}, t) := t.$$

In fact,

¹This is an expanded version of the proof on p. 65 of GURTIN's An Introduction to Continuum Mechanics, which he attributes to MARTINS.

²In the usual way, we calculate on the interior and get to the closure via the appropriate generalization of the Extension Theorem. (Theorem B1.2.2).

$$D\Gamma(\mathbb{X}, t)[(\underline{u}, \tau)] = (D\mathbb{F}(\mathbb{X}, t)[(\underline{u}, \tau)], D\gamma(\mathbb{X}, t)[(\underline{u}, \tau)])$$

Now using the product structure of the domain, we see from Theorem A5.12.6 that

$$\begin{aligned} D\mathbb{F}(\mathbb{X}, t)[(\underline{u}, \tau)] &= D_1 \mathbb{F}(\mathbb{X}, t)[\underline{u}] + D_2 \mathbb{F}(\mathbb{X}, t)[\tau] \\ &= D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + D_2 \mathbb{X}_n(\mathbb{X}, t)[\tau] \quad (\text{substitution}) \\ &= D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + \tau \mathbb{X}_{n,2}(\mathbb{X}, t) \quad (\text{Theorem A5.3.1}) \\ &= D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + \tau \mathbb{V}_n(\mathbb{X}, t) \quad (\text{Theorem B1.13.3}), \end{aligned}$$

Similarly, since $\gamma(\mathbb{X}, t) = t$

$$D\gamma(\mathbb{X}, t)[(\underline{u}, \tau)] = D_1 \gamma(\mathbb{X}, t)[\underline{u}] + D_2 \gamma(\mathbb{X}, t)[\tau] = \tau.$$

$\therefore$

$$D\Gamma(\mathbb{X}, t)[(\underline{u}, \tau)] = (D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + \tau \mathbb{V}_n(\mathbb{X}, t), \tau). \quad (*)$$

Thus, the differentiability of Γ is determined by that of $\mathbb{X}_n$, and by Theorem A5.13.2

$$\Gamma \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{D}, \mathcal{Y}).$$

Next suppose that

$$\Gamma(\mathbb{X}_1, t_1) = \Gamma(\mathbb{X}_2, t_2)$$

$$\Rightarrow (\mathbb{X}_n(\mathbb{X}_1, t_1), t_1) = (\mathbb{X}_n(\mathbb{X}_2, t_2), t_2) \quad (\text{definition of } \Gamma)$$

$$\Rightarrow \mathbb{X}_n(\mathbb{X}_1, t_1) = \mathbb{X}_n(\mathbb{X}_2, t_2) \text{ and } t_1 = t_2 \quad (\text{equality of lists})$$

$$\Rightarrow \mathbb{X}_n(\mathbb{X}_1, t_1) = \mathbb{X}_n(\mathbb{X}_2, t_1) \quad (\text{substitution})$$

$$\Rightarrow \mathbb{X}_1 = \mathbb{X}_2 \quad (\mathbb{X}_n(\cdot, t) \text{ is one-to-one}).$$

$\therefore$

$$\Gamma(\mathbb{X}_1, t_1) = \Gamma(\mathbb{X}_2, t_2) \Rightarrow (\mathbb{X}_1, t_1) = (\mathbb{X}_2, t_2);$$

i.e., the function Γ is one-to-one, and by Theorem A1.2.2
 $\exists$ an inverse function

$$\Gamma^{-1}: \mathcal{U} \rightarrow \mathcal{U}(\mathcal{B}) \times \mathcal{D} \subset \mathcal{E}_3 \times \mathbb{R}.$$

In fact,

$$\Gamma^{-1}(\mathcal{X}, t) = (\mathbb{X}_n^{-1}(\mathcal{X}, t), t); \quad (**)$$

for

$$\Gamma^{-1}(\Gamma(\mathbb{X}, t)) = \Gamma^{-1}(\mathbb{X}_n(\mathbb{X}, t), t) \quad (\text{definition of } \Gamma)$$

$$= \Gamma^{-1}(\mathcal{X}, t) \quad (\text{substitution})$$

$$= (\mathbb{X}_n^{-1}(\mathcal{X}, t), t) \quad (\text{definition of trial } \Gamma^{-1})$$

$$= (\mathbb{X}, t) \quad (\text{substitution}).$$

Thus, Γ is class C^N and one-to-one. It then follows from a corollary¹ of the inverse function theorem that Γ^{-1} is C^N provided that at each $(\mathbb{X}, t) \in \mathcal{M}(\mathcal{B}) \times \mathbb{R}$ the derivative

$$D\Gamma(\mathbb{X}, t) : \mathcal{V}_3 \times \mathbb{R} \rightarrow \mathcal{V}_3 \times \mathbb{R}$$

is invertible. Now by Theorems A5.12.1 and 2, we can view $\mathcal{V}_3 \times \mathbb{R}$ as a 4-dimensional linear space, say $\mathcal{V}_4$. $\therefore$ the linear transformation $D\Gamma(\mathbb{X}, t)$ is simply a second-order tensor on $\mathcal{V}_4$. By Theorem A3.9.9, $D\Gamma(\mathbb{X}, t)$ is invertible iff

$$\det D\Gamma(\mathbb{X}, t) \neq 0;$$

and by Theorem A3.8.7 this is equivalent to

$$D\Gamma(\mathbb{X}, t)[\underline{u}, \tau] = \underline{0} \Rightarrow (\underline{u}, \tau) = (\underline{0}, 0). \quad (\#)$$

By (*),

$$D\Gamma(\mathbb{X}, t)[\underline{u}, \tau] = \underline{0}$$

$\Rightarrow$

$$(D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + \tau \underline{v}_n(\mathbb{X}, t), \tau) = (\underline{0}, 0)$$

$\Rightarrow$

$$D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] + \tau \underline{v}_n(\mathbb{X}, t) = \underline{0} \text{ and } \tau = 0 \quad (\text{equality of lists})$$

¹Theorem A5.11.2.

$$\Rightarrow D_1 \mathbb{X}_n(\mathbb{X}, t)[\underline{u}] = \underline{0} \quad (\text{substitution})$$

$$\Rightarrow \underline{u} = \underline{0} \quad (\text{Theorems B1.13.1 and A3.8.7}).$$

Thus, the implication (#) is satisfied, so $D\Gamma(\mathbb{X}, t)$ is invertible and

$$\Gamma^{-1} \in C^N(\mathcal{J}, \mathcal{U}(\mathcal{B}) \times \mathcal{Q}).$$

But by (**), $\Gamma^{-1}(\mathbb{X}, t) = (\mathbb{X}_n^{-1}(\mathbb{X}, t), t);$

and by Theorem A5.13.2, this requires

$$\mathbb{X}_n^{-1} \in C^N(\mathcal{J}, \mathcal{U}(\mathcal{B})). \quad \square$$

In view of the next theorem, we take Axiom (M3) as a replacement for Axioms (M1) and (M2) from this point on.

Theorem B1.13.5, $(M3) \Rightarrow (M1) \text{ and } (M2).$

Proof. Let (M3) be satisfied; i.e., $\mathbb{X}_x \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{E}_3)$, $N \geq 2$.

Since

$$\mathbb{X} = \mathbb{X}(X, t) = \mathbb{X}_n(\mathbb{X}, t) = \mathbb{X}_n(\mathcal{U}(X), t),$$

it follows immediately that $\mathbb{X}(X, \cdot) \in C^2(\mathcal{Q}, \mathcal{E}_3)$. Thus, (M1) is satisfied.

Next use the inverse of the trajectory map to write

$$\underline{v}_\alpha(x, t) = \underline{v}_n(\mathbb{X}, t) = \underline{v}_n(\Gamma^{-1}(x, t)). \quad (*)$$

By Theorem B1.13.3,

$$\underline{v}_n = \mathbb{X}_{n,2},$$

so that (M3) $\Rightarrow \underline{v}_n \in C^{N-1}(\mathcal{M}(\mathcal{B}) \times \mathcal{Q}, \mathcal{V}_3)$. By Theorem B1.13.4,

$$\Gamma^{-1} \in C^N(\mathcal{Y}, \mathcal{M}(\mathcal{B}) \times \mathcal{Q}).$$

Then application of the chain rule in the form of Theorem A5.10.7 to (*) $\Rightarrow$

$$\underline{v}_\alpha \in C^{N-1}(\mathcal{Y}, \mathcal{V}_3);$$

i.e., (M2) is met. Of course, in the above demonstration we work on the interior and get to boundary points via the extension theorem. $\square$

INSERT material on pp. B1.13.14 a,b

Since $\mathbb{X}_n(\cdot, t)$ is a deformation, it follows from the geometrical considerations of §B1.5 that its gradient is an important object.

Definition B1.13.4. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{M}$.

INSERT for p. B1.13.14 (after proof of Thm. B1.13.5) B1.13.14a

More generally, we have

Theorem B1.13.6. Let $\underline{\Psi}_n \in C^M(\mathcal{N}(\mathcal{B}) \times \mathcal{Q}, \mathcal{V})$, where
 $\mathcal{V}$ is a finite-dimensional normed linear space. Then $\underline{\Psi}_s: \mathcal{J} \rightarrow \mathcal{V}$
defined by

$$\underline{\Psi}_s(x, t) = \underline{\Psi}_n(\mathbb{X}_n^{-1}(x, t), t) \quad)^1$$

belongs to the function class $C^P(\mathcal{J}, \mathcal{V})$, where $P = \min\{M, N\}$
with $N \geq 2$ being the smoothness index of Axiom (N3).

Conversely, if $\underline{\Psi}_s \in C^M(\mathcal{J}, \mathcal{V})$, then $\underline{\Psi}_n: \mathcal{N}(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathcal{V}$
defined by

$$\underline{\Psi}_n(\mathbb{X}, t) = \underline{\Psi}_s(\mathbb{X}_n(\mathbb{X}, t), t)$$

belongs to $C^P(\mathcal{N}(\mathcal{B}) \times \mathcal{Q}, \mathcal{V})$.

Proof. Suppose $\underline{\Psi}_n \in C^M(\mathcal{N}(\mathcal{B}) \times \mathcal{Q}, \mathcal{V})$. Then using
the machinery developed in the proof of Theorem B1.13.4, we
can write

$$\underline{\Psi}_s(x, t) = \underline{\Psi}_n(\Gamma^{-1}(x, t)).$$

Since $\Gamma^{-1} \in C^N(\mathcal{J}, \mathcal{N}(\mathcal{B}) \times \mathcal{Q})$, it follows from the chain rule in
the form of Theorem A5.10.7 that

$$\underline{\Psi}_s = \underline{\Psi}_n \circ \Gamma^{-1} \in C^P(\mathcal{J}, \mathcal{V}).$$

¹Cf. Definition B1.13.2.

As usual, the above analysis is on the interior of $\mathcal{D}$, and we get to the closure via the Extension Theorem.

The converse is established in the same way; the details are left as Exercise B1.13.1. $\square$

Of course, the significance of Theorem B1.13.6 is that to the extent that the motion is smooth (class C^N by Axiom (M3)) we can move between the referential and spatial descriptions without losing smoothness.

return to p. B1.13.14

Then

$$\underline{F}_n(\mathbb{X}, t) = D_1 \mathbb{X}_n(\mathbb{X}, t)$$

is the deformation gradient relative to $\mathcal{U}$ at the place $\mathbb{X} \in \mathcal{U}(\mathcal{B})$ and the time $t \in \mathcal{Q}$.

From (M3) and Theorem B1.13.1, we have

Theorem B1.13.7. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{U}$. Then

$$(i) \quad \underline{F}_n \in C^{N-1}(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \text{Inv } \mathcal{V}_3)^1$$

and

$$(ii) \quad J_n = |\det \underline{F}_n| \neq 0 \text{ on } \mathcal{U}(\mathcal{B}) \times \mathcal{Q}.$$

Obviously, Definition B1.5.1 carries over to the present case. Formally, we make

Definition B1.13.5. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{U}$. Then at each $(\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q}$, $\underline{F}_n(\mathbb{X}, t)$ admits the polar decompositions

$$\underline{F}_n(\mathbb{X}, t) = \underline{R}_n(\mathbb{X}, t) \underline{U}_n(\mathbb{X}, t) = \underline{V}_n(\mathbb{X}, t) \underline{R}_n(\mathbb{X}, t),$$

where $\underline{R}_n(\mathbb{X}, t) \in \text{Orth } \mathcal{V}_3$ and $\underline{U}_n(\mathbb{X}, t), \underline{V}_n(\mathbb{X}, t) \in \text{Psym } \mathcal{V}_3$.

¹Of course, this N is the one from (M3): $\mathbb{X}_n \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{E}_3)$, $N \geq 2$.

We use the following terminology:

$\underline{R}_n(\mathbb{X}, t)$ --- the rotation tensor relative to $\mathcal{M}$ at $(\mathbb{X}, t)$;

$\underline{U}_n(\mathbb{X}, t)$ --- the right stretch tensor relative to $\mathcal{M}$ at $(\mathbb{X}, t)$;

$\underline{V}_n(\mathbb{X}, t)$ --- the left stretch tensor relative to $\mathcal{M}$ at $(\mathbb{X}, t)$;

$\underline{C}_n(\mathbb{X}, t) := [\underline{F}_n(\mathbb{X}, t)]^T \underline{F}_n(\mathbb{X}, t)$ --- the right deformation tensor relative to $\mathcal{M}$ at $(\mathbb{X}, t)$;

and

$\underline{B}_n(\mathbb{X}, t) := \underline{F}_n(\mathbb{X}, t) [\underline{F}_n(\mathbb{X}, t)]^T$ --- the left deformation tensor relative to $\mathcal{M}$ at $(\mathbb{X}, t)$.

Of course, the relationships given in Theorem B1.5.1 apply to the above tensors. For their smoothness properties we have

Theorem B1.13.8. Let $\{\mathbb{X}_n(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to a reference configuration $\mathcal{M}$. Then

$$\underline{C}_n, \underline{B}_n, \underline{U}_n, \underline{V}_n \in C^{N-1}(\mathcal{M}(\mathcal{B}) \times \mathcal{Q}, \text{Sym } \mathcal{V}_3)$$

and

$$\underline{R}_n \in C^{N-1}(\mathcal{M}(\mathcal{B}) \times \mathcal{Q}, \text{Orth } \mathcal{V}_3).$$

Proof. The starting point is Theorem B1.13.7:

$$\underline{F}_n \in C^{N-1}(\mathcal{M}(\mathcal{B}) \times \mathcal{Q}, \text{Inv } \mathcal{V}_3).$$

Since $\underline{C}_n = \underline{F}_n^T \underline{F}_n$ and $\underline{B}_n = \underline{F}_n \underline{F}_n^T$, it follows from the differentiability of the transpose (Theorem A5.5.4), the differentiability of the product of tensors (Theorem A5.6.7), and the chain rule (Theorem A5.7.1) that the functions $\underline{C}_n$ and $\underline{B}_n$ inherit the smoothness of $\underline{F}_n$.

Next, since $\underline{U}_n = \sqrt{\underline{C}_n}$ and $\underline{V}_n = \sqrt{\underline{B}_n}$, the differentiability properties of $\underline{U}_n$ and $\underline{V}_n$ follow from the smoothness of the tensor square root (Theorem A5.11.3) and the chain rule in the form of Theorem A5.10.7.

Finally, we get at $\underline{R}_n$ from $\underline{R}_n = \underline{F}_n \underline{U}_n^{-1}$. The only concern here is the differentiability of the tensor inverse function $\underline{F}: \text{Inv } \mathcal{V}_n \rightarrow \text{Inv } \mathcal{V}_n$ defined by

$$\underline{F}(\underline{X}) = \underline{X}^{-1},$$

But $\underline{F}$ is of class C^∞ by Theorem A5.10.6. Then the differentiability properties of $\underline{R}_n$ follow from those of $\underline{F}_n$ and $\underline{U}_n$ via the product rule and the chain rule.

As usual, the above considerations are on the interior of $\mathcal{X}(\mathcal{B}) \times \mathcal{Q}$; and we get results on the closure by the Extension Theorem. $\square$

Next, we recall from Definition B1.8.6 that

$$\underline{L}_s(\mathbf{x}, t) = D_{\mathbf{x}} \underline{V}_s(\mathbf{x}, t) \quad , \quad (\mathbf{x}, t) \in \mathcal{I}$$

is the spatial velocity gradient. Since $\mathbf{x} = \mathbf{x}(X, t) = \mathbf{x}_n(\mathbf{x}, t)$, we can use Definitions B1.8.5 and B1.13.2 to write

$$\underline{L}_m(X, t) = \underline{L}_s(\mathbf{x}(X, t), t) \quad , \quad (X, t) \in \mathcal{B} \times \mathcal{Q}$$

and

$$\underline{L}_n(\mathbf{x}, t) = \underline{L}_s(\mathbf{x}_n(\mathbf{x}, t), t) \quad , \quad (\mathbf{x}, t) \in \mathcal{M}(\mathcal{B}) \times \mathcal{Q}.$$

Our next result provides a formula for $\underline{L}_n$ directly in terms of the referential description without going through the spatial

velocity field,

Theorem B1.13.9. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion and $\mathcal{U}$ a reference configuration of a body $\mathcal{B}$. Then the spatial velocity gradient $\underline{L}_s = D_1 \underline{V}_s$ is given in the referential description by

$$\underline{L}_n(\mathbb{X}, t) = \underline{F}_{n,2}(\mathbb{X}, t) [\underline{F}_n(\mathbb{X}, t)]^{-1}, \quad (\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q}.$$

Proof. By Definition B1.13.2,

$$\underline{V}_n(\mathbb{X}, t) = \underline{V}_s(\mathbb{X}_n(\mathbb{X}, t), t).$$

$\therefore$ by the chain rule (Theorem A5.7.1),

$$\begin{aligned} D_1 \underline{V}_n(\mathbb{X}, t) &= D_1 \underline{V}_s(\mathbb{X}_n(\mathbb{X}, t), t) D_1 \mathbb{X}_n(\mathbb{X}, t) \\ &= \underline{L}_s(\mathbb{X}_n(\mathbb{X}, t), t) \underline{F}_n(\mathbb{X}, t) \\ &= \underline{L}_n(\mathbb{X}, t) \underline{F}_n(\mathbb{X}, t) \quad (*) \end{aligned}$$

at any interior point of $\mathcal{U}(\mathcal{B}) \times \mathcal{Q}$. But by Theorems B1.13.3 and A5.3.1, we also have

$$\underline{V}_n(\mathbb{X}, t) = \dot{\mathbb{X}}_{n,2}(\mathbb{X}, t) = D_2 \mathbb{X}_n(\mathbb{X}, t) [1].$$

$$\begin{aligned}
D_1 \underline{v}_n(\mathbb{X}, t) [\underline{u}] &= D_1 (D_2 \mathbb{X}_n(\mathbb{X}, t) [1]) [\underline{u}] \\
&= D_1 D_2 \mathbb{X}_n(\mathbb{X}, t) [1, \underline{u}] \quad (\text{notation, see p. A5.12.11}) \\
&= D_2 D_1 \mathbb{X}_n(\mathbb{X}, t) [\underline{u}, 1] \quad (\text{Theorem A5.12.7})^1 \\
&= D_2 (D_1 \mathbb{X}_n(\mathbb{X}, t) [\underline{u}]) [1] \quad (\text{notation}) \\
&= D_2 (\underline{F}_n(\mathbb{X}, t) \underline{u}) [1] \quad (\text{Definition B1.13.4}) \\
&= D_2 \underline{f}(\mathbb{X}, t) [1] \quad (\underline{f}(\mathbb{X}, t) := \underline{F}_n(\mathbb{X}, t) \underline{u}) \\
&= \underline{f}_{,2}(\mathbb{X}, t) \quad (\text{Theorem A5.3.1}) \\
&= \underline{F}_{n,2}(\mathbb{X}, t) \underline{u} \quad (\underline{f} := \underline{F}_n \underline{u})
\end{aligned}$$

or

$$D_1 \underline{v}_n(\mathbb{X}, t) = \underline{F}_{n,2}(\mathbb{X}, t),$$

which when combined with (*) yields

$$\underline{F}_{n,2} = \underline{L}_n \underline{F}_n$$

on the interior of $\mathbb{W}(\mathbb{B}) \times \mathbb{Q}$. We get the result on the closure by the Extension Theorem (Theorem B1.2.2). $\square$
(a generalization of)

¹ This is a place where we need differentiability (in fact class C^2) of $\mathbb{X}_n$ in $\mathbb{X}$ and t jointly, cf. Axiom (M3).

A useful corollary of the above result is

Theorem B1.13.10. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion and $\mathcal{U}$ a reference configuration of a body $\mathcal{B}$. Then the time rate of the right deformation tensor $\underline{C}_n = \underline{F}_n^T \underline{F}_n$ is given by

$$\underline{C}_{n,2}(\mathbb{X}, t) = 2[\underline{F}_n(\mathbb{X}, t)]^T \underline{D}_n(\mathbb{X}, t) \underline{F}_n(\mathbb{X}, t), \quad (\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q},$$

where $\underline{D}_n = \text{sym } \underline{L}_n$ is the stretching tensor.

Proof. Exercise B1.13.2. $\square$

We have already introduced the common notation

$$J_n(\mathbb{X}, t) = |\det \underline{F}_n(\mathbb{X}, t)|, \quad (\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q}. \quad)^1$$

As was alluded to in § B1.1, in many treatments of continuum mechanics, it is assumed that $\det \underline{F}_n > 0$; and then the absolute value above is superfluous. The next theorem gives the justification for such an assumption,

Theorem B1.13.11. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Let $\mathcal{U}_0$ be the configuration that the body occupies at the time $t_0 \in \mathcal{Q}$, i.e.,

$$\mathcal{U}_0 = \mathbb{X}(\cdot, t_0).$$

¹cf. Theorem B1.13.1 and Definition B1.13.4.

Then

$$\det \tilde{F}_{n_0}(\mathbb{X}, t) > 0 \quad \forall (\mathbb{X}, t) \in \mathcal{U}_0(\mathcal{B}) \times \mathcal{Q},$$

Proof. For each $t \in \mathcal{Q}$, $\mathbb{X}_{n_0}(\cdot, t)$ is a deformation;
and $\therefore$

$$\det \tilde{F}_{n_0}(\mathbb{X}, t) \neq 0 \quad \forall (\mathbb{X}, t) \in \mathcal{U}_0(\mathcal{B}) \times \mathcal{Q}$$

By Definition B1.1.1. Now for each $\mathbb{X} \in \mathcal{U}_0(\mathcal{B})$, the real-valued function $\det \tilde{F}_{n_0}(\mathbb{X}, \cdot)$ is continuous¹ on $\mathcal{Q}$, and since this function never vanishes, it is either always > 0 or always < 0 .

Exercise B1.13.3. Prove this claim. I.e. show that if $f: \mathcal{Q} \rightarrow \mathbb{R}$ is continuous on $\mathcal{Q}$ and if $f \neq 0$ on $\mathcal{Q}$, then either $f > 0$ on $\mathcal{Q}$ or $f < 0$ on $\mathcal{Q}$. In other words, because it is continuous and never vanishes, f cannot change sign. (Hint: Suppose f does change sign and use the intermediate value theorem.)

Under the present hypotheses, $\mathbb{X}_{n_0}(\cdot, t_0)$ is the identity mapping:

$$\mathbb{X}_{n_0}(\mathbb{X}, t_0) = \mathbb{X}(\mathcal{U}_0^{-1}(\mathbb{X}), t_0) = \mathcal{U}_0(\mathcal{U}_0^{-1}(\mathbb{X})) = \mathbb{X}, \quad \mathbb{X} \in \mathcal{U}_0(\mathcal{B}).$$

$\therefore$

$$\tilde{F}_{n_0}(\mathbb{X}, t_0) = D_1 \mathbb{X}_{n_0}(\mathbb{X}, t_0) = \underline{I}, \quad)^2$$

¹ $\tilde{F}_{n_0}(\mathbb{X}, \cdot)$ is continuous by Theorem B1.13.7, and the function $\det: \text{Lin } \mathcal{V}_3 \rightarrow \mathbb{R}$ is continuous since it is differentiable (Theorems A5.2.4 and A5.8.1).

² Cf. Theorem A5.5.4.

and

$$\det \underline{F}_{\underline{n}_0}(\underline{x}, t_0) = 1 \quad)^1$$

Then, in view of the discussion above, $\det \underline{F}_{\underline{n}_0}$ is strictly positive on $\mathcal{M}_0(\mathcal{B}) \times \mathcal{Q}$, $\square$

Generally, we will not assume that the body ever occupies the reference configuration at some time during the motion under consideration; and, therefore, all that we generally can say about $\det \underline{F}_{\underline{n}}$ is that it never changes sign.

INSERT material on p. B1.13.23a

The following theorem will be very important to our development of continuum mechanics. Because of the tools we built in Part A, its proof is just a routine calculation.

Theorem B1.13.13. Let $\{\underline{x}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion and $\underline{n}$ a reference configuration of a body $\mathcal{B}$. Then the time rate of $J_{\underline{n}} = |\det \underline{F}_{\underline{n}}|$ is given by

$$J_{\underline{n},2}(\underline{x}, t) = J_{\underline{n}}(\underline{x}, t) \operatorname{div}_1 \underline{v}_{\underline{n}}(\underline{x}_{\underline{n}}(\underline{x}, t), t), \quad (\underline{x}, t) \in \mathcal{M}(\mathcal{B}) \times \mathcal{Q},$$

where $\operatorname{div}_1 \underline{v}_{\underline{n}}(\underline{x}, t)$ stands for the divergence of the vector field $\underline{v}_{\underline{n}}(\cdot, t)$ evaluated at the point $\underline{x}$.)²

¹ Cf. Theorem A3.8.2 (ii).

² By Definition A6.6.1, $\operatorname{div}_1 \underline{v}_{\underline{n}}(\underline{x}, t) = \operatorname{tr} D_1 \underline{v}_{\underline{n}}(\underline{x}, t)$.

INSERT for p. B1.13.23

Theorem B1.13.12. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a
motion and $\mathcal{U}$ a reference configuration of a body $\mathcal{B}$.
Then

$$J_n \in C^{N-1}(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathbb{R}),$$

where $N \geq 2$ is the smoothness exponent of Axiom (M3)
 $(\mathbb{X}_n \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{E}_3))$.

Proof. As we saw in the proof of the previous theorem,
 $\det \underline{F}_n$ never changes sign. Suppose $\det \underline{F}_n > 0$. Then

$$J_n(\mathbb{X}, t) = \det \underline{F}_n(\mathbb{X}, t).$$

Now by Theorem B1.13.7, $\underline{F}_n \in C^{N-1}(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \text{Inv } \mathcal{V}_3)$;
 and from Theorem A5.8.1, it is clear that $\det \in C^\infty(\text{Lin } \mathcal{V}_3, \mathbb{R})$.
 The smoothness claimed for the composite function $J_n = \det \underline{F}_n$
 then follows immediately from Theorem A5.10.7. If $\det \underline{F}_n < 0$,
 then $J_n = -\det \underline{F}_n$, which belongs to the same differentiability
 class as $\det \underline{F}_n$. $\square$

return to p. B1.13.23

Proof. Let (x, t) be an interior point of $\mathcal{M}(\mathcal{B}) \times \mathcal{Q}$. First, suppose that $\det \tilde{F}_n(x, t) > 0$; then as we saw in the proof of Theorem B1.13.11, $\det \tilde{F}_n > 0$ everywhere, and we can write

$$J_n = \det \tilde{F}_n = \iota_3(F_n) \text{ on } \mathcal{M}(\mathcal{B}) \times \mathcal{Q}. \quad (*)$$

By the previous theorem and the chain rule, J_n is differentiable on $\mathcal{M}(\mathcal{B}) \times \mathcal{Q}$; and

$$D_2 J_n(x, t)[\alpha] = D \iota_3(\tilde{F}_n(x, t)) [D_2 \tilde{F}_n(x, t)[\alpha]],$$

which by Theorem A5.3.1 and the linearity of the derivative becomes

$$\begin{aligned} \nabla J_{n,2}(x, t) &= \nabla D \iota_3(\tilde{F}_n(x, t)) [\tilde{F}_{n,2}(x, t)] \\ &= \iota_3(\tilde{F}_n(x, t)) \nabla (F_n^{-1}(x, t) \tilde{F}_{n,2}(x, t)) \quad (\text{Theorem A5.8.1}) \\ &= \iota_3(\tilde{F}_n(x, t)) \nabla (F_{n,2}(x, t) \tilde{F}_n^{-1}(x, t)) \quad (\text{Theorem A3.7.2}) \\ &= \iota_3(\tilde{F}_n(x, t)) \nabla (\tilde{L}_n(x, t)) \quad (\text{Theorem B1.13.9}) \\ &= \iota_3(\tilde{F}_n(x, t)) \nabla (\tilde{L}_n(x(x, t), t)) \quad (\text{Definition B1.13.2}) \\ &= \iota_3(\tilde{F}_n(x, t)) \nabla (D_1 \tilde{V}_n(x(x, t), t)) \quad (\text{Definition B1.8.6}) \\ &= J_n(x, t) \nabla \tilde{V}_n(x(x, t), t) \quad (\text{Definition A6.6.1 and } (*)). \end{aligned}$$

The extension to $\mathcal{M}(\mathcal{B}) \times \mathcal{Q}$ is by Theorem B1.2.2.

If $\det \underline{F}_n(\underline{x}, t) < 0$, then $J_n(\underline{x}, t) = -\det \underline{F}_n(\underline{x}, t)$; and

$$J_{n,2}(\underline{x}, t) = -\det \underline{F}_n(\underline{x}, t) \operatorname{div}_{\underline{v}_n}(\underline{x}_n(\underline{x}, t), t)$$

from the analysis above, Hence, in either case,

$$J_{n,2}(\underline{x}, t) = J_n(\underline{x}, t) \operatorname{div}_{\underline{v}_n}(\underline{x}_n(\underline{x}, t), t). \quad \square$$

This result plays a key role in the proof of the following theorem, which is a major tool of continuum mechanics.

Theorem B1.13.14. (The Transport Theorem)¹ Let
 $\{\underline{x}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Suppose
 $f \in C^1(\mathcal{Q}, \mathbb{R})$ and $\underline{f} \in C^1(\mathcal{Q}, \mathcal{V}_3)$. For each part $P \subset \mathcal{B}$,
define $j : \mathcal{Q} \rightarrow \mathbb{R}$ and $\underline{j} : \mathcal{Q} \rightarrow \mathcal{V}_3$ by

$$j(t) = \int_{\mathcal{Q}_t} f(\underline{x}, t) dV_{\underline{x}} \quad \text{and} \quad \underline{j}(t) = \int_{\mathcal{Q}_t} \underline{f}(\underline{x}, t) dV_{\underline{x}}.$$

Then $j \in C^1(\mathcal{Q}, \mathbb{R})$ and $\underline{j} \in C^1(\mathcal{Q}, \mathcal{V}_3)$, and their ordinary derivatives are given by

$$\dot{j}(t) = \int_{\mathcal{Q}_t} \left\{ \underline{f}_{,2}(\underline{x}, t) + D_1 f(\underline{x}, t) [\underline{v}_n(\underline{x}, t)] + f(\underline{x}, t) \operatorname{div}_{\underline{v}_n}(\underline{x}, t) \right\} dV_{\underline{x}}$$

¹ Often, - REYNOLDS' transport theorem.

$$= \int_{\mathcal{Q}_t} [f_{,2}(\underline{x}, t) + \nabla_1 f(\underline{x}, t) \cdot \underline{v}_s(\underline{x}, t) + f(\underline{x}, t) \operatorname{div}_1 \underline{v}_s(\underline{x}, t)] dV_{\underline{x}} \quad)'$$

$$= \int_{\mathcal{Q}_t} [f_{,2}(\underline{x}, t) + (\operatorname{div}_1 (f \underline{v}_s))(\underline{x}, t)] dV_{\underline{x}}$$

$$= \int_{\mathcal{Q}_t} f_{,2}(\underline{x}, t) dV_{\underline{x}} + \int_{\partial \mathcal{Q}_t} f(\underline{x}, t) \underline{v}_s(\underline{x}, t) \cdot \underline{\eta}(\underline{x}, t) dA_{\underline{x}}$$

and

$$\dot{\underline{j}}(t) = \int_{\mathcal{Q}_t} \left\{ \underline{f}_{,2}(\underline{x}, t) + D_1 \underline{f}(\underline{x}, t) [\underline{v}_s(\underline{x}, t)] + (\operatorname{div}_1 \underline{v}_s(\underline{x}, t)) \underline{f}(\underline{x}, t) \right\} dV_{\underline{x}}$$

$$= \int_{\mathcal{Q}_t} [\underline{f}_{,2}(\underline{x}, t) + (\operatorname{div}_1 (\underline{f} \otimes \underline{v}_s))(\underline{x}, t)] dV_{\underline{x}}$$

$$= \int_{\mathcal{Q}_t} \underline{f}_{,2}(\underline{x}, t) dV_{\underline{x}} + \int_{\partial \mathcal{Q}_t} (\underline{v}_s(\underline{x}, t) \cdot \underline{\eta}(\underline{x}, t)) \underline{f}(\underline{x}, t) dA_{\underline{x}},$$

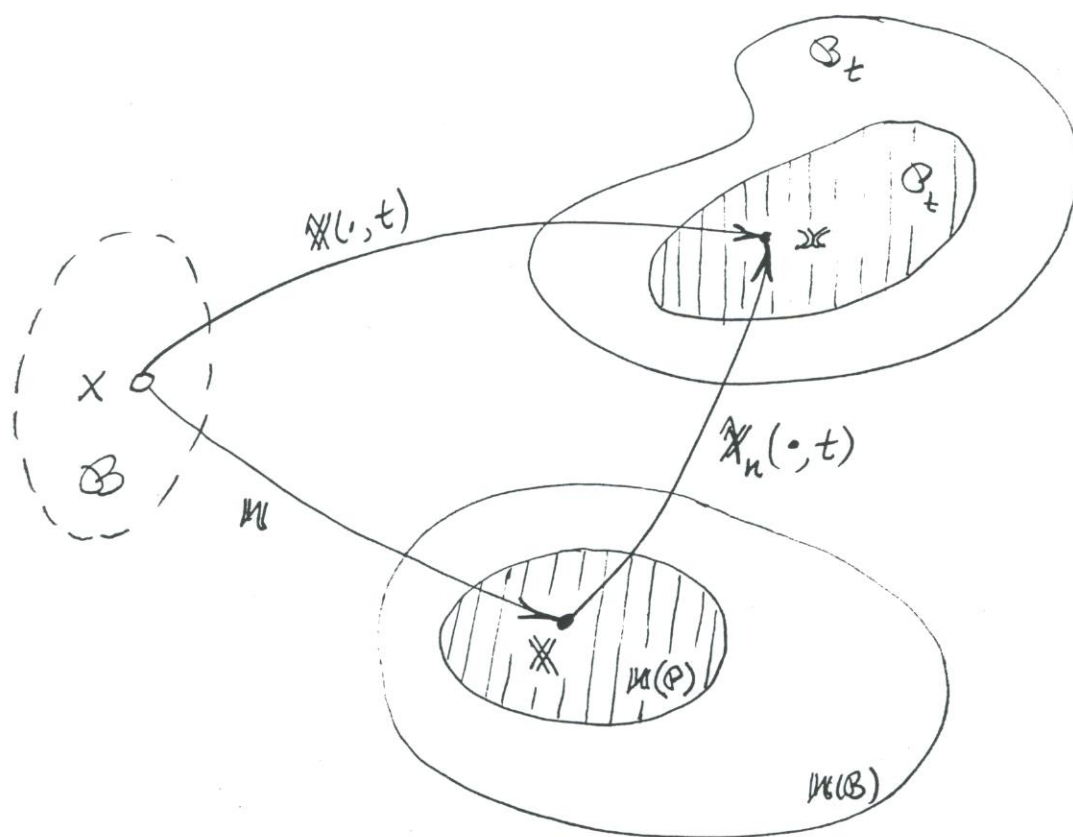
where $\underline{\eta}(\underline{x}, t)$ is the outward unit normal to $\partial \mathcal{Q}_t$ at point $\underline{x}$.

Proof. The main difficulty in the calculation of $\dot{\underline{j}}$ is that the region of integration $\mathcal{Q}_t$ in the definition of $\underline{j}(t)$ depends

¹ $\nabla_1 f(\underline{x}, t)$ stands for the gradient of the scalar field $f(\cdot, t)$ evaluated at point $\underline{x}$.

on t . We overcome this by introducing a reference configuration $\mathcal{R}$ for $\mathcal{B}$ and using the Change of Variable Theorem for volume integrals (Theorem A6.10.2) to convert the defining integral to one over a time independent region in $\mathcal{R}(\mathcal{B})$:

$$j(t) = \int_{\mathcal{Q}_t} f(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\mathcal{R}(\mathcal{B})} f(\mathcal{X}_n(\mathcal{X}, t), t) J_n(\mathcal{X}, t) dV_{\mathcal{X}}.$$



By the assumed smoothness of f , the known smoothness of J_n ¹, and the time independence of the region $\mathcal{R}(\mathcal{B})$, the second integral above defines a smooth function of t . In fact, we can differentiate under the integral to get

¹G. Theorem B1.13.13.

$$j(t) = \int_{\mu(P)} [g_{,2}(\mathbb{X}, t) J_n(\mathbb{X}, t) + f(\mathbb{X}_n(\mathbb{X}, t), t) J_{n,2}(\mathbb{X}, t)] dV_X, \quad (*)'$$

where

$$g(\mathbb{X}, t) = f(\mathbb{X}_n(\mathbb{X}, t), t).$$

By the chain rule for functions of several variables (Theorem A5.13.3),

$$D_2 g(\mathbb{X}, t)[\alpha] = D_1 f(\mathbb{X}_n(\mathbb{X}, t), t)[D_2 \mathbb{X}_n(\mathbb{X}, t)[\alpha]] + D_2 f(\mathbb{X}_n(\mathbb{X}, t), t)[D_2 h(\mathbb{X}, t)[\alpha]],$$

where $h(\mathbb{X}, t) = t$. By Theorem A5.3.1, this reduces to

$$\alpha g_{,2}(\mathbb{X}, t) = D_1 f(\mathbb{X}_n(\mathbb{X}, t), t)[\alpha \mathbb{X}_{n,2}(\mathbb{X}, t)] + \alpha h_{,2}(\mathbb{X}, t) f'_{,2}(\mathbb{X}_n(\mathbb{X}, t), t);$$

whence,

$$g_{,2}(\mathbb{X}, t) = D_1 f(\mathbb{X}_n(\mathbb{X}, t), t)[v_n(\mathbb{X}, t)] + f'_{,2}(\mathbb{X}_n(\mathbb{X}, t), t), \quad (**)$$

where Theorem B1.13.3 has been employed. Placing (**) into (*) and using Theorem B1.13.13 on $J_{n,2}$, we obtain

$$j(t) = \int_{\mu(P)} \left\{ f'_{,2}(\mathbb{X}_n(\mathbb{X}, t), t) + D_1 f(\mathbb{X}_n(\mathbb{X}, t), t)[v_n(\mathbb{X}, t)] + f(\mathbb{X}_n(\mathbb{X}, t), t) \operatorname{div}_1 v_n(\mathbb{X}_n(\mathbb{X}, t), t) \right\} J_n(\mathbb{X}, t) dV_X,$$

which by the change of variable theorem becomes

¹The elementary product rule used here is covered by Theorems A5.6.3 and A5.3.1,

$$\dot{j}(t) = \int_{\mathcal{Q}_t} \left\{ f_{,2}(\underline{x}, t) + D_1 f(\underline{x}, t) [\underline{v}_a(\underline{x}, t)] + f(\underline{x}, t) \operatorname{div}_1 \underline{v}_a(\underline{x}, t) \right\} dV_{\underline{x}}.$$

To complete the proof, we note that since f is scalar-valued,

$$D_1 f(\underline{x}, t) [\underline{v}_a(\underline{x}, t)] = \nabla_1 f(\underline{x}, t) \cdot \underline{v}_a(\underline{x}, t).$$

For the next step, we recall from Exercise A5.6.4 that

$$D_1 (f \underline{v}_a) = \underline{v}_a \otimes \nabla_1 f + f D_1 \underline{v}_a,$$

the trace of which is the identity

$$\operatorname{div}_1 (f \underline{v}_a) = \underline{v}_a \cdot \nabla_1 f + f \operatorname{div}_1 \underline{v}_a.$$

Finally, by the divergence theorem (Theorem A6.10.1),

$$\int_{\mathcal{Q}_t} \operatorname{div}_1 (f \underline{v}_a) dV = \int_{\partial \mathcal{Q}_t} f \underline{v}_a \cdot \underline{n} dA.$$

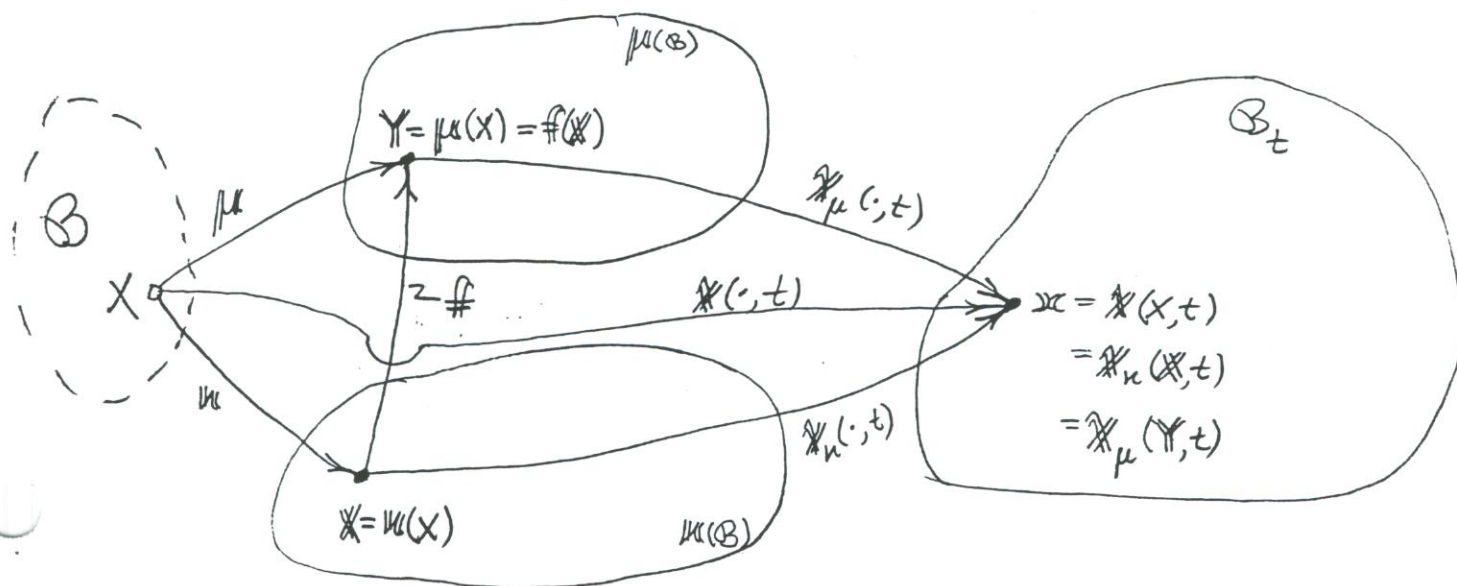
The proof of the vector case is left as Exercise B1.13.4, in which you may wish to make use of Theorem A6.9.5. $\square$

It is interesting to note that despite the above proof, reference configurations do not enter into the statement of the transport theorem.

Finally, it will be important for us to know what happens to the deformation gradient if we change from one reference configuration to another.

Theorem B1.13.15. Let κ and μ be two reference configurations
of a body $\mathcal{B}$. Let $\underline{G} = D\sharp$, where $\sharp = \mu \circ \kappa^{-1}$ is the
deformation that connects κ and μ .¹ Let $\{\chi(\cdot, t) : t \in \mathcal{Q}\}$ be
a motion of $\mathcal{B}$. Then

$$\underline{F}_{\kappa}(\chi, t) = \underline{F}_{\mu}(\sharp(\chi), t) \underline{G}(\chi), \quad (\chi, t) \in \kappa(\mathcal{B}) \times \mathcal{Q}.$$



¹ G. (B1) of Definition B1.7.1.

Proof. Referring to the figure, we see that

$$\mathcal{X} = \mathcal{X}_\kappa(\mathcal{X}, t) = \mathcal{X}(\kappa^{-1}(\mathcal{X}), t) = \mathcal{X}(X, t) = \mathcal{X}(\mu^{-1}(\Psi), t) = \mathcal{X}_\mu(\Psi, t) = \mathcal{X}_\mu(\#(\mathcal{X}), t).$$

Thus,

$$\mathcal{X}_\kappa(\cdot, t) = \mathcal{X}_\mu(\cdot, t) \circ \# ;$$

and by the chain rule (Theorem A5.7.1) and the Extension Theorem (Theorem B1.2.2),

$$D_1 \mathcal{X}_\kappa(\cdot, t) = D_1 \mathcal{X}_\mu(\cdot, t) \circ D\#$$

or

$$F_{\sim \kappa}(\mathcal{X}, t) = F_{\sim \mu}(\#(\mathcal{X}), t) \circ G(\mathcal{X}), \quad (\mathcal{X}, t) \in \kappa(\mathcal{B}) \times \mathcal{Q}, \quad \square$$

Supplementary Reading

Same as for § B1.1

B1.14. The Referential Description of Rigid and Isochoric Motions

First, we consider the special case where the reference configuration actually is occupied by the body at some time during the rigid motion under consideration.

Theorem B1.14.1. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Let the reference configuration $\mathcal{K}_0$ be the configuration of $\mathcal{B}$ at the time $t_0 \in \mathcal{Q}$; i.e., $\mathcal{K}_0 = \mathbb{X}(\cdot, t_0)$. Then the following statements are equivalent:

- (i) the motion is rigid;
- (ii) each member of the one-parameter family of deformations $\{\mathbb{X}_{n_0}(\cdot, t) : t \in \mathcal{Q}\}$ is rigid;
- (iii) $\mathbb{X}_{n_0} : \mathcal{K}_0(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathcal{E}_3$ has the form

$$\mathbb{X}_{n_0}(\mathbb{X}, t) = \mathbb{X}_{n_0}(\mathring{\mathbb{X}}, t) + \underline{\underline{Q}}(t)(\mathbb{X} - \mathring{\mathbb{X}}),$$

where $\mathring{\mathbb{X}}$ is an arbitrary point in $\mathcal{K}_0(\mathcal{B})$, $\underline{\underline{Q}} \in C^N(\mathcal{Q}, \text{Orth}^+(\mathbb{V}_3))$;

$$(iv) \quad \underline{\underline{C}}_{n_0} = \underline{\underline{F}}_{n_0}^T \underline{\underline{F}}_{n_0} = \underline{\underline{I}} \quad \text{on } \mathcal{K}_0(\mathcal{B}) \times \mathcal{Q},$$

¹ $N \geq 2$ is the smoothness index of Axiom (M3).

Proof. (i) $\Leftrightarrow$ (ii): For each $t \in \mathcal{Q}$, the deformation $\mathbb{X}_{n_0}(\cdot, t)$ is identical with the deformation $\mathbb{F}(\cdot, t)$ of Theorem B1.10.1. The claimed result is then simply Theorem B1.10.2.

(ii) $\Leftrightarrow$ (iii): Suppose that the deformation $\mathbb{X}_{n_0}(\cdot, t)$ is rigid. It follows from the proof of Theorem B1.10.4 ((i) $\Rightarrow$ (ii)) that

$$\mathbb{X}_{n_0}(\mathbb{X}, t) = \mathbb{X}_{n_0}(\dot{\mathbb{X}}, t) + \underline{\underline{Q}}(t)(\mathbb{X} - \dot{\mathbb{X}}) \text{ on } \mathbb{N}_0(\mathcal{B}) \times \mathcal{Q}, \quad (*)$$

where $\underline{\underline{Q}}(t) \in \text{Orth } \mathcal{V}_3$ and $\underline{\underline{Q}} \in C^N(\mathcal{Q}, \text{Lin } \mathcal{V}_3)$. That the values of $\underline{\underline{Q}}$ must, in fact, be proper orthogonal is a consequence of Theorem B1.13.11. Thus, (ii) $\Rightarrow$ (iii). Conversely, suppose that $\mathbb{X}_{n_0}$ has the form (*). Then for each $t \in \mathcal{Q}$, the deformation $\mathbb{X}_{n_0}(\cdot, t)$ is rigid by Theorem B1.2.1, i.e., (iii) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (iv): Theorem B1.2.1. $\square$

In the general case where the body does not necessarily ever occupy the reference configuration, we have

Theorem B1.14.2. Let $\{\mathbb{X}(\cdot, t): t \in \mathcal{Q}\}$ and n_0 be a
motion and a reference configuration of a body $\mathcal{B}$. Then the
following statements are equivalent:

(i) the motion is rigid;

(ii) the referential velocity field $\underline{\underline{L}}_{n_0}: \mathbb{N}(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathcal{V}_3$ has

the form

$$\underline{v}_n(\underline{x}, t) = \underline{v}_n(\underline{x}^0, t) + \underline{\Omega}(t) [\underline{x}_n(\underline{x}, t) - \underline{x}_n(\underline{x}^0, t)] \quad)',$$

where $\underline{x}^0$ is an arbitrary point in $\mathcal{N}(\mathcal{B})$ and $\underline{\Omega} \in C^{N-1}(\mathcal{Q}, \text{Skw } \mathcal{V}_3)$ ²;

$$(iii) \quad \underline{D}_n = \underline{0} \quad \text{on } \mathcal{N}(\mathcal{B}) \times \mathcal{Q};$$

$$(iv) \quad \underline{C}_{n,2} = \underline{0} \quad \text{on } \mathcal{N}(\mathcal{B}) \times \mathcal{Q};$$

$$(v) \quad \exists \text{ a function } \hat{\underline{C}}_n : \mathcal{N}(\mathcal{B}) \rightarrow \text{Psym } \mathcal{V}_3 \quad \ni$$

$$\underline{C}_n(\underline{x}, t) = \hat{\underline{C}}_n(\underline{x}) \quad \forall (\underline{x}, t) \in \mathcal{N}(\mathcal{B}) \times \mathcal{Q}.$$

Proof: Apart from the smoothness of $\underline{\Omega}$, the equivalency of (i), (iii), and (iii') follows immediately from Theorem B1.10.4 on converting to the referential description. The smoothness demonstration is left as Exercise B1.14.1 (Hint: Review the proof of $\underline{\Omega} \in C^2(\mathcal{Q}, \text{Orth } \mathcal{V}_3)$ on pp. B1.10.6-7.).

(iii) $\iff$ (iv): This is an easy consequence of Theorem B1.13.10; the details are left as Exercise B1.14.2.

(iv) $\iff$ (v): By the fundamental theorem of integral calculus, for each $\underline{x} \in \mathcal{N}(\mathcal{B})$ and any $t_1 \in \mathcal{Q}$,

¹ Of course, the skew tensor $\underline{\Omega}(t)$ can be replaced by its axial vector as in Theorem B1.10.6.

² $N \geq 2$ is the smoothness index of Axiom (M3).

$$\underline{C}_n(\mathbb{X}, t) - \underline{C}_n(\mathbb{X}, t_1) = \int_{t_1}^t \underline{C}_{n,2}(\mathbb{X}, \tau) d\tau, \quad)^1$$

Suppose that $\underline{C}_{n,2} = \underline{0}$ on $\mathbb{R}(\mathcal{B}) \times \mathcal{Q}$. Then from above

$$\underline{C}_n(\mathbb{X}, t) = \underline{C}_n(\mathbb{X}, t_1) =: \hat{\underline{C}}_n(\mathbb{X});$$

i.e., (iv) $\Rightarrow$ (v). Conversely, suppose that $\underline{C}_n(\mathbb{X}, t) = \hat{\underline{C}}_n(\mathbb{X})$ on $\mathbb{R}(\mathcal{B}) \times \mathcal{Q}$. Then obviously

$$\underline{C}_{n,2} = \underline{0} \text{ on } \mathbb{R}(\mathcal{B}) \times \mathcal{Q};$$

i.e., (v) $\Rightarrow$ (iv). $\square$

Of course, it is important to realize that all of the results of the above theorem for a general reference configuration $\mathbb{R}$ necessarily apply to the occupied reference configuration $\mathbb{R}_0 = \mathbb{X}(\cdot; t_0)$ of Theorem B1.14.1.

Next, we turn to the consideration of isochoric motions in the referential description. We follow the pattern laid out above for rigid motions.

Theorem B1.14.3, Let $\{\mathbb{X}(\cdot; t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Let the reference configuration $\mathbb{R}_0$ be the configuration of $\mathcal{B}$ at the time $t_0 \in \mathcal{Q}$; i.e., $\mathbb{R}_0 = \mathbb{X}(\cdot; t_0)$. Then the following

¹ This calculation makes sense because of the connectedness of the time interval $\mathcal{Q}$; cf. Theorem A5.9.3.

statements are equivalent:

(i) the motion is isochoric;

(ii) each member of the one-parameter family of deformations $\{\chi_{n_0}(\cdot, t) : t \in \mathcal{D}\}$ is isochoric;

(iii) $J_{n_0} = 1$ on $n_0(\mathcal{B}) \times \mathcal{D}$,

Proof. (i) $\Leftrightarrow$ (ii): Theorem B1.12.1.

(ii) $\Leftrightarrow$ (iii): Exercise B1.14.3. $\square$

We now are in position to establish the following important result.

Theorem B1.14.4. A motion $\{\chi(\cdot, t) : t \in \mathcal{D}\}$ of a body $\mathcal{B}$ is isochoric iff

$$\operatorname{div}_\# v_\# = 0 \text{ on } \mathcal{I}.$$

Proof. Recall Theorem B1.13.13, i.e.,

$$J_{n_0, 2}(\chi, t) = J_{n_0}(\chi, t) \operatorname{div}_\# v_\#(\chi_{n_0}(\chi, t), t) \text{ on } n_0(\mathcal{B}) \times \mathcal{D}, (*)$$

where we have chosen the reference configuration n_0 to be the one actually occupied by the body at some time $t_0 \in \mathcal{D}$. Suppose that the motion is isochoric. Then by the previous

theorem, $J_{n_0} = 1$ on $\mathcal{K}_0(\mathcal{B}) \times \mathcal{Q}$; and $(*) \Rightarrow$

$$\operatorname{div}_* \underline{V}_* (\mathbb{X}_{n_0}(\mathbb{X}, t), t) = 0 \text{ on } \mathcal{K}_0(\mathcal{B}) \times \mathcal{Q}$$

$\Rightarrow$

$$\operatorname{div}_* \underline{V}_* (\mathbb{X}, t) = 0 \text{ on } \mathcal{I} \quad (\mathbb{X} = \mathbb{X}_{n_0}(\mathbb{X}, t)).$$

Conversely, suppose that $\operatorname{div}_* \underline{V}_* = 0$ on $\mathcal{I}$. Then $(*) \Rightarrow$
 $J_{n_0, 2} = 0$ on $\mathcal{K}_0(\mathcal{B}) \times \mathcal{Q}$; and by the fundamental theorem
of integral calculus,

$$J_{n_0}(\mathbb{X}, t) - J_{n_0}(\mathbb{X}, t_0) = \int_{t_0}^t J_{n_0, 2}(\mathbb{X}, \tau) d\tau = 0. \quad (**)$$

But the deformation from $\mathcal{K}_0$ corresponding to the time t_0 is the
identity map

$$\mathbb{X}_{n_0}(\mathbb{X}, t_0) = \mathbb{X} \text{ on } \mathcal{K}_0(\mathcal{B}).$$

Hence,

$$J_{n_0}(\mathbb{X}, t_0) = 1 \text{ on } \mathcal{K}_0(\mathcal{B}); \quad)'$$

and $(**) \Rightarrow J_{n_0} = 1$ on $\mathcal{K}_0(\mathcal{B}) \times \mathcal{Q}$. Then we conclude from
the previous theorem that the motion is isochoric. $\square$

It is interesting to note that the statement of the
above theorem, i.e.

the motion is isochoric iff $\operatorname{div}_* \underline{V}_* = 0$ on $\mathcal{I}$,

¹cf. Theorems A5.5.4 and A3.B.2.

has nothing to do with reference configurations; consequently, it is especially useful in fluid mechanics. In solid mechanics, generally the most convenient equivalency is (i) $\Leftrightarrow$ (iii) of Theorem B1.14.3, i.e.

the motion is isochoric iff $J_{n_0} = 1$ on $\kappa_0(B) \times \mathcal{Q}$,

which requires the body to actually occupy the reference configuration κ_0 at some time t_0 during the motion. If we do not choose the reference configuration so that this is the case, then the counterpart of Theorem B1.14.3 is

Theorem B1.14.5. Let $\{\kappa(\cdot, t); t \in \mathcal{Q}\}$ and κ be a motion and a reference configuration of a body B . Then the following statements are equivalent:

(i) the motion is isochoric;

(ii) $J_{n,2} = 0$ on $\kappa(B) \times \mathcal{Q}$;

(iii) $\exists$ a function $\hat{J}_n: \kappa(B) \rightarrow \mathbb{R}^+ \ni$

$$J_n(\kappa, t) = \hat{J}_n(\kappa) \quad \forall (\kappa, t) \in \kappa(B) \times \mathcal{Q}.$$

Proof. (i) $\Leftrightarrow$ (ii): This result follows immediately from Theorems B1.14.4 and B1.13.13.

(ii) $\Leftrightarrow$ (iii): Exercise B1.14.4. $\square$

B1.14.8

Supplementary Reading

Same as for § B1.1

B1.15. Coordinate Analysis in the Referential Description

The mathematical machinery that we have set up so far for dealing with configurations, deformations, and motions is well suited to developing the general theory of continuum mechanics, but it does not always lend itself to work with particular problems, especially in the mechanics of solids. Often in setting up problems by the inverse method, we will know the coordinates of the point occupied by a typical particle in the reference configuration and also the coordinates (perhaps w.r.t. a different coordinate system) of the point occupied by this same particle in a different configuration. For example, the bending, stretching, and shearing of a rectangular block might be described by a deformation of the form

$$r = \sqrt{2AX}, \quad \theta = BY, \quad z = \frac{Z}{AB} - BCY,$$

where the particle initially at the point with rectangular Cartesian coordinates (X, Y, Z) is deformed to the point with circular cylindrical coordinates (r, θ, z) ; here A, B , and C are constants.

With this example in mind, let $\{\underline{e}_i\}$ and $\{\underline{e}_A\}$ denote two bases for an n -dimensional inner product space $\mathcal{V}_n$. We distinguish between them by using lower case letter indices for one basis and upper case letter indices for the other. In the example above,

$$\{\underline{e}_i\} = \{\underline{e}_r, \underline{e}_\theta, \underline{e}_z\} \text{ and } \{\underline{e}_A\} = \{\underline{e}_X, \underline{e}_Y, \underline{e}_Z\}.$$

Obviously, numerical indices, e.g.,

$$\{\underline{e}_i\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\},$$

do not distinguish between different bases. Nonetheless, we shall treat these letter indices numerically with regard to the summation convention. For example,

$$v^i \underline{e}_i = v^r \underline{e}_r + v^\theta \underline{e}_\theta + v^z \underline{e}_z,$$

when $\{\underline{e}_i\} = \{\underline{e}_r, \underline{e}_\theta, \underline{e}_z\}$. The bases dual to $\{\underline{e}_i\}$ and $\{\underline{e}_A\}$ are denoted by $\{\underline{e}^i\}$ and $\{\underline{e}^A\}$, respectively.

Of course, a vector $\underline{u} \in V_n$ admits the representations

$$\underline{u} = u^i \underline{e}_i = u_i \underline{e}^i = u^A \underline{e}_A = u_A \underline{e}^A. \quad)^1$$

The representation possibilities for a second-order tensor are richer. To see this, let $\underline{T} \in \text{Lin } V_n$ and consider the equation $\underline{T}\underline{u} = \underline{v}$ for $\underline{u} \in V_n$. Since

$$\begin{aligned} \underline{T}\underline{u} &= \underline{T}(u^i \underline{e}_i) && \text{(substitution)} \\ &= u^i \underline{T}\underline{e}_i && \text{(linearity of } \underline{T} \text{),} \end{aligned}$$

¹cf. the developments of § A2.2.

$$v_A = e_A \cdot v \quad (\text{Theorem A2.5.8})$$

$$= e_A \cdot T u \quad (\text{substitution})$$

$$= e_A \cdot (u^i T e_i) \quad (\text{substitution})$$

$$= (e_A \cdot T e_i) u^i \quad (\text{linearity of inner product})$$

$$= (e_A \cdot T e_i) (e^i \cdot u) \quad (\text{Theorem A2.5.8});$$

and $\therefore$

$$T u = v_A e^A \quad (\text{substitution})$$

$$= (e_A \cdot T e_i) (e^i \cdot u) e^A \quad (\text{substitution})$$

$$= (e_A \cdot T e_i) (e^A \otimes e^i) u \quad (\text{Theorem A3.1.6}).$$

Since u is an arbitrary vector and since the other cases are handled in the same manner, we have proven

Theorem B1.15.1. Let $\{e_i\}$, $\{e^i\}$ and $\{e_A\}$, $\{e^A\}$ be bases and their duals for V_n . Then any second-order tensor $T \in \text{Lin } V_n$ admits the double representations ¹

¹ Double in the sense that two distinct basis systems. (basis plus its dual) are employed.

$$\begin{aligned}
\tilde{T} &= T_{Ai} \tilde{e}^A \otimes \tilde{e}^i = T_{iA} \tilde{e}^i \otimes \tilde{e}^A, \\
&= T^{Ai} \tilde{e}_A \otimes \tilde{e}_i = T^{iA} \tilde{e}_i \otimes \tilde{e}_A, \\
&= T^A_{\cdot i} \tilde{e}_A \otimes \tilde{e}^i = T^{\cdot A}_i \tilde{e}^i \otimes \tilde{e}_A, \\
&= T^{\cdot i}_A \tilde{e}^A \otimes \tilde{e}_i = T^i_{\cdot A} \tilde{e}_i \otimes \tilde{e}^A,
\end{aligned}$$

where

$$\begin{aligned}
T_{Ai} &= \tilde{e}_A \cdot \tilde{T} \tilde{e}_i, & T_{iA} &= \tilde{e}_i \cdot \tilde{T} \tilde{e}_A, \\
T^{Ai} &= \tilde{e}^A \cdot \tilde{T} \tilde{e}^i, & T^{iA} &= \tilde{e}^i \cdot \tilde{T} \tilde{e}^A, \\
T^A_{\cdot i} &= \tilde{e}^A \cdot \tilde{T} \tilde{e}_i, & T^{\cdot A}_i &= \tilde{e}_i \cdot \tilde{T} \tilde{e}^A, \\
T^{\cdot i}_A &= \tilde{e}_A \cdot \tilde{T} \tilde{e}^i, & T^i_{\cdot A} &= \tilde{e}^i \cdot \tilde{T} \tilde{e}_A.
\end{aligned}$$

Of course, we still have the single representations of Theorem A3.2.2; e.g.,

$$\tilde{T} = T_{ij} \tilde{e}^i \otimes \tilde{e}^j = T_{AB} \tilde{e}^A \otimes \tilde{e}^B,$$

where

$$T_{ij} = \tilde{e}_i \cdot \tilde{T} \tilde{e}_j, \quad T_{AB} = \tilde{e}_A \cdot \tilde{T} \tilde{e}_B.$$

As an example, we consider the identity tensor.

Theorem B1.15.2. The double components of the identity tensor are the inner products of the corresponding base vectors; e.g.,

$$(\underline{\underline{I}})_{Ai} = \underline{e}_A \cdot \underline{e}_i =: g_{Ai}. \quad)^1$$

Proof. Exercise B1.15.1. $\square$

Obviously, these new double g symbols are symmetric; e.g.,

$$g_{iA} = g_{Ai}.$$

However, in general,

$$g_A^i := \underline{e}^i \cdot \underline{e}_A \neq \delta_A^i.$$

Also, we have

Theorem B1.15.3. The double g symbols can be used to raise and lower indices as before²; e.g.,

$$g^{Ai} v_A = v^i, \quad g^{Ai} T_{Aj} = T^i_j, \text{ etc.}$$

Proof. Exercise B1.15.2. $\square$

¹ Cf. Definition A2.5.3 and Theorem A3.2.7.

² Cf. Theorems A2.5.11 and A3.2.8.

As before, the double components of the transpose $\tilde{T}$ are obtained by interchanging the order of the indices on the components of T .¹

Theorem B1.15.4. Let $T \in \text{Lin } V_n$. Then

$$(\tilde{T}^T)^{iA} = T^{Ai}, \quad (\tilde{T}^T)_{iA} = T_{Ai},$$

$$(\tilde{T}^T)^{\cdot i}_{\cdot A} = T^{\cdot i}_{\cdot A}, \quad (\tilde{T}^T)^{\cdot A}_{\cdot i} = T^{\cdot A}_{\cdot i}.$$

Proof. Exercise B1.15.3. $\square$

Multiplication of second-order tensors can be handled in terms of double components by

Theorem B1.15.5. In terms of double components, tensor multiplication follows the rules of matrix multiplication; e.g., for $T, S \in \text{Lin } V_n$,

$$(\tilde{T}\tilde{S})^{\cdot i}_{\cdot j} = T^{\cdot i}_{\cdot A} S^{\cdot A}_{\cdot j}, \quad (\tilde{T}\tilde{S})^{\cdot i}_{\cdot A} = T^{\cdot iB} S_{BA}, \text{ etc. })^2$$

Proof. See the next two exercises. $\square$

¹ Cf. Theorem A3.6.4,

² Cf. Theorem A3.5.4.

Exercise B1.15.4. Prove Theorem B1.15.5 directly by the method which was used to prove Theorem A3.6.4.

Exercise B1.15.5. Prove Theorem B1.15.5 by starting with the known result

$$(\underline{T}\underline{S})^i_j = T^{ik} S_{kj}$$

(Theorem A3.5.4) and getting to the desired results by using the g symbols to move the indices.

We now have enough background to return to the task at hand which is the investigation of the motion relative to a reference configuration κ . Thus, we consider

$$\underline{x} = \underline{\chi}_\kappa(\underline{X}, t), \quad \underline{X} \in \kappa(\mathcal{B}).$$

Since the time t will not play an essential role in this section and since the label κ makes for awkward notation, we simply write

$$\underline{x} = \underline{\chi}(\underline{X}), \quad \underline{X} \in \kappa(\mathcal{B}).$$

As was noted earlier, in the practice of solid mechanics, such deformations are often specified by giving the coordinates of point $\underline{x}$ in terms of the coordinates of point $\underline{X}$, and generally the coordinate system used to locate $\underline{x}$ is different from the one used to locate $\underline{X}$. Thus, let there be two

coordinate systems for E_3 defined by the correspondences

$$\mathbb{X} = \hat{\mathbb{X}}(X^1, X^2, X^3), \quad X^A = \hat{X}^A(\mathbb{X}), \quad A=1,2,3$$

and

$$\mathfrak{x} = \hat{\mathfrak{x}}(x^1, x^2, x^3), \quad x^i = \hat{x}^i(\mathfrak{x}), \quad i=1,2,3.$$

Note that upper case indices are employed for the coordinate system used to describe $\mathbb{X}$, and lower case indices for the coordinate system for $\mathfrak{x}$ — a pattern that is easy to keep straight. ¹

Since $\mathfrak{x} = \mathbb{X}(\mathbb{X})$, we can write

$$x^i = \hat{x}^i(\mathfrak{x}) = \hat{x}^i(\mathbb{X}(\mathbb{X})) = \hat{x}^i(\mathbb{X}(\hat{\mathbb{X}}(X^1, X^2, X^3)));$$

or, as is customary,

$$x^i = x^i(X^A).$$

By the chain rule, ²

$$\frac{\partial x^i}{\partial X^A}(X^1, X^2, X^3) = \nabla \hat{x}^i(\mathfrak{x}) \cdot \left[D\mathbb{X}(\mathbb{X}) \hat{\mathbb{X}}_{,A}(X^1, X^2, X^3) \right].$$

¹ TRUESDELL and NOLL in "The Non-linear Field Theories of Mechanics" use Greek indices in place of our upper case Latin indices

² Work out the details as Exercise B1.15.6.

But by Theorems A6.11.1 and 2,

$$\nabla \hat{x}^i(x) = \underline{e}^i(x) \text{ and } \hat{x}_A(\hat{x}^1(X), \hat{x}^2(X), \hat{x}^3(X)) = \underline{e}_A(X);)^1$$

and by Definition B1.13.4,

$$D\mathbb{X}(X) = \underline{F}(X).$$

is the deformation gradient. Hence, we are led to

Theorem B1.15.6. Let the deformation $\mathbb{X}$ ($x = \mathbb{X}(X)$)
relative to the reference configuration $\mathbb{K}$ ($X = \mathbb{K}(x)$) be
specified in terms of coordinates by

$$x^i = x^i(X^A) \text{ (with inverse } X^A = X^A(x^i))$$

in accordance with the scheme outlined above, then the
double components of the deformation gradient $\underline{F} = D\mathbb{X}$
and its inverse are given by

¹Again note the crucial role played by the upper and lower case indices in keeping the two coordinate systems straight. It is important to realize that even if the two coordinate systems are the same, e.g., both spherical, we still will generally have $\{\underline{e}_i\} \neq \{\underline{e}_A\}$, since $\underline{e}_i$ is evaluated at x and $\underline{e}_A$ at X . The distinction vanishes iff the natural bases are independent of position; i.e., iff x and X are referred to the same Cartesian coordinate system.

$$F^i_{\bullet A} = \underline{e}^i(x) \cdot \underset{\sim}{F}(\mathbb{X}) \underset{\sim}{e}_A(\mathbb{X}) = \frac{\partial x^i}{\partial X^A} \quad)^1$$

and

$$(\underset{\sim}{F}^{-1})^A_{\bullet \bar{i}} = \frac{\partial X^A}{\partial x^{\bar{i}}}.$$

Proof. Only the last assertion remains to be established, and it follows from applying the above procedure to the inverse deformation $\mathbb{X}^{-1}$ ($\mathbb{X} = \mathbb{X}^{-1}(x)$).

Alternatively, one could start with the statement

$$\underset{\sim}{F} \underset{\sim}{F}^{-1} = \underline{I},$$

which in terms of double components is equivalent to

$$F^i_{\bullet A} (\underset{\sim}{F}^{-1})^A_{\bullet j} = (\underline{I})^i_{\bullet j} = \delta^i_j.$$

Then by the first part of the present theorem,

$$\frac{\partial x^i}{\partial X^A} (\underset{\sim}{F}^{-1})^A_{\bullet j} = \delta^i_j,$$

which $\Rightarrow$

$$\frac{\partial X^B}{\partial x^i} \frac{\partial x^i}{\partial X^A} (\underset{\sim}{F}^{-1})^A_{\bullet j} = \frac{\partial X^B}{\partial x^i} \delta^i_j = \frac{\partial X^B}{\partial x^j}.$$

¹ Since $x = \mathbb{X}(\mathbb{X})$, $x = \hat{x}(x^1, x^2, x^3)$, etc., $F^i_{\bullet A}$ can be viewed as a function of $\mathbb{X}$ or X^A or x or x^i — whichever is most convenient. Because of the roles played here by both x and $\mathbb{X}$, though $\underline{e}^i(x)$ and $\underline{e}_A(\mathbb{X})$, respectively, $F^i_{\bullet A}$ is sometimes called a two point tensor field.

But by the chain rule, ¹

$$\frac{\partial x^B}{\partial x^i} \frac{\partial x^i}{\partial x^A} = \frac{\partial x^B}{\partial x^A} = \delta_A^B;$$

and $\therefore$

$$(\underline{F}^{-1})^B_{\cdot j} = \frac{\partial x^B}{\partial x^j} \cdot \square$$

It is interesting to note that covariant differentiation, CHRISTOFFEL symbols, etc. do not show up when one calculates the table components of $\underline{F} = D\mathbf{x}$, even though both coordinate systems are entirely general.

Any of the other components of $\underline{F}$ can be obtained readily by utilizing the g symbols to raise and lower indices, e.g.,

$$\begin{aligned} F^{ij} &= g^{Aj} F^i_{\cdot A} \quad)^2 \\ &= g^{Aj} \frac{\partial x^i}{\partial x^A} \end{aligned}$$

Collecting such results together, we have

¹ Work out the details as Exercise B1.15.7.

² Since these double g symbols convert tensor components to those associated with a basis attached to a different point, they are sometimes called shifters.

Theorem B1.15.7. In the terminology of Theorem B1.15.6, the single components of the deformation gradient
 $E = D\mathbf{x}$ are given by

$$F^{ij} = g^{Aj} \frac{\partial x^i}{\partial X^A} \quad , \quad F_{ij} = g_{ki} g_j^A \frac{\partial x^k}{\partial X^A} \quad ,$$

$$F^{AB} = g_i^A g^{BC} \frac{\partial x^i}{\partial X^C} \quad , \quad F_{AB} = g_{Ai} \frac{\partial x^i}{\partial X^B} \quad ,$$

Proof. Exercise B1.15.8. $\square$

We also shall need expressions for the right and left deformation tensors

$$\underline{C} = \underline{F}^T \underline{F} \quad \text{and} \quad \underline{B} = \underline{F} \underline{F}^T \quad . \quad)^1$$

¹g. Definition B1.13.5,

Theorem B1.15.B. In the terminology of Theorem B1.15.6,
the components of the right and left deformation tensors are
given by

$$C_{AB} = g_{kl} \frac{\partial x^k}{\partial x^A} \frac{\partial x^l}{\partial x^B}, \quad C^{AB} = g^{AK} g^{BL} g_{kl} \frac{\partial x^k}{\partial x^A} \frac{\partial x^l}{\partial x^B},$$

$$C_{ij} = g^A_i g^B_j g_{kl} \frac{\partial x^k}{\partial x^A} \frac{\partial x^l}{\partial x^B}, \quad C^{ij} = g^{iA} g^{jB} g_{kl} \frac{\partial x^k}{\partial x^A} \frac{\partial x^l}{\partial x^B},$$

$$B^{ij} = g^{KL} \frac{\partial x^i}{\partial x^K} \frac{\partial x^j}{\partial x^L}, \quad B_{ij} = g_{ik} g_{jl} g^{KL} \frac{\partial x^i}{\partial x^K} \frac{\partial x^j}{\partial x^L},$$

$$B^{AB} = g^A_i g^B_j g^{KL} \frac{\partial x^i}{\partial x^K} \frac{\partial x^j}{\partial x^L}, \quad B_{AB} = g_{Ai} g_{Bj} g^{KL} \frac{\partial x^i}{\partial x^K} \frac{\partial x^j}{\partial x^L},$$

$$(\tilde{B}^{-1})_{ij} = g_{KL} \frac{\partial x^K}{\partial x^i} \frac{\partial x^L}{\partial x^j}, \quad (\tilde{B}^{-1})^{ij} = g^{ik} g^{jl} g_{KL} \frac{\partial x^K}{\partial x^k} \frac{\partial x^L}{\partial x^l},$$

$$(\tilde{B}^{-1})_{AB} = g^i_A g^j_B g_{KL} \frac{\partial x^K}{\partial x^i} \frac{\partial x^L}{\partial x^j}, \quad (\tilde{B}^{-1})^{AB} = g^{Ai} g^{Bj} g_{KL} \frac{\partial x^K}{\partial x^i} \frac{\partial x^L}{\partial x^j}.$$

Proof. Only the first (and simplest) expression in each grouping need be calculated from "first principles"; then the rest of the expressions in that grouping follow from straight forward raising, lowering, and shifting of indices with the g symbols. We will work out $(\tilde{B}^{-1})_{ij}$ here. The

calculation of C_{AB} and B'^i is left as Exercise B1.15.9.

First of all by Theorems A3.9.7(iii) and A3.9.12,

$$\underline{B}^{-1} = (\underline{F}\underline{F}^T)^{-1} = (\underline{F}^T)^{-1}\underline{F}^{-1} = (\underline{F}^{-1})^T \underline{F}^{-1}.$$

$\therefore$

$$(\underline{B}^{-1})_{ij} = [(\underline{F}^{-1})^T]_{\cdot i}^{\cdot K} (\underline{F}^{-1})_{Kj} \quad (\text{Theorem B1.15.5})$$

$$= (\underline{F}^{-1})_{\cdot i}^{\cdot K} (\underline{F}^{-1})_{Kj} \quad (\text{Theorem B1.15.4})$$

$$= (\underline{F}^{-1})_{\cdot i}^{\cdot K} g_{KL} (\underline{F}^{-1})_{\cdot j}^{\cdot L} \quad (\text{Theorem B1.15.3})$$

$$= g_{KL} \frac{\partial x^K}{\partial x^i} \frac{\partial x^L}{\partial x^j} \quad (\text{Theorem B1.15.6}). \quad \square$$

This scheme of working with the components of tensor fields in terms of two coordinate systems attached to two different points has an elementary but messy theory known as two-point tensor fields or double tensor fields. We could easily work out transformation rules, covariant differentiation, higher order tensors, etc.; but we will have no need of these things.

Supplementary Reading

ERICKSEN, "Tensor fields", Appendix to Truesdell and Toupin's CFT (below).

LEIGH, Nonlinear Continuum Mechanics.

TRUESDELL and NOLL, "The non-linear field theories of mechanics", Handbuch der Physik, vol. III/3 (edit. Flügge) 1965.

TRUESDELL and TOUPIN, "The classical field theories", Handbuch der Physik, vol III/1 (edit. Flügge) 1960.

B1.16. The Present Configuration as the Reference Configuration. The Relative Description

In some cases (especially for fluids with long-range memory), it is convenient to use the configuration that the body is in at time t as a reference configuration. Generally, t is the present time, and earlier times are denoted by τ .

As usual, $\mathbf{x}$ denotes the position of the particle $X \in \mathcal{B}$ at time t ; and now we write $\mathbf{\xi}$ for the place occupied by X at time τ . Thus, if $\{\mathbf{x}(\cdot, t)\}$ is the motion of $\mathcal{B}$,

$$\mathbf{x} = \mathbf{x}(X, t) \quad \text{and} \quad \mathbf{\xi} = \mathbf{x}(X, \tau).$$

Solving the first equation for X ¹ and substituting the result into the second equation yields

$$\mathbf{\xi} = \mathbf{x}(\mathbf{x}^{-1}(\mathbf{x}, t), \tau) =: \mathbf{x}_{(t)}(\mathbf{x}, \tau)$$
²

for the position at time τ of the particle that is at $\mathbf{x}$ at time t . By Axiom (B1) of Definition B1.7.1,

¹Cf. Definition B1.8.2 and the surrounding discussion.

²Often, the notation $\mathbf{x}_t$ is used in place of $\mathbf{x}_{(t)}$; but, for us, $\mathbf{x}_t$ means $\mathbf{x}(\cdot, t)$; cf. Definition B1.8.2 and the discussion that precedes it. Another, common notation is to write $\mathbf{\xi} = \mathbf{x}(\mathbf{x}, \tau; t)$.

$$\mathbb{X}_{(t)}(\cdot, \tau) = \mathbb{X}(\cdot, \tau) \circ \mathbb{X}^{-1}(\cdot, t) : \mathbb{B}_t \rightarrow \mathbb{E}_3$$

is a deformation on the region $\mathbb{B}_t$ for each $\tau \in \mathcal{Q}$. On comparing this arrangement with Definition B1.13.1, we see that here we have a special case of motion relative to a reference configuration with $\mathbb{X}(\cdot, t)$ (the configuration of the body at time t) being the reference configuration. Of course, as the time t varies, this reference configuration varies. Since it was assumed in § B1.13 that the reference configuration $\mathbb{B}$ was fixed, not many of the results obtained there will carry over to the present section. Formally, we make

Definition B1.16.1. Given a motion $\{\mathbb{X}(\cdot, t), t \in \mathcal{Q}\}$ of a body $\mathbb{B}$, the one-parameter family of deformations $\{\mathbb{X}_{(t)}(\cdot, \tau) : \tau \in \mathcal{Q}\}$ on $\mathbb{B}_t$ defined by

$$\mathbb{X}_{(t)}(\cdot, \tau) = \mathbb{X}(\cdot, \tau) \circ \mathbb{X}^{-1}(\cdot, t) \quad [\mathbb{X}_{(t)}(\mathbf{x}, \tau) = \mathbb{X}(\mathbb{X}^{-1}(\mathbf{x}, t), \tau), \mathbf{x} \in \mathbb{B}_t]$$

is the motion of $\mathbb{B}$ relative to the present configuration $\mathbb{X}(\cdot, t)$. The function $\mathbb{X}_{(t)} : \mathbb{B}_t \times \mathcal{Q} \rightarrow \mathbb{E}_3$ is the relative deformation¹.

Since $\mathbf{x} = \mathbb{X}(\mathbf{X}, \tau)$, ordinary partial differentiation w.r.t. the second variable yields

$$\frac{\partial \mathbf{x}}{\partial \tau} = \mathbb{X}_{,2}(\mathbf{X}, \tau)$$

¹ Sometimes, relative transplacement.

$$= \underline{v}_m(X, \tau) \quad (\text{Defn. B1.8.3 and First Notational Agreement})$$

$$= \underline{v}_m(\mathbb{X}^{-1}(\xi, \tau), \tau) \quad (\text{substitution from Defn. B1.8.2})$$

$$= \underline{v}_s(\xi, \tau) \quad (\text{Defn. B1.8.5}).$$

In many flow situations, the spatial velocity field $\underline{v}_s$ is known, and the relative deformation $\mathbb{X}_{(t)}$ can be found by solving the ordinary differential equation

$$\frac{\partial \xi}{\partial \tau} = \underline{v}_s(\xi, \tau)$$

subject to the initial condition

$$\xi \Big|_{\tau=t} = x,$$

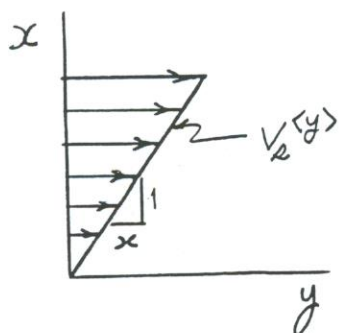
and then recalling that

$$\xi = \mathbb{X}_{(t)}(x, \tau).$$

Exercise B1.16.1. Relative to a fixed rectangular Cartesian coordinate system and in an obvious unabridged notation, the spatial velocity field of simple shearing flow is given by

$$\underline{v}_s^{(x)} = 0, \quad \underline{v}_s^{(y)} = \kappa x, \quad \underline{v}_s^{(z)} = 0,$$

where the constant κ is called the rate of shear.



Carry out the procedure outlined above to find that the relative deformation corresponding to simple shearing flow is given by

$$\xi = \chi_{(t)}^x(x, \tau) = x, \quad \eta = \chi_{(t)}^y(x, \tau) = y + x(x(\tau - t)), \quad \zeta = \chi_{(t)}^z(x, \tau) = z.$$

By Definition B1.9., the pathline of a particle is the space curve traced out by the particle as time progresses. Therefore, for fixed x and t ,

$$\xi = \chi_{(t)}(x, \tau), \quad \tau \in \mathcal{D}$$

is a parametric description of the pathline of that particle which will be at the place x at time t .

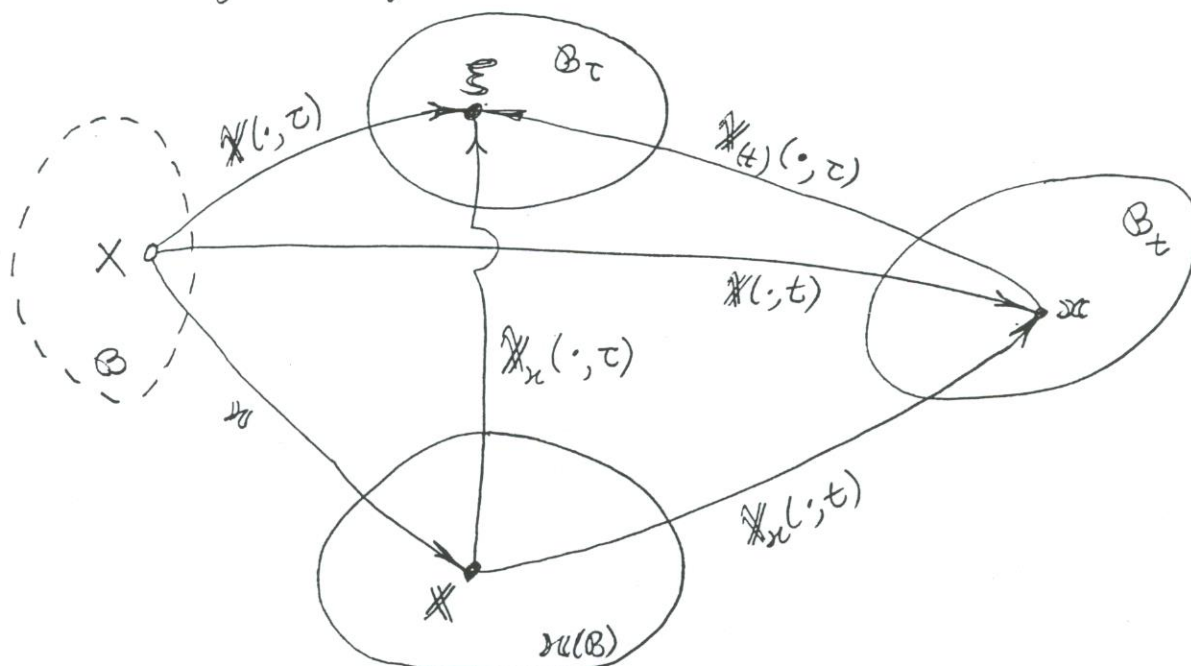
In analogy with Definition B1.13.4, the spatial derivative of the relative deformation $\chi_{(t)}$ is called the relative deformation gradient. Formally, we make

Definition B1.16.2. Let $\{\chi_{(t)}(\cdot, \tau) : \tau \in \mathcal{D}\}$ be a motion of a body $\mathcal{B}$ relative to the present configuration $\chi(\cdot, t)$. Then

$$F_{(t)}(x, \tau) = D_1 \chi_{(t)}(x, \tau)$$

is the relative deformation gradient at the place $x \in B_t$ and the time $\tau \in \mathcal{Q}$.

We need to know the relationship between the relative deformation gradient $F_{(t)}$ and the deformation gradient relative to a reference configuration x , i.e. F_x . To this end the following diagram will be helpful.



Since

$$X_x(\cdot, \tau) = X_{(t)}(\cdot, \tau) \circ X_x(\cdot, t),$$

the chain rule (Theorem A5.7.1) implies that for $x \in \overline{x(B)}$

$$D_1 X_x(x, \tau) = D_1 X_{(t)}(x_tilde, \tau) D_1 X_x(x_tilde, t)$$

or

$$F_{\sim x}(\cdot, \tau) = F_{\sim(t)}(\cdot, \tau) F_{\sim x}(\cdot, t)$$

by Definitions B1.13.4 and B1.16.2. As usual, we can extend

to the boundary via the Extension Theorem (Theorem B1.2.2).
 Actually, we also could have gotten this result by careful application of Theorem B1.13.15. Thus, we are lead to

Theorem B1.16.1. Let $\mathcal{x}$ be a reference configuration of a
body $\mathcal{B}$, and let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of $\mathcal{B}$. Then

$$F_{\sim n}(\mathbb{X}, \tau) = F_{\sim(t)}(\mathcal{x}, \tau) F_{\sim n}(\mathbb{X}, t), \quad \mathbb{X} \in \mathbb{X}(\mathcal{B}), \quad t \in \mathcal{Q}, \tau \in \mathcal{Q},$$

where $\mathcal{x} = \mathbb{X}_n(\mathbb{X}, t)$. Also, if $\mathcal{x}' = \mathbb{X}(X, t') = \mathbb{X}_n(\mathbb{X}, t')$, then

$$F_{\sim(t')}(\mathcal{x}', \tau) = F_{\sim(t)}(\mathcal{x}, \tau) F_{\sim(t')}(\mathcal{x}', t), \quad t, t', \tau \in \mathcal{Q}.$$

Proof. Only the last assertion remains to be proven, and this is left as Exercise B1.16.2. $\square$

Since $\mathbb{X} = \mathbb{X}_n^{-1}(\mathcal{x}, t)$ and since the tensor $F_{\sim n}(\mathbb{X}, t)$ is invertible¹, the above result can be written in the form

$$F_{\sim(t)}(\mathcal{x}, \tau) = F_{\sim n}(\mathbb{X}_n^{-1}(\mathcal{x}, t), \tau) [F_{\sim n}(\mathbb{X}_n^{-1}(\mathcal{x}, t), t)]^{-1}; \quad (*)$$

and this enables us to read off the smoothness properties of $F_{\sim(t)}$ as a function on $\mathcal{B}_t \times \mathcal{Q}$ for each $t \in \mathcal{Q}$. By Theorems B1.13.4 and B1.13.7,

$$\mathbb{X}_n^{-1} \in C^N(\mathcal{I}, \mathcal{N}(\mathcal{B})) \text{ and } F_{\sim n} \in C^{N-1}(\mathcal{N}(\mathcal{B}) \times \mathcal{Q}, \text{Inv } V_3),$$

¹ Cf. Theorems B1.13.7 (ii) and A3.9.9.

where N is the smoothness index of Axiom (M3):

$$\mathbb{X}_n \in C^N(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \mathcal{E}_3), \quad N \geq 2.$$

Consider the first factor of $(*)$ for arbitrarily fixed $t \in \mathcal{Q}$.¹

$$\tilde{F}_n(\mathbb{X}_n^{-1}(\mathbb{X}, t), \tau) = \tilde{F}_n(\tilde{g}(\mathbb{X}, \tau)) =: \tilde{H}(\mathbb{X}, \tau),$$

where

$$\tilde{g}(\mathbb{X}, \tau) = (\mathbb{X}_n^{-1}(\mathbb{X}, t), \tau).$$

Since $\mathbb{X}_n^{-1} \in C^N(\mathcal{Y}, \mathcal{U}(\mathcal{B}))$, it is not difficult to show that

$$\tilde{g} \in C^N(\mathcal{B}_t \times \mathcal{Q}, \mathcal{U}(\mathcal{B}) \times \mathcal{Q})$$

the details are left as Exercise B1.16.3. Then since

$\tilde{F}_n \in C^{N-1}(\mathcal{U}(\mathcal{B}) \times \mathcal{Q}, \text{Inv } \mathcal{V}_3)$, it follows from the Chain Rule in the form of Theorem A5.10.7 that the composite function

$$\tilde{H} \in C^{N-1}(\mathcal{B}_t \times \mathcal{Q}, \text{Inv } \mathcal{V}_3).$$

Similarly, for the second factor of $(*)$,

$$\tilde{F}_n(\mathbb{X}_n^{-1}(\mathbb{X}, t), t) = \tilde{F}_n(\tilde{j}(\mathbb{X}, \tau)) =: \tilde{K}(\mathbb{X}, \tau),$$

where

$$\tilde{j}(\mathbb{X}, \tau) = (\mathbb{X}_n^{-1}(\mathbb{X}, t), t).$$

¹As usual, we calculate on the interior and extend the results to the boundary via the Extension Theorem (Theorem B1.2.2).

From the smoothness properties of $\mathbb{X}_n^{-1}$ and F_n , it follows readily (Exercise B1.16.A) that

$$\underline{j} \in C^N(\mathbb{B}_t \times \mathcal{Q}, \mathcal{N}(\mathbb{B}) \times \mathcal{Q}) \text{ and } \underline{K} \in C^{N-1}(\mathbb{B}_t \times \mathcal{Q}, \text{Inv } \mathcal{V}_3).$$

But our interest here is in

$$[F_{\underline{j}}(\mathbb{X}_n^{-1}(\underline{x}, t), t)]^{-1} = [\underline{K}(\underline{x}, \tau)]^{-1} =: \underline{M}(\underline{x}, \tau).$$

Hence,

$$\underline{M}(\underline{x}, \tau) = \underline{F}(\underline{K}(\underline{x}, \tau)),$$

where $\underline{F}$ is the tensor inverse function defined on $\text{Inv } \mathcal{V}_3$ to $\text{Inv } \mathcal{V}_3$ by $\underline{F}(\underline{X}) = \underline{X}^{-1}$.

By Theorem A5.10.6,

$$\underline{F} \in C^\infty(\text{Inv } \mathcal{V}_3, \text{Inv } \mathcal{V}_3);$$

and $\therefore$ by Theorem A5.10.7,

$$\underline{M} \in C^{N-1}(\mathbb{B}_t \times \mathcal{Q}, \text{Inv } \mathcal{V}_3).$$

Finally, by (*) the relative deformation gradient is defined on $\mathbb{B}_t \times \mathcal{Q}$ to $\text{Inv } \mathcal{V}_3$ by

$$\underline{F}_{\underline{j}}(\underline{x}, \tau) = \underline{H}(\underline{x}, \tau) \underline{M}(\underline{x}, \tau).$$

Then repeated use of the Product Rule in the form of Theorem A5.6.7 shows that

$$\underset{\sim}{F}_{(t)} \in C^{N-1}(\mathcal{B}_t \times \mathcal{Q}, \text{Inv } V_3).$$

Thus, we have established most of

Theorem B1.16.2. Let $\{\underset{\sim}{X}_{(t)}(\cdot, \tau) : \tau \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to the present configuration $\underset{\sim}{X}(\cdot, t)$. Then for each $t \in \mathcal{Q}$,

$$(i) \quad \underset{\sim}{F}_{(t)} \in C^{N-1}(\mathcal{B}_t \times \mathcal{Q}, \text{Inv } V_3) \quad)^1$$

and

$$(ii) \quad J_{(t)}(\underset{\sim}{x}, \tau) := \det \underset{\sim}{F}_{(t)}(\underset{\sim}{x}, \tau) > 0, \quad (\underset{\sim}{x}, \tau) \in \mathcal{B}_t \times \mathcal{Q}$$

Proof. Only (ii) remains to be proved, and that constitutes Exercise B1.16.5. $\square$

Rotation, stretch, and deformation tensors are based on the relative deformation gradient in the natural way.²

Definition B1.16.3. Let $\{\underset{\sim}{X}_{(t)}(\cdot, \tau) : \tau \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$ relative to the present configuration $\underset{\sim}{X}(\cdot, t)$. Then at each $(\underset{\sim}{x}, \tau) \in \mathcal{B}_t \times \mathcal{Q}$, the relative deformation gradient $\underset{\sim}{F}_{(t)}(\underset{\sim}{x}, \tau)$ admits the polar decompositions

¹ $N \geq 2$ is the smoothness index of Axiom (M3).

² Cf. Definition B1.13.5

$$\underset{\sim(t)}{F}(\underline{x}, \tau) = \underset{\sim(t)}{R}(\underline{x}, \tau) \underset{\sim(t)}{U}(\underline{x}, \tau) = \underset{\sim(t)}{V}(\underline{x}, \tau) \underset{\sim(t)}{R}(\underline{x}, \tau),$$

where $\underset{\sim(t)}{R}(\underline{x}, \tau) \in \text{Orth } \mathcal{V}_3$ and $\underset{\sim(t)}{U}(\underline{x}, \tau), \underset{\sim(t)}{V}(\underline{x}, \tau) \in \text{Psym } \mathcal{V}_3$.

We use the following terminology:

$\underset{\sim(t)}{R}(\underline{x}, \tau)$ --- the relative rotation tensor at $(\underline{x}, \tau)$;

$\underset{\sim(t)}{U}(\underline{x}, \tau)$ --- the relative right stretch tensor at $(\underline{x}, \tau)$;

$\underset{\sim(t)}{V}(\underline{x}, \tau)$ --- the relative left stretch tensor at $(\underline{x}, \tau)$;

$\underset{\sim(t)}{C}(\underline{x}, \tau) := [\underset{\sim(t)}{U}(\underline{x}, \tau)]^2 = [\underset{\sim(t)}{F}(\underline{x}, \tau)]^T \underset{\sim(t)}{F}(\underline{x}, \tau)$ --- the relative
right deformation tensor¹ at $(\underline{x}, \tau)$;

and

$\underset{\sim(t)}{B}(\underline{x}, \tau) := [\underset{\sim(t)}{V}(\underline{x}, \tau)]^2 = \underset{\sim(t)}{F}(\underline{x}, \tau) [\underset{\sim(t)}{F}(\underline{x}, \tau)]^T$ --- the relative
left deformation tensor² at $(\underline{x}, \tau)$.

The smoothness properties of the above fields are easily inferred from Theorem B1.16.2.³

¹ Usually, relative right CAUCHY-GREEN tensor.

² Usually, relative left CAUCHY-GREEN tensor.

³ Cf. the proof of Theorem B1.13.8.

Theorem B1.16.3. Let $\{\mathbb{X}_{(t)}(\cdot, \tau) : \tau \in \mathcal{Q}\}$ be a motion of
a body $\mathcal{B}$ relative to the present configuration $\mathbb{X}(\cdot, t)$. Then
for each $t \in \mathcal{Q}$,

$$\underline{U}_{(t)}, \underline{V}_{(t)}, \underline{C}_{(t)}, \underline{B}_{(t)} \in C^{N-1}(\mathcal{B} \times \mathcal{Q}, \text{Psym } \mathcal{V}_3)$$

and

$$\underline{R}_{(t)} \in C^{N-1}(\mathcal{B} \times \mathcal{Q}, \text{Orth}^+ \mathcal{V}_3).$$

Proof. Exercise B1.16.6. $\square$

Theorem B1.16.1 implies the relationship between $\underline{C}_{(t)}$ and $\underline{C}_x$,
 but it does not lead to one between $\underline{B}_{(t)}$ and $\underline{B}_x$.

Theorem B1.16.4. Let $\mathcal{K}$ be a reference configuration of
a body $\mathcal{B}$, and let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of $\mathcal{B}$. Then
for $t, \tau \in \mathcal{Q}$ and $\mathbb{X} \in \mathcal{K}(\mathcal{B})$,

$$(i) \quad \underline{C}_x(\mathbb{X}, \tau) = [\underline{F}_x(\mathbb{X}, t)]^T \underline{C}_{(t)}(\mathbb{X}, \tau) \underline{F}_x(\mathbb{X}, t),$$

$$(ii) \quad \underline{B}_x(\mathbb{X}, \tau) = \underline{F}_{(t)}(\mathbb{X}, \tau) \underline{B}_{(t)}(\mathbb{X}, t) [\underline{F}_{(t)}(\mathbb{X}, \tau)]^T,$$

and

$$(iii) \quad \underline{B}_{(t)}(\mathbb{X}, \tau) = \underline{F}_x(\mathbb{X}, \tau) [\underline{C}_x(\mathbb{X}, t)]^{-1} [\underline{F}_x(\mathbb{X}, \tau)]^T,$$

where $\mathbb{X} = \mathbb{X}_x(\mathbb{X}, t)$.

Proof. Exercise B1.16.7. $\square$

When $\tau = t$, $\mathbb{X}_{(t)}(\cdot, \tau)$ reduces to the identity map on $\mathcal{B}_t$, and sometimes it is useful to recognize this and its consequences.

Theorem B1.16.5. Let $\{\mathbb{X}_{(t)}(\cdot, \tau) : \tau \in \mathcal{I}\}$ be a motion of a body $\mathcal{B}$ relative to the present configuration $\mathbb{X}(\cdot, t)$. Then for each $(\mathbb{X}, t) \in \mathcal{I}$,

$$\mathbb{X}_{(t)}(\mathbb{X}, t) = \mathbb{X},$$

$$\underline{F}_{(t)}(\mathbb{X}, t) = \underline{R}_{(t)}(\mathbb{X}, t) = \underline{U}_{(t)}(\mathbb{X}, t) = \underline{V}_{(t)}(\mathbb{X}, t) = \underline{C}_{(t)}(\mathbb{X}, t) = \underline{B}_{(t)}(\mathbb{X}, t) = \underline{I},$$

and

$$\underline{J}_{(t)}(\mathbb{X}, t) = 1.$$

Proof. Exercise B1.16.8. $\square$

The relative deformation machinery can be used to obtain interesting interpretations of the spatial velocity gradient $\underline{L}$ ¹, the stretching tensor $\underline{D}$ ², and the spin tensor $\underline{W}$ ³. We start by differentiating the identity

$$\underline{F}_{(n)}(\mathbb{X}, \tau) = \underline{F}_{(t)}(\mathbb{X}, \tau) \underline{F}_{(n)}(\mathbb{X}, t) \quad)^4$$

w.r.t. τ to get)⁵

^{1,2,3} See Definition B1.8.6.

⁴ See Theorem B1.16.1.

⁵ See Theorems A5.5.4(iii) and A5.3.1.

$$\tilde{F}_{n,2}(\mathbb{X}, \tau) = \tilde{F}_{(t),2}(\mathbb{X}, \tau) \tilde{F}_n(\mathbb{X}, t),$$

or

$$\tilde{F}_{(t),2}(\mathbb{X}, \tau) = \tilde{F}_{n,2}(\mathbb{X}, \tau) [\tilde{F}_n(\mathbb{X}, t)]^{-1}.$$

Then evaluation at $\tau = t$ yields

$$\begin{aligned} \tilde{F}_{(t),2}(\mathbb{X}, t) &= \tilde{F}_{n,2}(\mathbb{X}, t) [\tilde{F}_n(\mathbb{X}, t)]^{-1} \\ &= \tilde{L}_n(\mathbb{X}, t) \quad (\text{Theorem B1.13.9}) \\ &= \tilde{L}_n(\mathbb{X}, t) \quad (\text{Definition B1.13.2}). \quad (*) \end{aligned}$$

On the other hand, if we start with the polar decomposition

$$\tilde{F}_{(t)}(\mathbb{X}, \tau) = \tilde{R}_{(t)}(\mathbb{X}, \tau) \tilde{U}_{(t)}(\mathbb{X}, \tau) = \tilde{V}_{(t)}(\mathbb{X}, \tau) \tilde{R}_{(t)}(\mathbb{X}, \tau) \quad)',$$

Differentiate w.r.t. τ , then set $\tau = t$, and use Theorem B1.16.5, we get

$$\tilde{F}_{(t),2}(\mathbb{X}, t) = \tilde{R}_{(t),2}(\mathbb{X}, t) + \tilde{U}_{(t),2}(\mathbb{X}, t) = \tilde{V}_{(t),2}(\mathbb{X}, t) + \tilde{R}_{(t),2}(\mathbb{X}, t). \quad (**)$$

Obviously, $\tilde{U}_{(t),2}(\mathbb{X}, t)$ and $\tilde{V}_{(t),2}(\mathbb{X}, t)$ are symmetric because of the symmetry of $\tilde{U}_{(t)}(\mathbb{X}, \tau)$ and $\tilde{V}_{(t)}(\mathbb{X}, \tau)$. Also it is easy to see that $\tilde{R}_{(t),2}(\mathbb{X}, t)$ is skew since $\tilde{R}_{(t)}(\mathbb{X}, \tau)$ is orthogonal; the details of this calculation are left as Exercise B1.16.9 (cf. the analysis of $\tilde{\Omega}$ on pp. B1.10.7-8.). By Theorem A3.6.5,

See Definition B1.16.3.

the decomposition of a tensor into symmetric and skew parts is unique; and, therefore, on comparison of (*) and (**) with Definition B1.8.6, we have

Theorem B1.16.6. Let $\{\chi(\cdot, t) : t \in \mathcal{I}\}$ be a motion of a
body $\mathcal{B}$. Then for each $(x, t) \in \mathcal{I}$,

$$(i) \quad \underline{F}_{(t),2}(x, t) = \underline{L}_s(x, t),$$

$$(ii) \quad \underline{U}_{(t),2}(x, t) = \underline{V}_{(t),2}(x, t) = \underline{D}_s(x, t),$$

and

$$(iii) \quad \underline{R}_{(t),2}(x, t) = \underline{W}_s(x, t).$$

Thus, we see that the spatial velocity gradient $\underline{L}(\cdot, t)$ is the rate of change of the gradient of the deformation that connects the configurations of the body at times t and $t + \epsilon$ as $\epsilon \rightarrow 0$. The stretching $\underline{D}$ and the spin $\underline{W}$ can be interpreted similarly.

The rates of the relative right deformation tensor $\underline{C}_{(t)}$ are useful in the modelling of certain fluids. Formally, we have

Definition B1.16.4. Let $\{\chi(\cdot, t) : t \in \mathcal{I}\}$ be a motion
of a body $\mathcal{B}$. Then

$$\underline{A}_{\sim n}(x, t) = \underline{C}_{(t), \underbrace{22 \dots 2}_{n \text{ times}}}(x, t)$$

is the n^{th} RIVLIN-ERICKSEN tensor at $(x, t) \in J$.)'

We can calculate $\underline{A}_n$ by repeatedly differentiating

$$\underline{C}_{(t)}(x, \tau) = [\underline{F}_n(x, t)]^{-T} \underline{C}_{(t)}(x, \tau) [\underline{F}_n(x, t)]^{-1} \quad)^2$$

w.r.t. τ and then evaluating at $\tau = t$. The first round gives

$$\begin{aligned} \underline{C}_{(t), 2}(x, t) &= [\underline{F}_n(x, t)]^{-T} \underline{C}_{n, 2}(x, t) [\underline{F}_n(x, t)]^{-1} \\ &= 2 \underline{D}_n(x, t) \quad (\text{Theorem B1.13.10}) \\ &= 2 \underline{D}_n(x, t) \quad (\text{Definition B1.13.2}). \end{aligned}$$

Thus,

$$\underline{A}_1(x, t) = 2 \underline{D}_n(x, t).$$

∴ stay tuned

$$n \leq N-1$$

It, of course, is possible to describe any field of interest relative to the present configuration $\mathbb{X}(\cdot, t)$. For example, consider the velocity of the particle X at time τ , $\underline{v}_m(X, \tau)$. We have

$$\begin{aligned}\underline{v}_m(X, \tau) &= \underline{v}_n(\mathbb{X}, \tau) && \text{(Definition B1.13.2)} \\ &= \underline{v}_n(\mathbb{X}_n^{-1}(\mathbb{X}, \tau), \tau) && \text{(substitution)} \\ &= \underline{v}_n(\mathbb{X}_n^{-1}(\mathbb{X}_{(t)}(x, \tau), \tau), \tau) && \text{(substitution)} \\ &=: \underline{v}_{(t)}(x, \tau) && \text{(labeling),}\end{aligned}$$

and this name calling is appropriate for $\underline{v}_{(t)}(x, \tau)$ is, indeed, the velocity at time τ of that particle, namely x , which is at the place x at time t . However, we will not formalize ¹this relative description because we shall use it only for a few kinematical fields such as $\underline{c}_{(t)}$ and then only in the context of certain fluids with long-range memory. Furthermore, Theorem B1.16.4 (i) makes it clear that, in general,

$$\underline{c}_{\sim x}(\mathbb{X}, \tau) \neq \underline{c}_{\sim(t)}(x, \tau)$$

— a quite unattractive feature of the relative description.

Supplementary Reading

COLEMAN, MARKOVITZ, and NOLL, Viscometric Flows of Non-Newtonian Fluids (1966)

TRUESDELL, A First Course in Rational Continuum Mechanics (1977)

TRUESDELL, The Elements of Continuum Mechanics (1966)

TRUESDELL and NOLL, "The non-linear field theories of mechanics", Handbuch der Physik, vol III/3 (edit. S. Flügge) 1965