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Part C

Examples of Simple Materials

In this part, we consider some particular examples of simple materials and thereby introduce several of the more successful continuum mechanical theories of material behavior.

Chapter C1

Some Fluids

In the first two sections of this chapter, we consider some special constitutive equations for fluids that lead to relatively "easy" theories. In the third section, we consider the most general incompressible fluid (within the context of simple materials) but limit the class of flows so as to keep the mathematics tractable.

C1.1. REINER-RIVLIN Fluids

Here we consider a special kind of nonlinear fluid response that includes viscous effects but not long-range memory. The results of §B4.6 on material symmetry and physical experience suggest that the constitutive equation for any fluid should not have the stress depending on quantities that refer in an essential way to a particular reference configuration. In order to have the stress depend on changes of shape (actually, rate of change), we include the spatial velocity gradient $^1 \underline{L}$. Also, we expect the stress in most fluids to depend on the mass density ρ (actually, vice versa). Thus, we are lead to

Definition C1.1.1. A particle X of a simple body $\mathcal{B}$ is a viscous fluid without long-range memory if $\exists$ a function

$$\underline{H}^X: \mathbb{R}^+ \times \text{Lin } \mathcal{V}_3 \rightarrow \text{Sym } \mathcal{V}_3$$

$\ni$ the stress tensor at (x, t) corresponding to a motion $\{ \mathbf{x}(\cdot, t) : t \in \mathbb{R} \}$ of $\mathcal{B}$ is given by the constitutive equation

$$(VF) \quad \underline{T}(x, t) = \underline{H}^X(\rho(x, t), \underline{L}(x, t)), (x, t) \in J.$$

Of course, if each particle of a body is a viscous fluid without long-range memory, then we say that the body is a viscous fluid without long-range memory. If this is the case and, in addition,

¹See Definition B1.8.6.

the response function H^X is the same $\forall$ particles X , then the body is homogeneous¹ w.r.t. stress response.

Definition C1.1.1 is a bit presumptuous in that incorporates the previously defined technical concepts "simple body" and "fluid" in a nonobvious way. To see that such terminology is, indeed, justified, we need to consider the referential description. Accordingly, let $\mathfrak{x}$ be an arbitrary reference configuration of $\mathcal{B}$. Then by Theorem B1.13.9,

$$\underline{\underline{L}}(\mathfrak{x}, t) = \underline{\underline{L}}(\mathfrak{x}_x(\mathfrak{x}, t), t) = \underline{\underline{L}}_x(\mathfrak{x}, t) = \underline{\underline{F}}_{x,2}(\mathfrak{x}, t) [\underline{\underline{F}}_x(\mathfrak{x}, t)]^{-1};$$

and by the referential density equation (Theorem B2.1.5),

$$\rho(\mathfrak{x}, t) = \rho(\mathfrak{x}_x(\mathfrak{x}, t), t) = \frac{\rho_x(\mathfrak{x})}{J_x(\mathfrak{x}, t)}.$$

Substitution into (VF) yields

Theorem C1.1.1, Let the particle X of a simple body $\mathcal{B}$ be a viscous fluid without long-range memory. Then its constitutive equation admits the referential description.

$$(VF)_x \quad \underline{\underline{T}}_x(\mathfrak{x}, t) = H^X\left(\frac{\rho_x(\mathfrak{x})}{J_x(\mathfrak{x}, t)}, \underline{\underline{F}}_{x,2}(\mathfrak{x}, t) [\underline{\underline{F}}_x(\mathfrak{x}, t)]^{-1}, (\mathfrak{x}, t) \in \mathcal{M}(\mathcal{B}) \times \mathbb{R}\right)$$

The constitutive equation $(VF)_x$ makes it clear that we are dealing with a simple material in the sense of

¹ Cf. Definition B1.4.3.

Definition B1.4.2 since the history of $\underline{F}_n(\mathbb{X}, \cdot)$ up to time t determines $\underline{F}_{n,2}(\mathbb{X}, t)$. Of course, one does not need much of the history, and this is the reason for the terminology viscous fluid "without long-range memory". It will be convenient to symbolically write

$$\underline{J} = \underline{H}^x(\rho, \underline{L}) = \underline{H}^x\left(\frac{\rho_n}{J_n}, \underline{F}_{n,2} \underline{F}_n^{-1}\right) = \underline{g}_n^x(\underline{F}_n^t).$$

To investigate the material symmetry¹ of a viscous fluid without long-range memory, we consider the transformation

$$\underline{F}_n \rightarrow \underline{F}_n \underline{H}$$

for arbitrary $\underline{H} \in \text{Unim } V_3$. Then

$$J_n = |\det \underline{F}_n| \rightarrow |\det(\underline{F}_n \underline{H})| = |\det \underline{F}_n| |\det \underline{H}| = J_n \quad (\text{Theorem A3.8.4}),$$

$$\underline{F}_{n,2} \rightarrow (\underline{F}_n \underline{H})_{,2} = \underline{F}_{n,2} \underline{H} \quad (\text{Exercise B1.10.1}),$$

$$\underline{F}_n^{-1} \rightarrow (\underline{F}_n \underline{H})^{-1} = \underline{H}^{-1} \underline{F}_n^{-1} \quad (\text{Theorem A3.9.7 (iii)}),$$

$$\underline{F}_{n,2} \underline{F}_n^{-1} \rightarrow (\underline{F}_{n,2} \underline{H})(\underline{H}^{-1} \underline{F}_n^{-1}) = \underline{F}_{n,2} \underline{F}_n^{-1} \quad (\text{substitution and associativity of tensor mult.})$$

and $\therefore$

$$\underline{g}_n^x((\underline{F}_n \underline{H})^t) = \underline{g}_n^x(\underline{F}_n^t),$$

Thus, replacing the history of $\underline{F}_n$ with the history of $\underline{F}_n \underline{H}$ does not

¹See Definition B4.5.2 and the surrounding discussion.

alter the stress, and we conclude from Definition B4.5.2 that $\underline{H} \in \mathcal{D}_n^X$. Since $\underline{H}$ is an arbitrary element of $\text{Unim } \mathcal{V}_3$, $\text{Unim } \mathcal{V}_3 \subset \mathcal{D}_n^X$; but necessarily $\mathcal{D}_n^X \subset \text{Unim } \mathcal{V}_3$. Hence,

$$\mathcal{D}_n^X = \text{Unim } \mathcal{V}_3;$$

and by Definition B4.6.2, we have

Theorem C1.1.2. Let the particle X of a simple body $\mathcal{B}$ be a viscous fluid without long-range memory. Then X is a fluid particle in the sense of NOLL's classification.

To see what restrictions, if any, are placed on $(VF)_n$ by the Principle of Material Indifference¹, we consider the transformation

$$F_n(X, t) \rightarrow \underline{F}_n^*(X, t) = Q(t) \underline{F}_n(X, t)$$

for arbitrary $Q \in C^{N+1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$,² Then

$$J_n = |\det F_n| \rightarrow \underline{J}_n^* = |\det \underline{F}_n^*| = |\det(Q \underline{F}_n)| = |\det Q| |\det \underline{F}_n| = J_n \text{ (Thm. A3.8.4)}$$

$$\underline{F}_{n,2} \rightarrow \underline{F}_{n,2}^* = (Q \underline{F}_n)_{,2} = \dot{Q} \underline{F}_n + Q \underline{F}_{n,2} \text{ (Exercise B1.10.1),}$$

$$\underline{F}_n^{-1} \rightarrow \underline{F}_n^{*-1} = (Q \underline{F}_n)^{-1} = \underline{F}_n^{-1} Q^{-1} = \underline{F}_n^{-1} Q^T \text{ (Thms. A3.9.7(ii) and A3.10.3,}$$

¹ See § B4.2.

² Here, $N \geq 2$ is the smoothness index of Axiom (M3).

$$\begin{aligned} \underline{F}_{n,2} \underline{F}_n^{-1} &\rightarrow (\dot{\underline{Q}} \underline{F}_n + \underline{Q} \underline{F}_{n,2}) \underline{F}_n^{-1} \underline{Q}^T \quad (\text{substitution}), \\ &= \dot{\underline{Q}} \underline{Q}^T + \underline{Q} \underline{F}_{n,2} \underline{F}_n^{-1} \underline{Q}^T \quad (\text{distributivity and associativity of tensor multiplication}); \end{aligned}$$

and $\therefore$

$$\underline{I} \rightarrow \underline{I}^* = \underline{H}^x(\underline{F}_n^*) = \underline{H}^x\left(\frac{\rho_n}{J_n}, \dot{\underline{Q}} \underline{Q}^T + \underline{Q} \underline{F}_{n,2} \underline{F}_n^{-1} \underline{Q}^T\right).$$

Converting back to the spatial description, we have

$$\underline{I}^* = \underline{H}^x(\rho, \dot{\underline{Q}} \underline{Q}^T + \underline{Q} \underline{L} \underline{Q}^T).$$

By the Principle of Material Indifference,

$$\underline{I}^* = \underline{Q} \underline{I} \underline{Q}^T;$$

and $\therefore$ we have the requirement that

$$\underline{H}^x(\rho, \dot{\underline{Q}} \underline{Q}^T + \underline{Q} \underline{L} \underline{Q}^T) = \underline{Q} \underline{H}^x(\rho, \underline{L}) \underline{Q}^T \quad \forall \underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } V_3). \quad ^1$$

As the first step in "solving" this functional equation, we decompose $\underline{L}$ into its symmetric and skew parts, recalling from Definition 1.8.6 the terminology

$$\underline{D} = \text{sym } \underline{L} \quad \dots \text{ the stretching field}$$

and

$$\underline{W} = \text{skw } \underline{L} \quad \dots \text{ the spin tensor field.}$$

¹ Again, as was noted in the discussion following Theorem B4.2.6, the density ρ is a harmless variable as far as Material Indifference is concerned.

Then the above identity becomes

$$\underline{H}^x(p, \dot{\underline{Q}}\underline{Q}^T + \underline{Q}\underline{D}\underline{Q}^T + \underline{Q}\underline{W}\underline{Q}^T) = \underline{Q}\underline{H}^x(p, \underline{L})\underline{Q}^T. \quad (*)$$

Arbitrarily fix the time t , and choose $\underline{Q}$ to be the function defined on $\mathbb{R}$ by

$$\underline{Q}(\tau) = \exp[(t-\tau)\underline{W}(x, t)].^1$$

Then by Theorems

$$\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } V_3)$$

Moreover, by Theorem

$$\underline{Q}(t) = \underline{I} \quad ;$$

and by Theorem

$$\dot{\underline{Q}}(\tau) = -\underline{W}(x, t) \exp[(t-\tau)\underline{W}(x, t)] \quad ;$$

so that by Theorem

$$\dot{\underline{Q}}(t) = -\underline{W}(x, t).$$

¹This does not look right because of the presence of the variable x in $\underline{W}(x, t)$. Strictly, we should write $\underline{W}(x, t) = \underline{W}(\mathbb{X}(X, t), t) = \underline{W}_m(X, t)$, and X is fixed in that our deliberations here are for a given particular particle X .

With this choice of $\underline{Q}$, $(*) \Rightarrow$

$$\underline{I} = \underline{H}^x(\rho, \underline{L}) = \underline{H}^x(\rho, \underline{D}) ;$$

i.e., the dependence on the spin tensor drops out.

This important result does not exhaust the implications of Material Indifference. Next, we apply the principle to the constitutive equation $\underline{I} = \underline{H}^x(\rho, \underline{D})$. The analysis on pp. C1.1.4-5 showed that when

$$\underline{F} \rightarrow \underline{F}^* = \underline{Q} \underline{F},$$

then

$$\rho \rightarrow \rho^* = \rho$$

and

$$\underline{L} \rightarrow \underline{L}^* = \dot{\underline{Q}} \underline{Q}^T + \underline{Q} \underline{L} \underline{Q}^T ;$$

consequently,

$$\underline{D} = \text{sym } \underline{L} \rightarrow \underline{D}^* = \text{sym } \underline{L}^* = \underline{Q} \underline{D} \underline{Q}^T \quad (\text{Exercise C1.1.1}).$$

Thus, in this case, the Principle of Material Indifference is the requirement that

$$\underline{H}^x(\rho, \underline{Q} \underline{D} \underline{Q}^T) = \underline{Q} \underline{H}^x(\rho, \underline{D}) \underline{Q}^T \quad \forall \underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3).$$

Hence, the symmetric tensor-valued function $\underline{H}^x(\rho, \cdot)$ of a symmetric tensor is isotropic (Definition A8.5.1); and by representation Theorem A8.5.8 we are led to

Theorem C1.1.3. If the constitutive equation

$$(VF) \quad \underline{T}(\underline{x}, t) = \underline{H}^X(\rho(\underline{x}, t), \underline{L}(\underline{x}, t)), \quad (\underline{x}, t) \in \mathcal{I}$$

for a viscous fluid without long-range memory satisfies the Principle of Material Indifference, then it can be written in the reduced form

$$\underline{T}(\underline{x}, t) = \alpha_0^X(\underline{l}_1(\underline{D}(\underline{x}, t)), \underline{l}_2(\underline{D}(\underline{x}, t)), \underline{l}_3(\underline{D}(\underline{x}, t)), \rho(\underline{x}, t)) \underline{I} \\ + \alpha_1^X(\cdot) \underline{D}(\underline{x}, t) + \alpha_2^X(\cdot) [\underline{D}(\underline{x}, t)]^2, \quad (\underline{x}, t) \in \mathcal{I},$$

where $\alpha_i^X: \mathcal{Q}_D \times \mathbb{R}^+ \rightarrow \mathbb{R}$ with $\mathcal{Q}_D = \{(\underline{l}_1(\underline{D}), \underline{l}_2(\underline{D}), \underline{l}_3(\underline{D})) : \underline{D} \in \text{Sym}_3^V\}$

Conversely, this reduced constitutive equation satisfies the Principle of Material Indifference $\forall$ choices of the response functions α_0^X , α_1^X , and α_2^X .

Proof. Only the converse assertion remains to be established, and this is left as Exercise C1.1.2. $\square$

Definition C1.1.2. A particle X of a simple body $\mathcal{B}$ with a constitutive equation of the reduced form given in Theorem C1.1.3, i.e.,

$$(RRF) \quad \underline{T} = \alpha_0^X(\underline{l}_1(\underline{D}), \underline{l}_2(\underline{D}), \underline{l}_3(\underline{D}), \rho) \underline{I} + \alpha_1^X(\cdot) \underline{D} + \alpha_2^X(\cdot) \underline{D}^2,$$

is a REINER-RIVLIN fluid particle.

In the incompressible case, we know from Theorem B4.10.6 that the constraint does not interfere with material symmetry considerations. Thus, for an incompressible REINER-RIVLIN fluid, the determinate stress is given by (RRF), the constraint stress is a hydrostatic pressure¹

$$\underline{\underline{N}}(\underline{x}, t) = -p(\underline{x}, t) \underline{\underline{I}},$$

and, of course, we have constraint of incompressibility, which for a fluid is most conveniently written as²

$$\text{tr } \underline{\underline{D}} = 0.$$

Since $\ell_1(\underline{\underline{D}}) = \text{tr } \underline{\underline{D}}$, the coefficients α_i^x in (RRF) do not depend on $\ell_1(\underline{\underline{D}})$ in the incompressible case. Moreover, in this case, the second fundamental invariant reduces to

$$\ell_2(\underline{\underline{D}}) = -\frac{1}{2} \text{tr } (\underline{\underline{D}}^2).$$

In addition, by Theorem B2.2.1,

$$\rho(\underline{x}, t) = \hat{p}(X)$$

when the material is incompressible. Setting

¹ See Theorem B4.9.2.

² See Theorem B1.14.4 and Definitions A6.6.1 and B1.8.6.

$$\alpha_i^x(0, \underline{v}_2(\underline{D}), \underline{v}_3(\underline{D}), \hat{p}(x)) = \bar{\alpha}_i^x(\underline{v}_2(\underline{D}), \underline{v}_3(\underline{D})),$$

we get

$$\underline{T} = -p \underline{I} + \bar{\alpha}_0^x(\underline{v}_2(\underline{D}), \underline{v}_3(\underline{D})) \underline{I} + \bar{\alpha}_1^x(\cdot) \underline{D} + \bar{\alpha}_2^x(\cdot) \underline{D}^2$$

as the constitutive equation for an incompressible REINER-RIVLIN fluid. But the term $\bar{\alpha}_0^x(\underline{v}_2(\underline{D}), \underline{v}_3(\underline{D})) \underline{I}$ may as well be absorbed by the indeterminate hydrostatic pressure, which leaves

$$\underline{T} = -p \underline{I} + \bar{\alpha}_1^x(\cdot) \underline{D} + \bar{\alpha}_2^x(\cdot) \underline{D}^2.$$

This last step is a normalization of the determinate stress — albeit a different normalization than the one of Theorem B4.9.3, which requires that the determinate part of the stress be traceless. It is left for the reader as Exercise C1.1.3 to show that the normalization of Theorem B4.9.3 leads to

$$\bar{\alpha}_0^x(\underline{v}_2(\underline{D}), \underline{v}_3(\underline{D})) = \frac{2}{3} \underline{v}_2(\underline{D}) \bar{\alpha}_2^x(\underline{v}_2(\underline{D}), \underline{v}_3(\underline{D})),$$

and consequently

$$\underline{T} = -p \underline{I} + \frac{2}{3} \underline{v}_2(\underline{D}) \bar{\alpha}_2^x(\cdot) \underline{I} + \bar{\alpha}_1^x(\cdot) \underline{D} + \bar{\alpha}_2^x(\cdot) \underline{D}^2.$$

But still the $\frac{2}{3} \underline{v}_2(\underline{D}) \bar{\alpha}_2^x(\cdot) \underline{I}$ term may as well be absorbed into the indeterminate hydrostatic pressure term, and we are left with the same result as before.

Hence, we have

Definition C1.1.3. A particle X of a simple body B is an incompressible REINER-RIVLIN fluid if its constitutive equation is of the form

$$\begin{aligned} \underline{T}(\underline{x}, t) = & -p(\underline{x}, t) \underline{I} + \bar{\alpha}_1^X(\underline{l}_2(\underline{D}(\underline{x}, t)), \underline{l}_3(\underline{D}(\underline{x}, t))) \underline{D}(\underline{x}, t) \\ \text{(IRRF)} \quad & + \bar{\alpha}_2^X(\underline{l}_2(\underline{D}(\underline{x}, t)), \underline{l}_3(\underline{D}(\underline{x}, t))) \underline{D}^2(\underline{x}, t), \quad (\underline{x}, t) \in \mathcal{V}, \end{aligned}$$

where $\bar{\alpha}_i^X: \bar{\mathcal{I}}_D \rightarrow \mathbb{R}$ with $\bar{\mathcal{I}}_D = \{(\underline{l}_2(\underline{D}), \underline{l}_3(\underline{D})) : \underline{D} \in \text{Sym } \mathcal{V}_3 \text{ and } \text{tr } \underline{D} = 0\}$.

REINER-RIVLIN fluids were introduced in the late 1940's in an attempt to explain some of the phenomena observed in non-Newtonian fluids such as polymer solutions. In the special case of $\bar{\alpha}_2^X \equiv 0$, the theory has proved somewhat useful in the design of apparatus in which non-Newtonian fluids are used. For reasons which will become clear in the next section, the terminology generalized Newtonian fluid is sometimes used in the $\bar{\alpha}_2^X = 0$ case. Apart from this limited success, REINER-RIVLIN fluids now have been relegated largely to the pedagogical role of being the simplest fluid model with both viscosity and a nonlinear constitutive equation. In the next section, we shall see that REINER-RIVLIN fluids also provide a rational basis for the theory of NEWTONIAN fluids.

Supplementary Reading

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C1.2. NEWTONIAN Fluids

The theory of REINER-RIVLIN fluids was anticipated in a sense by STOKES in 1845 when he proposed the constitutive equation

$$\underline{T} = -p(\rho) \underline{I} + \underline{H}(\underline{D}) , \quad \underline{H}(\underline{0}) = \underline{0} . \quad)^1$$

Here, $-p(\rho) \underline{I}$ is the stress in the fluid at rest at density ρ , and $\underline{H}(\underline{D})$ represents the increase in the stress due to motion. However, STOKES immediately assumed that $\underline{H}$ was linear and isotropic, and thus he was lead² to the familiar

$$\underline{T} = -p(\rho) \underline{I} + \lambda (\text{tr } \underline{D}) \underline{I} + 2\mu \underline{D} .$$

We can obtain this result in perhaps a more meaningful way by supposing that we have a REINER-RIVLIN fluid flowing in such a way that $\underline{D}$ is small.³ By Definition

¹ BOUSSINESQ's 1868 suggestion that this be generalized to $\underline{T} = -p(\rho) \underline{I} + \underline{H}(\underline{D}, \underline{W})$ is seen now, in light of the Principle of Material Indifference (see Theorem C1.1.3), to be vacuous.

² Cf. Theorem A8.5.9.

³ Since $\underline{D}$ has the units of $1/\text{time}$, any $\underline{D} \neq \underline{0}$ can be made arbitrarily small (or large) by the choice of the time unit. Therefore, to give the present development meaning, we assume that the theory has been nondimensionalized.

C1.1.2,

$$\underline{I} = \alpha_0^X(\underline{y}_1(\underline{D}), \underline{y}_2(\underline{D}), \underline{y}_3(\underline{D}), \rho) \underline{I} + \alpha_1^X(\cdot) \underline{D} + \alpha_2^X(\cdot) \underline{D}^2.$$

To take advantage of the smallness of $\underline{D}$, we linearize in $\underline{D}$. To simplify the analysis of the coefficients, we notationally suppress X and ρ and write

$$\hat{\alpha}_i(\underline{D}) = \alpha_i^X(\underline{y}_1(\underline{D}), \underline{y}_2(\underline{D}), \underline{y}_3(\underline{D}), \rho).$$

Assuming differentiability of $\hat{\alpha}_i$ at $\underline{D} = \underline{0}$,¹ we have

$$\hat{\alpha}_i(\underline{D}) = \hat{\alpha}_i(\underline{0}) + D\hat{\alpha}_i(\underline{0})[\underline{D}] + o(\underline{D}) \text{ as } \underline{D} \rightarrow \underline{0}.$$

The fundamental invariants of $\underline{D}$ are given in Theorem A3.11.8; and since they all vanish with $\underline{D}$,

$$\hat{\alpha}_i(\underline{0}) = \alpha_i^X(0, 0, 0, \rho).$$

Of course, the differentiability of $\hat{\alpha}_i$ is determined by α_i^X ; and by Theorem A5.13.3,

$$D\hat{\alpha}_i(\underline{0})[\underline{D}] = \sum_{j=1}^3 D_j \alpha_i^X(0, 0, 0, \rho) [Dy_j(\underline{0})[\underline{D}]]$$

By Theorem A5.8.1,

$$Dy_1(\underline{0})[\underline{D}] = \underline{t} \underline{D}, \quad Dy_2(\underline{0})[\underline{D}] = Dy_3(\underline{0})[\underline{D}] = 0.$$

¹See Definition A5.2.2.

Then

$$\begin{aligned} D \hat{\alpha}_i(\underline{Q})[\underline{D}] &= D_i \alpha_i^x(0,0,0,\rho)[t \underline{D}] \\ &= \alpha_{i,1}^x(0,0,0,\rho) t \underline{D} \end{aligned}$$

by Theorem A5.3.1. Hence,

$$\underline{T} = \alpha_0^x(0,0,0,\rho) \underline{I} + \alpha_{0,1}^x(0,0,0,\rho) (t \underline{D}) \underline{I} + \alpha_1^x(0,0,0,\rho) \underline{D} + o(\underline{D}) \text{ as } \underline{D} \rightarrow \underline{Q}.$$

Discarding the $o(\underline{D})$ term and renaming the coefficients, we get

Definition C1.2.1. A particle X of a simple body $\mathcal{B}$ with a constitutive equation of the form

$$(LVF) \quad \underline{T} = -\hat{p}^x(\rho) \underline{I} + \hat{\lambda}^x(\rho) (t \underline{D}) \underline{I} + 2\hat{\mu}^x(\rho) \underline{D},$$

where $\hat{p}^x$, $\hat{\lambda}^x$, and $\hat{\mu}^x$ are functions on $\mathbb{R}^+$ to $\mathbb{R}$, is a linearly viscous fluid particle.

It is important to realize that despite our small $\underline{D}$ motivation of the definition of a linearly viscous fluid, (LVF) is an exact special case of (RFF); and as such, it is in complete accord with the Principle of Material Indifference. Thus, the linearly viscous fluid is a theoretically possible model for any size $\underline{D}$.

The linearly viscous fluid model contains several important special cases, which are obtained by placing various

restrictions on the response functions $\hat{p}^X$, $\hat{\lambda}^X$, and $\hat{\mu}^X$. The most important is when the functions $\hat{\lambda}^X$ and $\hat{\mu}^X$ are constant.

Definition C1.2.2. A particle X of a simple body $\mathcal{B}$
with a constitutive equation of the form

$$(NF) \quad \underline{T} = -\hat{p}^X(\rho) \underline{I} + \lambda^X(\text{tr} \underline{D}) \underline{I} + 2\mu^X \underline{D},$$

where $\hat{p}^X$ is a function on $\mathbb{R}^+$ to $\mathbb{R}$ and λ^X and μ^X are
constants, called the bulk modulus and the viscosity,
respectively, is a NEWTONIAN¹ fluid particle.

If the bulk modulus and viscosity of a Newtonian fluid vanish, we have

Definition C1.2.3. A particle X of a simple body $\mathcal{B}$
with a constitutive equation of the form

$$(EF) \quad \underline{T} = -\hat{p}^X(\rho) \underline{I},$$

where $\hat{p}^X$ is a function on $\mathbb{R}^+$ to $\mathbb{R}$, is an elastic²
fluid particle.

¹ Sometimes, and probably with more justification, STOKES or NAVIER-STOKES.

² Sometimes, barotropic.

If the response function $\hat{p}^x$ of an elastic fluid has the constant value p^x , then for any p

$$\begin{aligned}\hat{p}^x(p) &= p^x \\ &= f(x) \quad (\text{labeling}) \\ &= f(\mathcal{X}^{-1}(x, t)) \quad (\text{substitution}^1) \\ &= p(x, t) \quad (\text{labeling});\end{aligned}$$

and we have

Definition C1.2.4. A particle x of a simple body $\mathcal{B}$ with a constitutive equation of the form

$$(PF) \quad \underline{T}(x, t) = -p(x, t) \underline{I},$$

where p is a function on the trajectory $\underline{\gamma}$ to $\mathbb{R}$, is a perfect fluid particle.

Elastic (and necessarily perfect) fluids are often referred to as being inviscid or nonviscous because the shear stress on any plane is always zero.

In the incompressible case,²

¹ See Definition B1.8.2.

² See the development on p. C1.1.9 for additional detail.

$$\text{tr } \underline{D} = 0,$$

$p(\underline{x}, t) = \hat{p}(X)$, and we must add on the constraint stress which is an indeterminate hydrostatic pressure

$$\underline{N}(\underline{x}, t) = -p(\underline{x}, t) \underline{I},$$

Of course, any additive terms in the determinate stress of a hydrostatic form may as well be absorbed into the indeterminate hydrostatic pressure. In this case, the constitutive equation (LVF) for a linearly viscous fluid becomes

$$\underline{T}(\underline{x}, t) = -p(\underline{x}, t) \underline{I} + 2\mu^* \underline{D}(\underline{x}, t),$$

which is identical in form with the constitutive equation for an incompressible NEWTONIAN fluid. It is left as Exercise C1.2.1 to show that normalization to $\underline{T}^D = 0$ of Theorem B4.9.3 leads to the same result.

Hence, we have

Definition C1.2.5. A particle X of a simple body B is an incompressible NEWTONIAN¹ fluid if its constitutive equation is

² Sometimes, STOKES or NAVIER-STOKES or linearly viscous.

of the form

$$(INF) \quad \underline{T}(\underline{x}, t) = -p(\underline{x}, t) \underline{I} + 2\mu^x D(\underline{x}, t), \quad (\underline{x}, t) \in \mathcal{I},$$

where μ^x is a constant, called the viscosity.

Similarly, in the incompressible case, the constitutive equations for elastic and perfect fluids coalesce, and we have

Definition C1.2.6. A particle x of a simple body B is an incompressible perfect¹ fluid if its constitutive equation is of the form

$$(IPF) \quad \underline{T}(\underline{x}, t) = -p(\underline{x}, t), \quad (\underline{x}, t) \in \mathcal{I},$$

In the practice of fluid mechanics, we usually have uniform bodies in the sense that every particle of the body is a fluid of the same kind and homogeneity in the sense that the response functions and/or material constants do not vary from particle to particle. In this spirit, we have

Definition C1.2.7. An ideal fluid body is one for which each particle is an incompressible perfect fluid and which has uniform mass density in some (and hence every²) occupied configuration.

¹Sometimes, elastic or barotropic.

²See Theorem B2.2.1 (iii).

Combination of certain of the above constitutive equations with the first stress equation of motion¹ results in some of the best known and most studied partial differential equations of mathematical physics. The most famous two are given in the next two exercises. It is a remarkable fact that, despite its linearity, the NEWTONIAN model describes the behavior of water (incompressible case) and air (compressible case) quite well under a wide range of circumstances.

Exercise C1.2.2. Let the body $\mathcal{B}$ be a perfect fluid. Show that its motion is governed (in part) by

$$-\nabla_1 p + \rho_0 \underline{b} = \rho (\underline{v}_{0,2} + D_1 \underline{v}_0 [\underline{v}_0]) \quad \text{on } \mathcal{V}$$

— EULER's equation.

Exercise C1.2.3. Let the body $\mathcal{B}$ be a homogeneous incompressible NEWTONIAN fluid with uniform density ρ_0 . Show that its motion is governed (in part) by

$$-\nabla_1 p + \mu \Delta_1 \underline{v}_0 + \rho_0 \underline{b} = \rho_0 (\underline{v}_{0,2} + D_1 \underline{v}_0 [\underline{v}_0]) \quad \text{on } \mathcal{V}$$

— the NAVIER-STOKES equation. (Here, $\Delta_1 := (\nabla_1)_1 \nabla_1$ is the spatial LAPLACIAN operator.)

¹See Theorem B3.4.3.

Supplementary Reading

Same as for § C1.1 plus

GURTIN, An Introduction to Continuum Mechanics (1981)

C1.4. Viscometric Flows of Incompressible Simple Fluids

By Theorem B4.7.2 and our First Notational Agreement (p.B1.8.7), the constitutive equation of a simple fluid body is given by

$$\underline{\underline{I}}(\underline{\underline{x}}, t) = \underline{\underline{F}}^x(\underline{\underline{C}}_{\sim(t)}^t(\underline{\underline{x}}, \cdot); \rho(\underline{\underline{x}}, t)),$$

where the response function $\underline{\underline{F}}^x$ is isotropic; i.e.,

$$\underline{\underline{F}}^x(\underline{\underline{Q}}\underline{\underline{C}}\underline{\underline{Q}}^T; \rho) = \underline{\underline{Q}}\underline{\underline{F}}^x(\underline{\underline{C}}; \rho)\underline{\underline{Q}}^T \quad \forall \underline{\underline{Q}} \in \text{Orth } \mathcal{V}_3, \underline{\underline{C}} \in \mathcal{C}^{N-1}([0, \infty), \text{Sym } \mathcal{V}_3), \rho \in \mathbb{R}^+.$$

We are interested in the case where the simple fluid is incompressible. Since the present configuration $\mathcal{K}(\cdot, t)$ is necessarily occupied, it follows from Theorem B1.14.3 that the constraint of incompressibility can be expressed as

$$\begin{aligned} \underline{\underline{J}}_{\sim(t)}(\underline{\underline{x}}, \tau) &= \det \underline{\underline{F}}_{\sim(t)}(\underline{\underline{x}}, \tau) = 1; \\ \text{equivalently,} \quad \det \underline{\underline{C}}_{\sim(t)}(\underline{\underline{x}}, \tau) &= 1, (\underline{\underline{x}}, \tau) \in \mathcal{B}_t \times (-\infty, t]. \end{aligned}$$

By Theorem B2.2.1 (v), the density in this case is constant in the sense

$$\rho(\underline{\underline{x}}, t) = \hat{\rho}(\underline{\underline{x}});$$

and $\hat{\rho}(\underline{\underline{x}})$ can be incorporated in the definition of $\underline{\underline{F}}^x$. By Theorem B4.9.2, the constraint stress is a hydrostatic pressure

$$\underline{\underline{T}}(\underline{x}, t) = -p(\underline{x}, t) \underline{\underline{I}}.$$

By Theorem B4.9.3, the determinate stress is subject to the normalization

$$\text{tr } \underline{\underline{T}}^D(\underline{x}, t) = 0.$$

Finally, we know from Theorem B4.10.7 that the constraint of incompressibility does not interfere with material symmetry considerations. Thus, we have

Theorem C1.4.1. Let the particle X of a simple body B be an incompressible fluid particle. Then its constitutive equation can be written as

$$(ISF) \quad \underline{\underline{T}}_X(\underline{x}, t) = -p(\underline{x}, t) \underline{\underline{I}} + \underline{\underline{H}}^X(\underline{\underline{C}}_{(t)}^t(\underline{x}, \cdot)),$$

where the response function $\underline{\underline{H}}^X: C^{N-1}([0, \infty), P_{\text{sym}} \mathcal{V}_3 \cap U_{\text{uni}}^+ \mathcal{V}_3) \rightarrow \text{Sym } \mathcal{V}_3$ ¹
is isotropic, i.e.,

$$\underline{\underline{H}}^X(\underline{\underline{Q}} \underline{\underline{C}} \underline{\underline{Q}}^T) = \underline{\underline{Q}} \underline{\underline{H}}^X(\underline{\underline{C}}) \underline{\underline{Q}}^T \quad \forall \underline{\underline{Q}} \in \text{Orth } \mathcal{V}_3 \text{ and } \forall \underline{\underline{C}} \in C^{N-1}([0, \infty), P_{\text{sym}} \mathcal{V}_3 \cap U_{\text{uni}}^+ \mathcal{V}_3)$$

and normalized by

$$\text{tr } \underline{\underline{H}}^X(\underline{\underline{C}}) = 0 \quad \forall \underline{\underline{C}} \in C^{N-1}([0, \infty), P_{\text{sym}} \mathcal{V}_3 \cap U_{\text{uni}}^+ \mathcal{V}_3).$$

Conversely, a particle with a constitutive equation of the above

¹ $N \geq 2$ is the smoothness index of Axiom (M3).

form (and subject to the constraint $\det \underline{C}_{(t)}(\underline{x}, \tau) = 1, \tau \in (-\infty, t]$)
is an incompressible simple fluid particle.

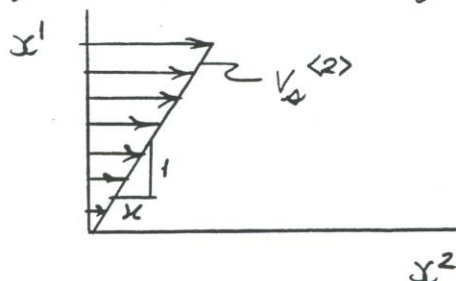
Moreover, the above constitutive equation satisfies the
Principle of Material Indifference for all response functions $\underline{H}^x$.

Proof. Only the last two assertions need attention, and this
 is left as Exercise C1.4.1. $\square$

We are interested in the case where every particle of the
 body under consideration is an incompressible simple fluid.
 Suppose such a body is undergoing a simple shearing flow,
 where the spatial velocity field is given with respect to a
 fixed rectangular Cartesian coordinate system by

$$\underline{V}_a^{(1)}(\underline{x}, t) = 0, \quad \underline{V}_a^{(2)}(\underline{x}, t) = \kappa x^1, \quad \underline{V}_a^{(3)}(\underline{x}, t) = 0.$$

with the rate of shear κ being a constant.



Since $\text{div}_a \underline{V}_a = 0$,¹ this flow is isochoric by Theorem B1.14.4.
 By Exercise B1.16.1, the corresponding relative deformation
 is

¹See Theorem A6.15.3.

$$\xi^1 = \chi_{(t)}^1(\underline{x}, \tau) = x^1, \quad \xi^2 = \chi_{(t)}^2(\underline{x}, \tau) = x^2 + x x^1(\tau - t), \quad \xi^3 = \chi_{(t)}^3(\underline{x}, \tau) = x^3,$$

where we are using the same rectangular Cartesian coordinate system at $\underline{\xi}$ as at $\underline{x}$. By Theorem B1.18., the components of the relative deformation gradient are

$$[F_{\sim(t)}(\underline{x}, \tau)] = \begin{bmatrix} 1 & 0 & 0 \\ x(\tau - t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

and therefore

$$[F_{\sim(t)}^t(\underline{x}, s)] = \begin{bmatrix} 1 & 0 & 0 \\ -xs & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$F_{\sim(t)}^t(\underline{x}, s) = \underline{I} - s \underline{M},$$

where relative to the given rectangular Cartesian coordinate system

$$[\underline{M}] = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (*)$$

Substitution of this history into the constitutive equation (ISF) yields

$$\underline{T}(\underline{x}, t) + p(\underline{x}, t) \underline{I} = \underline{T}^D(\underline{x}, t) = \underline{H}^X((\underline{I} - s \underline{M})(\underline{I} - s \underline{M})^T).$$

¹ See Theorem A3.2.5 (iii).

Thus,

$$\underline{T}^D(\underline{x}, t) = \underline{H}^X(\underline{M})$$

for some function $\underline{H}^X: \text{Lin } \mathcal{V}_3 \rightarrow \text{Sym } \mathcal{V}_3$; i.e., in the case of simple shearing flow, the function of the history of the relative right deformation tensor that gives the determinate stress degenerates into a function of the tensor $\underline{M}$, which is determined by the rate of shear $\underline{\kappa}$. The function $\underline{H}^X$ is not arbitrary. Obviously, the normalization of $\underline{H}^X$ carries over to

$$\text{to } \underline{H}^X(\underline{M}) = 0 \quad \forall \underline{M} \text{ of the form } (*).$$

Less obviously, the isotropy of $\underline{H}^X$ carries over as well, and we must have

$$\underline{H}^X(\underline{Q}\underline{M}\underline{Q}^T) = \underline{Q}\underline{H}^X(\underline{M})\underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3 \text{ and } \forall \underline{M} \text{ of the form } (*).$$

Exercise C1.4.2. Derive the above requirement. Since the isotropy of $\underline{H}^X$ is a result of Material Indifference, start with the constitutive equation in the form

$$\underline{T}^D(\underline{x}, t) = \underline{H}^X(\underline{F}_{(t)}^t(\underline{x}, \cdot) \underline{F}_{(t)}^t(\underline{x}, \cdot)),$$

Apply Material Indifference, and recognize that for simple shear flow

$$\underline{F}_{(t)}^t(\underline{x}, s) = \underline{I} - s\underline{M} \quad \text{and} \quad \underline{H}^X(\underline{F}_{(t)}^t(\underline{x}, \cdot) \underline{F}_{(t)}^t(\underline{x}, \cdot)) = \underline{H}^X(\underline{M}).$$

As usual, we "solve" the above invariance requirement by making special choices of $\underline{\underline{Q}} \in \text{Orth } V_3$. Our first choice is defined by

$$[\underline{\underline{Q}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where we are using the same rectangular Cartesian coordinate system as was employed in describing simple shearing flow. Given the particular form of $\underline{\underline{M}}$, an easy calculation yields

$$[\underline{\underline{Q}}][\underline{\underline{M}}][\underline{\underline{Q}}^T] = [\underline{\underline{M}}];$$

equivalently,

$$\underline{\underline{Q}}\underline{\underline{M}}\underline{\underline{Q}}^T = \underline{\underline{M}}.$$

Thus, for this $\underline{\underline{Q}}$, the invariance requirement yields

$$\underline{\underline{H}}^x(\underline{\underline{M}}) = \underline{\underline{H}}^x(\underline{\underline{Q}}\underline{\underline{M}}\underline{\underline{Q}}^T) = \underline{\underline{Q}}\underline{\underline{H}}^x(\underline{\underline{M}})\underline{\underline{Q}}^T,$$

or

$$\underline{\underline{T}}^D = \underline{\underline{Q}}\underline{\underline{T}}^D\underline{\underline{Q}}^T.$$

In terms of components,

$$[\underline{\underline{T}}^D] = \begin{bmatrix} T_D^{(11)} & T_D^{(12)} & T_D^{(13)} \\ \cdot & T_D^{(22)} & T_D^{(23)} \\ \cdot & \cdot & T_D^{(33)} \end{bmatrix} \text{ and } [\underline{\underline{Q}}^T \underline{\underline{T}}^D \underline{\underline{Q}}] = \begin{bmatrix} T_D^{(11)} & T_D^{(12)} & -T_D^{(13)} \\ \cdot & T_D^{(22)} & -T_D^{(23)} \\ \cdot & \cdot & T_D^{(33)} \end{bmatrix}.$$

Therefore,

$$T_D^{<13>} = T_D^{<23>} = 0,$$

In the present context, the normalization

$$\text{to } H^X(M) = 0$$

requires

$$\text{to } \underline{T}_D = T_D^{<11>} + T_D^{<22>} + T_D^{<33>} = 0,$$

Calling the normal stress differences

$$T_D^{<11>} - T_D^{<33>} = \sigma_1$$

and

$$T_D^{<22>} - T_D^{<33>} = \sigma_2,$$

we must have

$$T_D^{<33>} = -\frac{1}{3}(\sigma_1 + \sigma_2).$$

Now, in simple shearing flow,

$$\underline{T}_D = \underline{H}^X(\underline{M});$$

so the stress components are all functions of $\underline{M}$, which is determined by the rate of shear $\dot{\gamma}$. Thus, the above results can be summed up as

$$T_D^{<11>} - T_D^{<33>} = \sigma_1^X(\dot{\gamma}),$$

$$T_D^{<22>} - T_D^{<33>} = \sigma_2^X(\dot{\gamma}),$$

$$T_D^{<33>} = -\frac{1}{3} [\sigma_1^x(x) + \sigma_2^x(x)],$$

$$T_D^{<12>} = \tau^x(x),$$

$$T_D^{<13>} = T_D^{<23>} = 0,$$

where our notation emphasizes that the dependence on x will vary from particle to particle if the fluid is inhomogeneous. Alternatively, we could express the above information as

$$T_D^{<11>} = \frac{1}{3} [2\sigma_1^x(x) - \sigma_2^x(x)],$$

$$T_D^{<22>} = \frac{1}{3} [2\sigma_2^x(x) - \sigma_1^x(x)],$$

$$T_D^{<33>} = -\frac{1}{3} [\sigma_1^x(x) + \sigma_2^x(x)],$$

$$T_D^{<12>} = \tau^x(x),$$

$$T_D^{<13>} = T_D^{<23>} = 0.$$

We got to this point by making one special choice of Q in the invariance requirement

$$\underline{H}^x(\underline{Q}\underline{M}\underline{Q}^T) = \underline{Q} \underline{H}^x(\underline{M}) \underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } V_3$$

and for all $\underline{M}$ of the form (*). Next consider the orthogonal tensor defined by

$$[\underline{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

relative to the same rectangular Cartesian coordinate system used above. In this case,

$$[\underline{Q}][\underline{M}][\underline{Q}^T] = [-\underline{M}] \iff \underline{Q} \underline{M} \underline{Q}^T = -\underline{M},$$

and

$$[\underline{Q} \underline{T}^D \underline{Q}^T] = \begin{bmatrix} T_D^{<11>} & -T_D^{<12>} & 0 \\ \cdot & T_D^{<22>} & 0 \\ \cdot & 0 & T_D^{<33>} \end{bmatrix},$$

where we have already incorporated $T_D^{<13>} = T_D^{<23>} = 0$. But

$$\underline{Q} \underline{T}^D \underline{Q}^T = \underline{Q} \underline{H}^X(\underline{M}) \underline{Q}^T = \underline{H}^X(\underline{Q} \underline{M} \underline{Q}^T) = \underline{H}^X(-\underline{M})$$

in this case. Since $-\underline{M}$ corresponds to $-x$, we can calculate both $\underline{T}^D = \underline{H}^X(\underline{M})$ and $\underline{H}^X(-\underline{M})$ from the formulas on the middle of p. C1.4.B. Then $\underline{Q} \underline{T}^D \underline{Q}^T = \underline{H}^X(-\underline{M})$ results in

$$\frac{1}{3} [2\sigma_1^X(x) - \sigma_2^X(x)] = \frac{1}{3} [2\sigma_1^X(-x) - \sigma_2^X(-x)],$$

$$\frac{1}{3} [2\sigma_2^X(x) - \sigma_1^X(x)] = \frac{1}{3} [2\sigma_2^X(-x) - \sigma_1^X(-x)],$$

$$-\frac{1}{3} [\sigma_1^X(x) + \sigma_2^X(x)] = -\frac{1}{3} [\sigma_1^X(-x) + \sigma_2^X(-x)],$$

$$-\tau^X(x) = \tau^X(-x),$$

The last equation states that τ^x is an odd function. On subtracting the third equation successively from the first and second equations, we learn that σ_1^x and σ_2^x are each even functions.

This is all of the special choices of Q that we will make, but it is not clear that we have obtained all of the information implied by the invariance requirement.

A special case of simple shearing flow is the rest state, which corresponds to $x=0$. We leave it as Exercise C1.4.3 for the reader to show that Theorem B4.7.3 carries over to the incompressible case; i.e., in an incompressible simple fluid body which has always been at rest the stress field is hydrostatic. In particular, in this case we must have

$$T_D^{(11)} = T_D^{(22)} = T_D^{(33)}, \quad T_D^{(12)} = 0;$$

and the equations on the middle of p. C1.4.8 yield

$$2\sigma_1^x(0) - \sigma_2^x(0) = 2\sigma_2^x(0) - \sigma_1^x(0) = -\sigma_1^x(0) - \sigma_2^x(0), \quad \tau(0) = 0.$$

Whence,

$$\sigma_1^x(0) = \sigma_2^x(0) = \tau(0) = 0. \quad)^1$$

The above results are summarized in

¹Of course, we already knew $\tau(0) = 0$ from the fact that the function τ is odd.

Theorem C1.4.2. In the case of simple shearing flow,

$$V_a^{(1)}(x,t) = 0, \quad V_a^{(2)}(x,t) = x x', \quad V_a^{(3)}(x,t) = 0,$$

the constitutive equation of an incompressible simple fluid
body,

$$T_a(x,t) = -p(x,t) \underline{I} + \tilde{\Psi}^X(\tilde{C}_{(t)}^t(x, \cdot)), \quad (x,t) \in \mathcal{I},$$

reduces to

$$T_a^{(11)}(x,t) = -p(x,t) + \frac{1}{3} [2\sigma_1^X(x) - \sigma_2^X(x)],$$

$$T_a^{(12)}(x,t) = -p(x,t) + \frac{1}{3} [2\sigma_2^X(x) - \sigma_1^X(x)],$$

$$T_a^{(33)}(x,t) = -p(x,t) - \frac{1}{3} [\sigma_1^X(x) + \sigma_2^X(x)],$$

$$T_a^{(12)}(x,t) = \tau^X(x),$$

$$T_a^{(13)}(x,t) = T_a^{(23)}(x,t) = 0,$$

where the stress components are calculated w.r.t. the same rectangular
Cartesian coordinate system used to describe the simple shearing
flow. The functions σ_1^X, σ_2^X , and τ^X (on $\mathbb{R}$ to $\mathbb{R}$) are called the
viscometric functions; in particular, σ_1^X and σ_2^X are the normal
stress functions, while τ^X is the shear stress function. σ_1^X and
 σ_2^X are even functions, while τ^X is an odd function; moreover,
 $\sigma_1^X(0) = \sigma_2^X(0) = \tau(0) = 0.$

Thus, for simple shearing flow, we do not need full knowledge of the response function $\underline{\underline{H}}^X$ of (ISF); rather the three viscometric functions $\sigma_1^X, \sigma_2^X, \tau$ suffice.

Finally, but still in the context of simple shearing flow, we consider the First Stress Equation of Motion (Theorem B3.4.3),

$$\operatorname{div}_1 T_{\underline{\underline{a}}} + \rho \underline{\underline{b}} = \rho \underline{\underline{a}}_{\underline{\underline{a}}} \text{ on } \mathcal{I}.$$

By Theorem B1.8.2,

$$\underline{\underline{a}}_{\underline{\underline{a}}}(\underline{\underline{x}}, t) = D_1 \underline{\underline{v}}_{\underline{\underline{a}}}(\underline{\underline{x}}, t) [\underline{\underline{v}}_{\underline{\underline{a}}}(\underline{\underline{x}}, t)] + \underline{\underline{v}}_{\underline{\underline{a}}, 2}(\underline{\underline{x}}, t);$$

and it is an easy calculation (Exercise C1.4.4) to show that $\underline{\underline{a}}_{\underline{\underline{a}}} = \underline{\underline{0}}$ for simple shearing flow. Let us assume that the incompressible simple fluid body under consideration is homogeneous; then the viscometric functions do not depend on the particle, and we denote them simply by σ_1, σ_2 , and τ . In this case, the determinate stress is uniform and by Theorem A6.9.3,

$$\operatorname{div}_1 T_{\underline{\underline{a}}} = -\nabla_1 p.$$

Furthermore, let us assume that the body force is derivable from a potential in the sense that $\exists \psi: \mathcal{I} \rightarrow \mathbb{R} \ni$

$$\rho(\underline{\underline{x}}, t) \underline{\underline{b}}(\underline{\underline{x}}, t) = \nabla_1 \psi(\underline{\underline{x}}, t).$$

Then the equation of motion for simple shearing flow reduces to

$$\nabla_1(-p + \psi) = \mathcal{Q},$$

and we conclude¹ that $-p + \psi$ depends at most on time. In particular, in the absence of body force (i.e., $\mathcal{Q} \equiv 0$), there is no pressure gradient (i.e., $\nabla_1 p = 0$).

Since the above steps are reversible, we have

Theorem C1.4.3. In the case of simple shearing flow of a homogeneous incompressible simple fluid body subjected to a body force given by a potential $\psi: \mathcal{Y} \rightarrow \mathbb{R}$ according to $\rho \mathcal{Q} = \nabla_1 \psi$, the First Stress Equation of Motion is equivalent to

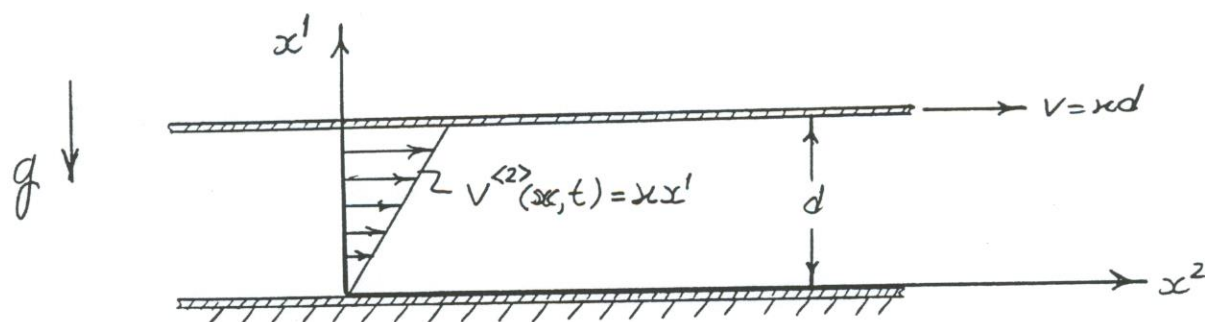
$$-p(x, t) = \psi(x, t) + f(t).$$

for some function $f: \mathbb{R} \rightarrow \mathbb{R}$.

Exercise C1.4.5. Develop the counterpart of Theorem C1.4.3 for the case where the body force is not given by a potential, cf. the discussion following Theorem B4.7.3.

Simple shearing flow can be interpreted (physically realized) as the flow between infinite parallel plates. The bottom plate is fixed, while the top plate moves at constant speed. The homogeneous incompressible simple fluid is assumed to adhere to the plates. If gravity acts as indicated in the figure, then

¹See Theorems A5.9.3 and A5.4.1.



$$\psi(x, t) = -g x^1 \quad)^1$$

and

$$p(x, t) = -g x^1 + f(t).$$

The preceding analysis provides a solution to this problem in terms of the viscometric functions of the homogeneous incompressible simple fluid between the plates.

Exercise C1.4.6. Solve the above problem for the case of a homogeneous incompressible Newtonian fluid. Compare the two solutions.

Actually, most of the foregoing analysis goes through even if x depends on x and t and also if the basis w.r.t. which M adopts its special form (*) depends on x and t . Of course, the dynamical analysis needs special attention in this more general setting. Specifically, we make

Definition C1.4.1. A flow is said to be viscometric if the history of its relative deformation gradient has the form

¹ See Theorem A6.14.1.

$$\tilde{F}_{(t)}^t(\underline{x}, s) = \tilde{R}(\underline{x}, t, s) [\underline{I} - s \tilde{M}(\underline{x}, t)],$$

where $\tilde{R}: \mathcal{I} \times (-\infty, t] \rightarrow \text{Orth } V_3$ with $\tilde{R}(\underline{x}, t, 0) = \underline{I}$ and $\tilde{M}: \mathcal{I} \rightarrow \text{Lin } V_3$ has the property that $\exists$ an orthonormal basis (generally depending on $\underline{x}$ and t) $\ni$ w.r.t. this basis

$$[\tilde{M}(\underline{x}, t)] = \begin{bmatrix} 0 & 0 & 0 \\ \kappa(\underline{x}, t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $\kappa: \mathcal{I} \rightarrow \mathbb{R}$.

In connection with this definition, the following result from linear algebra is relevant: Given $\tilde{M} \in \text{Lin } V_3$, $\exists$ an orthonormal basis $\ni$ w.r.t. this basis

$$[\tilde{M}] = \begin{bmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

iff $\tilde{M}^2 = \underline{0}$.¹ Such tensors are said to be nilpotent with index 2.

The significance of Definition C1.4.1 is revealed in

¹ See, e.g., NOMIZU's Linear Algebra (p.203) or SCHWALTER's Mechanics of Non-Newtonian Fluids (p.131).

Exercise C1.4.7. Carry out an analysis similar to the one just conducted for simple shearing flow to show that for a general viscometric flow of an incompressible simple fluid

$$\tilde{T}_s^D(x, t) = \tilde{H}^X(\tilde{C}_{(t)}^t(x, \cdot)) = \tilde{H}^X(\tilde{M}).$$

Hence, general viscometric flows can be handled in terms of the viscometric functions.

Most of the flows considered in classical viscometry are viscometric and can be handled for homogeneous incompressible simple fluids by the machinery of this section. The excellent monograph Viscometric Flows of Non-Newtonian Fluids by COLEMAN, MARKOVITZ, and NOLL should be consulted by anyone wishing to pursue this subject further.

Supplementary Reading

Same as for § C1.1 plus

COLEMAN, MARKOVITZ, and NOLL, Viscometric Flows of Non-Newtonian Fluids (1966)

WANG, Mathematical Principles of Mechanics and Electromagnetism, Part A: Analytical and Continuum Mechanics (1979)

Chapter C2

Elasticity

In this chapter, we specialize the constitutive equation for a simple material by suppressing its memory of the past and thereby obtain the theory of elasticity. We do not assume that the resulting response function is linear nor that the deformations considered are small. Consequently, our subject is often referred to as nonlinear elasticity or finite elasticity so as to distinguish it from the more highly developed linearized theory appropriate to infinitesimal deformations.

C2.1. The Elastic Response Function

The physical notion of elasticity which we wish to model is that the stress in an elastic body depends only on the present configuration of the body and is independent of how the body attains its present configuration. This is easily accomplished by starting with a simple material and dropping the history dependence.

Definition C2.1.1. A particle X of a simple body B is elastic¹ w.r.t. a reference configuration κ if $\exists$ a function

$$G_{\kappa}^X : \text{Inv } \mathcal{V}_3 \rightarrow \text{Sym } \mathcal{V}_3$$

$\exists$ the stress tensor at X corresponding to a motion $\{\kappa(\cdot, t) : t \in \mathbb{R}\}$ of B is given by

$$(E) \quad T_{\kappa}(X, t) = G_{\kappa}^X (F_{\kappa}(\kappa(X), t)), \quad t \in \mathbb{R}.$$

The function G_{κ}^X is called the elastic response function, and
(E) is called the elastic constitutive equation.

Thus, at an elastic particle, the stress at the present time is determined by the present value of the deformation gradient at the particle.

Obviously, the elastic constitutive equation (E) is a special

¹Sometimes, CAUCHY elastic.

case of the constitutive equation (SM) for a simple material.)¹
 Accordingly, all of the general theory for simple materials developed in the previous part applies to elasticity as a special case. In particular, we can read off from Theorem B4.1.2 that if a particle is elastic w.r.t. one reference configuration, then it is elastic w.r.t. all reference configurations.

Of course, if all of the particles of a body are elastic, then we say that the body is elastic; this is the case most often studied. If a body is elastic and if the response function is the same at each particle, then the body is elastically homogeneous, and the associated reference configuration is a homogeneous reference configuration.²

Elastic bodies often possess a configuration, called a natural state³, in which the stress is everywhere zero. Such a configuration would normally be chosen as the reference configuration $\mathbb{B}_0$, and this would require that

$$\underline{\underline{G}}^x_n(\underline{\underline{I}}) = \underline{\underline{0}} \quad \forall x \in \mathbb{B}.$$

We shall not assume the existence of a natural state except

¹ Cf. Definition B4.1.2.

² Cf. Definition B4.1.3.

³ Sometimes, placement at ease.

when explicitly stated to the contrary. The stress in the reference configuration; i.e.,

$$\overset{\circ}{T}_m(X) = \overset{\circ}{G}_n(\overset{\circ}{I})$$

is called the initial stress¹.

In practice, there are two important classes of materials which are observed to be well-modeled by elasticity (in their response to large deformations). They are solid rubbers and perfect fluids. Metals generally behave elastically under infinitesimal deformations, but as a rule they enter the "plastic" regime before their deformations become even moderately finite.

Elasticity is the most developed of the fully nonlinear (large deformations and nonlinear response) branches of continuum mechanics, and as such it provides a bench mark for the general theory. A more subtle reason to study elasticity can be seen by considering a simple body which has been at rest for all times up to and including the present time t . Then the history of the deformation gradient reduces to

$$\overset{\circ}{F}_n(X; \cdot) = \overset{\circ}{F}_n(X);$$

¹Often, residual stress, but we shall save this terminology for initial stress in the case where the body is free of external forces in the reference configuration.

and for this class of histories, the constitutive equation (SM) for a simple material reduces to the form (E) of a constitutive equation for an elastic material. In other words, as far as statics is concerned, simple materials behave elastically.

The elastic constitutive equation (E) is, of course, subject to the Principle of Material Indifference. From Theorem B4.2.3, we can read off that satisfaction of Material Indifference is equivalent to writing (E) in any of the following reduced forms:

$$(E-1) \quad \underline{T}_n(\mathbb{X}, t) = \underline{R}_n(\mathbb{X}, t) \underline{G}_n^X(\underline{U}_n(\mathbb{X}, t)) \underline{R}_n^T(\mathbb{X}, t);$$

$$(E-2) \quad \underline{T}_n(\mathbb{X}, t) = \underline{F}_n(\mathbb{X}, t) \underline{\hat{G}}_n^X(\underline{U}_n(\mathbb{X}, t)) \underline{F}_n^T(\mathbb{X}, t);$$

$$(E-3) \quad \underline{T}_n(\mathbb{X}, t) = \underline{F}_n(\mathbb{X}, t) \underline{\bar{G}}_n^X(\underline{C}_n(\mathbb{X}, t)) \underline{F}_n^T(\mathbb{X}, t);$$

where each of the three response functions $\underline{G}_n^X$, $\underline{\hat{G}}_n^X$, and $\underline{\bar{G}}_n^X$ is an $\text{Sym } \mathcal{V}_3$ to $\text{Sym } \mathcal{V}_3$.

Solid rubbers are generally modeled as being incompressible¹, and often this is an appropriate assumption for perfect fluids; therefore this constraint is of particular importance in elasticity. It follows from the Principle of Determinism for an Internally Constrained Simple Body² and Theorem B4.9.2 that the constitutive equation

¹ Cf. Definition B4.9.2.

² See pp. B4.8.15-16.

for an incompressible elastic particle is

$$(IE) \quad \underline{T}_m(X, t) = -p_m(X, t) \underline{I} + \underline{G}_n^X(F(\underline{x}(X), t)),$$

where the motions are subject to the constraint

$$\det \underline{F}_{n_0}(\underline{x}, t) = 1 \quad \forall t \in \mathbb{R},$$

and where the reference configuration $\underline{x}_0 = \underline{x}(\cdot, t_0)$.

By Theorems B4.8.7 and B4.9.1, an appropriate reduced constitutive equation for an incompressible elastic particle is

$$(IE-3) \quad \underline{T}_{n_0}(\underline{x}, t) = -p_{n_0}(\underline{x}, t) \underline{I} + \underline{F}_{n_0}(\underline{x}, t) \underline{J}_{n_0}^X(\underline{C}_{n_0}(\underline{x}, t)) \underline{F}_{n_0}^T(\underline{x}, t),$$

where the motions are subject to the constraint

$$\det \underline{C}_{n_0}(\underline{x}, t) = 1 \quad \forall t \in \mathbb{R}.$$

The assumption of incompressibility is made so often in elasticity that it has become commonplace to refer to an unconstrained elastic particle as being a compressible elastic particle.

Finally, we remark that there is another common concept of elasticity, called hyperelasticity¹, in which the constitutive equation has the form (in the unconstrained case)

¹ Sometimes, GREEN elasticity.

$$\underline{T}_n(\underline{x}, t) = \frac{\rho_n(\underline{x})}{J_n(\underline{x}, t)} \nabla \psi_n^x(\underline{F}_n(\underline{x}, t)) \underline{F}_n^T(\underline{x}, t),$$

where

$$\psi_n^x : \text{Inv } \mathcal{V}_3 \rightarrow \mathbb{R}$$

is the stored-energy function¹ per unit mass. This notion of elasticity will arise naturally in thermoelasticity, and we leave its motivation for there. Clearly, hyperelasticity is a special case of elasticity.

Supplementary Reading

ATKIN and FOX, An Introduction to the Theory of Elasticity

BEATTY, "Topics in finite elasticity: Hyperelasticity of rubber, elastomers, and biological tissues — with examples", Appl. Mech. Revs. 40, CHADWICK, Continuum Mechanics 1699-1734 (1987).

CIARLET, Mathematical Elasticity, Volume I: Three-Dimensional Elasticity

GREEN and ADKINS, Large Elastic Deformations

GREEN and ZERNA, Theoretical Elasticity

GURTIN, An Introduction to Continuum Mechanics

¹Often, strain-energy function or elastic potential.

GURTIN, Topics in Finite Elasticity, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 1981.

MARSDEN and HUGHES, Mathematical Foundations of Elasticity

OGDEN, Non-Linear Elastic Deformations

TRUESDELL, The Elements of Continuum Mechanics

TRUESDELL and NOLL, "The non-linear field theories of mechanics", Handbuch der Physik, vol III/3 (edit FLÜGGE) 1965.

VALENT, Boundary Value Problems of Finite Elasticity

WANG, Mathematical Principles of Mechanics and Electromagnetism,
Part A: Analytical and Continuum Mechanics

WANG and TRUESDELL, Introduction to Rational Elasticity

C2.2. Material Symmetry

Recall that the material symmetry group of a particle of a simple body corresponds to the set of all density preserving changes of reference configuration which leave the stress response of the particle to all subsequent motions unaltered. In particular, we see from Definition B4.5.2 that the material symmetry group of an elastic particle X relative to the reference configuration κ is

$$\mathcal{I}_\kappa^X = \{ \underline{H} \in \text{Unim } V_3 : \underline{G}_\kappa^X(\underline{F}\underline{H}) = \underline{G}_\kappa^X(\underline{F}) \quad \forall \underline{F} \in \text{Inv } V_3 \}.$$

We also recall NOLL's classification: An elastic particle X is

isotropic if $\text{Orth } V_3 \subset \mathcal{I}_\kappa^X$, κ undistorted;

solid if $\mathcal{I}_\kappa^X \subset \text{Orth } V_3$, κ undistorted;

a fluid crystal if $\mathcal{I}_\kappa^X \neq \text{Orth } V_3 \quad \forall$ configurations κ ;

fluid if $\mathcal{I}_\kappa^X = \text{Unim } V_3 \quad \forall$ configurations κ .

So as to incorporate the Principle of Material Indifference once and for all, we work with the reduced constitutive equation

$$(E-3) \quad \underline{T}_\kappa = \underline{F}_\kappa \underline{G}_\kappa^X(\underline{C}_\kappa) \underline{F}_\kappa^T.$$

As a special case of Theorem B4.5.7, we have that

$$\mathcal{I}_n^X = \{ \underline{H} \in \text{Unim } \mathcal{V}_3 : \underline{H} \underline{\bar{G}}_n^X (\underline{H}^T \underline{C} \underline{H}) \underline{H}^T = \underline{\bar{G}}_n^X (\underline{C}) \quad \forall \underline{C} \in \text{Psym } \mathcal{V}_3 \}.$$

From this we can read off

Theorem C2.2.1. A necessary and sufficient condition that an elastic particle X be isotropic (with n an undistorted reference configuration of $\mathcal{B}$) is that for each $\underline{C} \in \text{Psym } \mathcal{V}_3$,

$$\underline{\bar{G}}_n^X (\underline{Q}^T \underline{C} \underline{Q}) = \underline{Q}^T \underline{\bar{G}}_n^X (\underline{C}) \underline{Q} \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3 ;$$

i.e., the reduced response function $\underline{\bar{G}}_n^X$ is isotropic¹.

We proceed by following tradition and use representation Theorem A8.5.8. However, in view of the remarks following that theorem, representation Theorem A8.5.7 might be a more intelligent choice even though it would be considerably more cumbersome to work with.

Theorem C2.2.2. If the elastic particle X is isotropic (with n an undistorted reference configuration of $\mathcal{B}$), then $\exists$ functions (not necessarily unique)

$${}_i \Theta_n^X : \mathcal{I}(\text{Psym } \mathcal{V}_3) \rightarrow \mathbb{R} \quad (i=0,1,2)$$

and

$${}_i \Psi_n^X : \mathcal{I}(\text{Psym } \mathcal{V}_3) \rightarrow \mathbb{R} \quad (i=-1,0,1)$$

¹cf. Definition A8.5.1.

$\exists$ the reduced constitutive equation can be written in either of the following forms:

$$(IE-1) \quad \underline{T}_n(\underline{X}, t) = {}_0\Theta_n^X(l_1(\underline{B}_n(\underline{X}, t)), l_2(\underline{B}_n(\underline{X}, t)), l_3(\underline{B}_n(\underline{X}, t))) \underline{I} \\ + {}_1\Theta_n^X(\dots) \underline{B}_n(\underline{X}, t) + {}_2\Theta_n^X(\dots) \underline{B}_n^2(\underline{X}, t);$$

$$(IE-2) \quad \underline{T}_n(\underline{X}, t) = {}_0\Psi_n^X(l_1(\underline{B}_n(\underline{X}, t)), l_2(\underline{B}_n(\underline{X}, t)), l_3(\underline{B}_n(\underline{X}, t))) \underline{I} \\ + {}_1\Psi_n^X(\dots) \underline{B}_n(\underline{X}, t) + {}_{-1}\Psi_n^X(\dots) \underline{B}_n^{-1}(\underline{X}, t);$$

where $l_1(\underline{B}_n)$, $l_2(\underline{B}_n)$, and $l_3(\underline{B}_n)$ are the fundamental invariants of $\underline{B}_n$ ¹,

and $\mathcal{Q}(\text{Psym } \mathcal{V}_3) = \{(l_1(\underline{B}), l_2(\underline{B}), l_3(\underline{B})) : \underline{B} \in \text{Psym } \mathcal{V}_3\},$

$$\underline{B}_n = \underline{F}_n \underline{F}_n^T$$

is the left deformation tensor.

Conversely, given any functions

$${}_i\Theta_n^X, {}_i\Psi_n^X : \mathcal{Q}(\text{Psym } \mathcal{V}_3) \rightarrow \mathbb{R},$$

the constitutive equations (IE-1) and (IE-2) each satisfy the Principle of Material Indifference and each define an

¹ Cf. Theorem A3.11.8.

isotropic elastic particle (with an undistorted reference configuration).

Proof. Suppose that the elastic particle X defined by the reduced constitutive equation

$$(E-3) \quad \underline{T}_n = \underline{F}_n \underline{\bar{G}}_n^X(\underline{C}_n) \underline{F}_n^T$$

is isotropic. Then by Theorem C2.2.1 and A8.5.8, $\exists$ functions

$$\varphi_0, \varphi_1, \varphi_2 : \mathcal{D}(\text{Psym } V_3) \rightarrow \mathbb{R}$$

$\Rightarrow$

$$\underline{T}_n = \underline{F}_n \left[\varphi_0'(\iota_1(\underline{C}_n), \iota_2(\underline{C}_n), \iota_3(\underline{C}_n)) \underline{I} + \varphi_1(\dots) \underline{C}_n + \varphi_2(\dots) \underline{C}_n^2 \right] \underline{F}_n^T.$$

But

$$\underline{F}_n \underline{I} \underline{F}_n^T = \underline{F}_n \underline{F}_n^T \quad (\text{multiplicative identity property})$$

$$= \underline{B}_n \quad (\text{definition}),$$

$$\underline{F}_n \underline{C}_n \underline{F}_n^T = \underline{F}_n (\underline{F}_n^T \underline{F}_n) \underline{F}_n^T \quad (\text{substitution})$$

$$= (\underline{F}_n \underline{F}_n^T) (\underline{F}_n \underline{F}_n^T) \quad (\text{associativity})$$

$$= \underline{B}_n^2 \quad (\text{substitution}),$$

and similarly,

$$\underline{F}_n \underline{C}_n^2 \underline{F}_n^T = \underline{B}_n^3.$$

By the HAMILTON-CAYLEY theorem (Theorem A3.14.3), $\underline{B}_n^3$ can be replaced by

$$\underline{B}_n^3 = l_1(\underline{B}_n) \underline{B}_n^2 - l_2(\underline{B}_n) \underline{B}_n + l_3(\underline{B}_n) \underline{I}.$$

Next, we recall from Theorem B1.5.1 that

$$\underline{B}_n = \underline{R}_n \underline{C}_n \underline{R}_n^T.$$

Since the fundamental invariants are isotropic functions (Theorem A8.4.3),

$$l_i(\underline{C}_n) = l_i(\underline{R}_n \underline{C}_n \underline{R}_n^T) = l_i(\underline{B}_n), \quad i \in \mathbb{I}_3;$$

i.e., $\underline{B}_n$ and $\underline{C}_n$ have the same fundamental invariants. Consequently, they also have the same characteristic polynomial, and therefore $\underline{B}_n$ and $\underline{C}_n$ have the same eigenvalues. It then follows from Theorem A3.15.1 that $\underline{B}_n$ is also positive definite.

Assembling these remarks, we obtain the representation (IE-1). To get (IE-2), we use the HAMILTON-CAYLEY theorem again to write

$$\underline{B}_n^2 = l_1(\underline{B}_n) \underline{B}_n - l_2(\underline{B}_n) \underline{I} + l_3(\underline{B}_n) \underline{B}_n^{-1}.$$

Of course, $\underline{B}_n$ is invertible since $\underline{B}_n \in \text{Psym } \mathcal{V}_3$ (Theorem A3.15.2).

The proof of the converse assertion is left as Exercise C2.2.1. $\square$

Even though most people have a solid in mind when they speak of an elastic particle, such a restriction is not built into Definition C2.1.1. In fact, we have

Theorem C2.2.3.¹ If the elastic particle X is a fluid,
then $\exists$ a function

$$p^X: \mathbb{R}^+ \rightarrow \mathbb{R}$$

$\exists$ the reduced constitutive equation (E-3) can be written in the form

$$(EF) \quad \underline{T}_s(x, t) = p^X(\rho(x, t)) \underline{I},$$

where $\rho(x, t)$ is the mass density at the point $x = \chi(X, t)$ for the configuration that the body is in at time t ².

Conversely, given any function $p^X: \mathbb{R}^+ \rightarrow \mathbb{R}$, the constitutive equation (EF) satisfies the Principle of Material Indifference and defines a fluid elastic particle.

Proof. Suppose that the elastic particle X is fluid. Then it is necessarily isotropic (Theorem B4.6.3), and we can start with the representation (IE-1) of the previous

¹ Cf. Definition C1.2.3.

² Cf. Definition B2.2.1.

Theorem:

$$(IE-1) \quad \underline{T}_n = \Theta_n^x(\iota_1(\underline{B}_n), \iota_2(\underline{B}_n), \iota_3(\underline{B}_n)) \underline{I} + \Theta_n^x(\dots) \underline{B}_n + \Theta_n^x(\dots) \underline{B}_n^2.$$

Furthermore, since $\mathcal{Q}_n^x = \text{Unim } \mathcal{V}_3$ for a fluid particle, if we replace $\underline{F}_n$ by $\underline{F}_n \underline{H}$ in (IE-1), we must get the same stress $\forall \underline{H} \in \text{Unim } \mathcal{V}_3$. Now, $\underline{B}_n = \underline{F}_n \underline{F}_n^T$, and $\therefore$

$$\underline{F}_n \rightarrow \underline{F}_n \underline{H} \Rightarrow \underline{B}_n \rightarrow \underline{F}_n \underline{H} \underline{H}^T \underline{F}_n^T =: \underline{E}.$$

Then much as in the proof of Theorem B4.10.7, we choose

$$\underline{H} = |\det \underline{F}_n|^{1/3} \underline{F}_n^{-1} \Rightarrow \underline{E} = |\det \underline{F}_n|^{2/3} \underline{I}$$

and

$$\iota_1(\underline{E}) = 3 |\det \underline{F}_n|^{2/3}, \quad \iota_2(\underline{E}) = 3 |\det \underline{F}_n|^{4/3}, \quad \iota_3(\underline{E}) = |\det \underline{F}_n|^2.$$

Thus,

$$\underline{T}_n = \Theta_n^x(|\det \underline{F}_n|) \underline{I}.$$

To get the form claimed, we note that by the referential density equation (Theorem B2.1.5),

$$|\det \underline{F}_n(\mathbb{X}, t)| = \frac{\rho_n(\mathbb{X})}{\rho(\mathbb{X}_n(\mathbb{X}, t), t)}.$$

We view ρ_n as known as part of the prescription of the reference configuration $\mathbb{X}$, and then we define the function $p_n^x: \mathbb{R}^+ \rightarrow \mathbb{R}$

through

$$p_n^x(\rho(\mathbf{x}_n(\mathbf{x}, t), t)) = -\Theta_n^x \left(\frac{\rho_n(\mathbf{x})}{\rho(\mathbf{x}_n(\mathbf{x}, t), t)} \right).$$

Thus,

$$\underline{T}_n(\mathbf{x}, t) = -p_n^x(\rho(\mathbf{x}_n(\mathbf{x}, t), t)) \underline{\underline{I}}.$$

On conversion to the spatial description, we have

$$\underline{T}_s(\mathbf{x}, t) = -p_n^x(\rho(\mathbf{x}, t)) \underline{\underline{I}}.$$

Had we started we started with a different reference configuration, say μ , the we would have come out with

$$\underline{T}_\mu(\mathbf{x}, t) = -p_\mu^x(\rho(\mathbf{x}, t)) \underline{\underline{I}}.$$

Hence, the response function p_n^x is independent of n , and we simply write

$$\underline{T}_s(\mathbf{x}, t) = -p^x(\rho(\mathbf{x}, t)) \underline{\underline{I}}.$$

Conversely, suppose that the stress response of the particle X is defined by this constitutive equation. For the investigation of material indifference and material symmetry, we convert to the referential description via use of the referential density equation:

$$\underline{T}_n(\mathbf{x}, t) = -p^x \left(\frac{\rho_n(\mathbf{x})}{|\det \underline{F}_n(\mathbf{x}, t)|} \right) \underline{\underline{I}}.$$

When the motion $\{\chi(\cdot, t) : t \in \mathbb{R}\}$ is replaced by the equivalent motion $\{\chi^*(\cdot, t) : t \in \mathbb{R}\}$ defined by

$$\chi^*(X, \tilde{t}) = c(t) + Q(t) [\chi(X, t) - x], \quad \tilde{t} = t - a,$$

$$\tilde{F}_n^*(\chi, \tilde{t}) = Q(t) F_n(\chi, t); \quad)^1$$

and $\therefore$ the stress comes out to be

$$\tilde{T}_n^*(\chi, \tilde{t}) = -p^x \left(\frac{P_n(\chi)}{|\det \tilde{F}_n^*(\chi, \tilde{t})|} \right) \tilde{I}$$

$$= -p^x \left(\frac{P_n(\chi)}{|\det [Q(t) F_n(\chi, t)]|} \right) \tilde{I} \quad (\text{substitution})$$

$$= -p^x \left(\frac{P_n(\chi)}{|\det Q(t)| |\det F_n(\chi, t)|} \right) \tilde{I} \quad \left(\begin{array}{l} \det(AB) = (\det A)(\det B) \\ \text{and} \\ |ab| = |a||b| \end{array} \right)$$

$$= -p^x \left(\frac{P_n(\chi)}{|\det F_n(\chi, t)|} \right) \tilde{I} \quad (Q(t) \in \text{Orth } \mathcal{V}_3)$$

$$= -p^x \left(\frac{P_n(\chi)}{|\det F_n(\chi, t)|} \right) Q(t) Q^T(t) \quad (Q(t) \in \text{Orth } \mathcal{V}_3)$$

$$= Q(t) \left[-p^x \left(\frac{P_n(\chi)}{|\det F_n(\chi, t)|} \right) \tilde{I} \right] Q^T(t) \quad (\text{tensor multiplication})$$

¹ Cf. the proof of Theorem B4.2.2.

$$= \underline{Q}(t) \underline{T}_n(\underline{x}, t) \underline{Q}^T(t) \quad (\text{substitution}).$$

Hence, the constitutive equation satisfies the Principle of Material Indifference.

As noted at the beginning of the section, the material symmetry group of an elastic particle is given by

$$\mathcal{I}_n^X = \{ \underline{H} \in \text{Unim } \mathcal{V}_3 : \underline{G}_n^X(\underline{F}\underline{H}) = \underline{G}_n^X(\underline{F}) \quad \forall \underline{F} \in \text{Inv } \mathcal{V}_3 \}.$$

For the constitutive equation (EF),

$$\underline{G}_n^X(\underline{F}) = -p^X \left(\frac{\rho_n(\kappa(X))}{|\det \underline{F}|} \right) \underline{I};$$

and for any $\underline{H} \in \text{Unim } \mathcal{V}_3$,

$$\begin{aligned} \underline{G}_n^X(\underline{H}\underline{F}) &= -p^X \left(\frac{\rho_n(\kappa(X))}{|\det(\underline{F}\underline{H})|} \right) \underline{I} \\ &= -p^X \left(\frac{\rho_n(\kappa(X))}{|\det \underline{F}| |\det \underline{H}|} \right) \underline{I} \\ &= -p^X \left(\frac{\rho_n(\kappa(X))}{|\det \underline{F}|} \right) \underline{I} \\ &= \underline{G}_n^X(\underline{F}), \end{aligned}$$

Hence, $\mathcal{I}_n^X = \text{Unim } \mathcal{V}_3$, so (EF) defines a fluid particle. $\square$

Of course, the converse assertion is particularly important because it tells us that additional special choices of elements of the material symmetry group cannot lead to further reductions in the form of the constitutive equation for an elastic fluid.

Now, if an isotropic elastic particle is also solid, then its material symmetry group is the orthogonal group (Theorem B4.6.7). Thus, its constitutive equation can be taken to have either of the forms (IE-1) or (IE-2) of Theorem C2.2.2¹; there are no additional reductions of the kind that lead us to the constitutive equation for an elastic fluid.

We also know that in general an isotropic particle is either solid or fluid (Theorem B4.6.8). Hence, in the context of elastic isotropic particles, there are no more cases for us to explore.

Finally, we consider the material symmetry of an incompressible elastic particle. As noted on pp. C2.1.4-5, the basic constitutive equation in this case is

$$\underline{T}_m(x,t) = -p_m(x,t) \underline{I} + \underline{G}_n^x(\underline{F}_n(\underline{u}_m(x),t));$$

¹With due regard paid to the caveat which precedes the statement of this theorem.

and F_n must satisfy the constraint

$$\det F_n(X, t) = 1 \quad \forall t \in \mathbb{R}.$$

Here, we take the reference configuration κ to be a configuration that is occupied at some time in all motions of the body $\mathcal{B}$ so that we can draw freely on the results of § B4.10.

By Theorem B4.10.6 and Definition B4.10.3, the material symmetry group of an incompressible particle is just the symmetry group of the determinate stress. In the isotropic case, the representations of Theorem C2.2.2 can be used for the determinate stress, but $I_3(B_n)$ will be fixed at 1. Also, the $\Theta_n^X \mathbb{I}$ (or $\Psi_n^X \mathbb{I}$) term can be absorbed into the indeterminate pressure.¹ Thus, we have

Theorem C2.2.4. If the incompressible elastic particle X is isotropic (with κ an undistorted and occupied reference configuration of $\mathcal{B}$), then $\exists$ functions (not necessarily unique)

$$i \Theta_n^X : \mathcal{J}(P_{\text{sym}} V_3) \rightarrow \mathbb{R} \quad (i=1, 2)$$

and

$$i \Psi_n^X : \mathcal{J}(P_{\text{sym}} V_3) \rightarrow \mathbb{R} \quad (i=-1, 1)$$

$\exists$ the reduced constitutive equation can be written in either of the following forms:

¹This, of course, takes care of the normalization of the determinate stress; cf. the remarks on 2.1.1.10

$$(IE-1)' \quad \underline{T}_n(\underline{x}, t) = -p(\underline{x}, t) \underline{I} + {}_1\hat{\theta}_n^x(l_1(\underline{B}_n(\underline{x}, t)), l_2(\underline{B}_n(\underline{x}, t))) \underline{B}_n(\underline{x}, t) \\ + {}_2\hat{\theta}_n^x(\dots) \underline{B}_n^2(\underline{x}, t);$$

$$(IE-2)' \quad \underline{T}_n(\underline{x}, t) = -p(\underline{x}, t) \underline{I} + {}_1\hat{\psi}_n^x(l_1(\underline{B}_n(\underline{x}, t)), l_2(\underline{B}_n(\underline{x}, t))) \underline{B}_n(\underline{x}, t) \\ + {}_1\hat{\psi}_n^x(\dots) \underline{B}_n^{-1}(\underline{x}, t);$$

where $l_1(\underline{B}_n)$, $l_2(\underline{B}_n)$, and $l_3(\underline{B}_n) = 1$ are the fundamental
invariants of $\underline{B}_n$, and

$$\mathcal{I}(\text{Psym } \mathcal{V}_3) = \{(l_1(\underline{B}), l_2(\underline{B})) : \underline{B} \in (\text{Psym } \mathcal{V}_3) \cap (\text{Unim}^+ \mathcal{V}_3)\}.$$

Conversely, given any functions

$${}_1\hat{\theta}_n^x, {}_1\hat{\psi}_n^x : \mathcal{I}(\text{Psym } \mathcal{V}_3) \rightarrow \mathbb{R},$$

the constitutive equations $(IE-1)'$ and $(IE-2)'$ each satisfy
the Principle of Material Indifference and each define
an isotropic elastic particle (with $\underline{u}$ an undistorted and
occupied reference configuration).

In the case of an incompressible elastic fluid the determinate pressure ¹ may as well be absorbed into the indeterminate pressure. ² Thus, we have

¹ Cf. Theorem C2.2.3.

² Again, this takes care of the normalization of the determinate stress; cf. the remarks on p. C1.1.10.

Theorem C2.2.5, If the incompressible elastic particle X is fluid, then the reduced constitutive equation can be written in the form

$$\underline{T}_s(x, t) = -p(x, t) \underline{I} \quad . \quad)^1$$

Conversely, this constitutive equation satisfies the Principle of Material Indifference and defines a fluid incompressible elastic particle.

The above constitutive equation is sufficiently degenerate that it might be instructive to provide a formal proof of the converse assertion. This constitutes Exercise C2.2.2.

Supplementary Reading

Same as for § C2.1

¹ Thus, an incompressible elastic fluid particle is just an incompressible perfect fluid particle; cf. Definition C1.2.6.

C2.3. Some Equilibrium Solutions for Compressible, Isotropic Homogeneous Elastic Solids

The general formulation of boundary-value problems in finite elasticity is not as trivial a task as one might expect,¹ and even when they are properly formulated, they are generally intractable. Consequently, we will delay the study of boundary-value problems; and instead in this section and the next, we show how inverse techniques can be used to provide sample solutions and to suggest experimental programs for the determination of material response functions.

In the inverse method, we start with a given deformation, calculate the corresponding stress from the constitutive equation, and check to see if the stress satisfies equilibrium (generally for zero body force). If equilibrium is not satisfied, then the deformation is discarded. However, if equilibrium is met, then we attempt to interpret the deformation and the stress in a physically meaningful setting.

I.e., we consider various shapes for the deformed body, calculate the corresponding surface tractions, and hope to get something of physical interest.

The semi-inverse method is just the same, except that in the deformation one includes some arbitrary parameters or

¹ Cf. §44 of Truesdell and Noll's NFTM.

functions that can be adjusted so that equilibrium is met or the boundary data come out to be more interesting.

Since body force is difficult to apply physically, we prefer to consider situations where the body force is zero. This, of course, generally entails neglecting the weight force. Also we are particularly interested in deformations that give rise to equilibrated stress fields no matter what the elastic response function might be. These notions are sensible beyond elasticity theory, and we formalize them with

Definition C2.3.1. A deformation is controllable for a specific material if it can be maintained in equilibrium by surface tractions alone.¹ A deformation is universal for a class of materials if it is controllable for every material in that class.² ³

Universal deformations are of great importance in experimental work. When a universal deformation can be

¹ I.e., the stress corresponding to a controllable deformation (as calculated from the constitutive equation that defines the specific material under consideration) satisfies the equilibrium equation for zero body force.

² More often than not in the literature, controllable is used as a synonym for what we have called universal.

³ Adjustments appropriate to incompressible materials are considered in the next section.

interpreted in the context of a practically realizable body shape and loading, then comparison of laboratory measurements with the deformation and corresponding stress provides a means for the determination of the response function of the particular elastic body being tested. Moreover, the experimenter knows that the deformation that he or she is trying to produce in the laboratory is possible to maintain in equilibrium by surface tractions of known type for a test specimen made of any material in the class under consideration.

Note that in this scheme of things the theory comes before the testing, and, in fact, the theory tells us what experiments to set up. If this is not the case, then the test is just the collection of data, which usually has no subsequent use. Of course, we do not ignore the fact that some experimental observations are necessary beforehand to suggest that the material under consideration might be elastic (and perhaps isotropic, compressible, etc.); nor do we make light of the fact that great physical insight based on observation underlies the general development of mechanics.

It should be clear from the above remarks that the equations with which we must deal in this section are the constitutive equation and the equilibrium equation. With regard to the constitutive equation, we are interested in a compressible, isotropic, homogeneous, elastic solid; consequently, we draw on (IE-2) of Theorem C2.2.2 to write

the referential stress as

$$\begin{aligned} \underline{T}_x(\underline{X}) = & \psi_0(\underline{I}_1(\underline{B}(\underline{X})), \underline{I}_2(\underline{B}(\underline{X})), \underline{I}_3(\underline{B}(\underline{X}))) \underline{I} \\ & + \psi_1(\dots) \underline{B}(\underline{X}) + \psi_2(\dots) \underline{B}^{-1}(\underline{X}). \end{aligned} \quad (*)$$

Here, $\mathcal{R}$ is the underlying reference configuration with respect to which the body is both solid and isotropic. ¹ We describe the deformation relative to the reference configuration, by writing

$$\underline{x} = \underline{\chi}(\underline{X}), \quad \underline{X} = \underline{\chi}^{-1}(\underline{x}).$$

Here, of course,

$$\underline{X} = \underline{x}(X)$$

is the position that the typical particle X of the body $\mathcal{B}$ occupies in the reference configuration $\mathcal{R}$. We also write

$$\underline{F} = D\underline{\chi}, \quad \underline{B} = \underline{F}\underline{F}^T, \text{ etc.};$$

and we require the reference configuration to be occupied so that $\det \underline{F} > 0$.

We suppress explicit mention of $\mathcal{R}$ in our notation, except where it is needed for clarity. Since we are doing equilibrium theory without regard to how the particle X actually moves

¹Cf. Theorem B4.6.7a and the subsequent remark.

from $\mathbb{X} \in \mathcal{X}(\mathcal{B})$ to $\mathbb{X} \in \mathcal{X}(\mathcal{M}(\mathcal{B}))$, the relevant fields are all independent of time.

The response functions Ψ_0, Ψ_1 , and Ψ_{-1} are the same for all particles because of the homogeneity. Thus, they do not depend explicitly on $\mathbb{X}$. In order to proceed, we assume that they have continuous first partial derivatives. In view of the remarks on p. A8.5.22 concerning such a single formula representation of an isotropic function, this means that we are not dealing here with the most general compressible, isotropic, homogeneous, elastic solid which has a smooth response function. This point seems to have been largely overlooked in the literature; so that we too do not forget, we shall henceforth refer to the material defined by (*) with smooth $\Psi_0, \Psi_1, \Psi_{-1}$ as being "special isotropic."

By Theorem B3.4.3, the stress equation of equilibrium in the absence of body force is

$$\operatorname{div} \underline{\underline{T}}(\mathbb{X}) = \underline{\underline{0}}, \quad \mathbb{X} \in \mathcal{X}(\mathcal{M}(\mathcal{B})), \quad (**)$$

The symmetry of $\underline{\underline{T}}(\mathbb{X})$ is guaranteed by the constitutive equation (*), of course,

$$\underline{\underline{T}}(\mathbb{X}) = \underline{\underline{T}}_n(\mathbb{X}^{-1}(\mathbb{X})). \quad (***)$$

Since the body force and the acceleration are both zero, the density does not appear in our basic equations. Hence, we need not worry here about conservation of mass.

Theorem C2.3.1. For the class of all compressible, special isotropic, homogeneous, elastic solids, all homogeneous deformations are universal.

Proof. Let the deformation $\mathbb{X}$ be homogeneous. Then by Theorem B1.4.1, it has the form

$$\mathbb{X}(\mathbb{X}) = \mathbb{X}(\mathring{\mathbb{X}}) + \mathring{\mathbb{F}}(\mathbb{X} - \mathring{\mathbb{X}}),$$

where $\mathring{\mathbb{F}}$ is uniform and $\mathring{\mathbb{X}}$ is an arbitrarily fixed point in $\mathring{\mathcal{B}}$. By the argument on p. B1.2.4,

$$\mathbb{F} = \mathring{\mathbb{F}} \Rightarrow \mathbb{B} = \mathring{\mathbb{F}}\mathring{\mathbb{F}}^T,$$

so that $\mathbb{B}$ is uniform. Then the corresponding stress given by $(*)$ is also uniform (independent of the choice of Ψ_0, Ψ_1 , and Ψ_-), and the equilibrium equation $(**)$ is automatically satisfied. Hence, by Definition C2.3.1, the arbitrary homogeneous deformation $\mathbb{X}$ is universal. $\square$

The converse of Theorem C2.3.1 is also true, but before

considering it, let us look at some particular solutions.

Pure Stretch

Let $\{\mathcal{O}; \underline{e}_{\langle i \rangle}\}$ be a reference frame for $\mathcal{E}_3$ and denote the corresponding coordinates of $\mathbb{X}$ and $\underline{x}$ by X^i and x^i , respectively. Consider the pure stretch defined (on all of $\mathcal{E}_3$) by

$$x^i = \lambda_i X^i, \quad \lambda_i = \text{const.} > 0. \quad)^1$$

In direct notation this says that

$$\underline{x} = \mathcal{O} + \underline{U}(\mathbb{X} - \mathcal{O}) = \mathbb{X}(\mathbb{X}),$$

where

$$\underline{U} = \lambda_1 \underline{e}_{\langle 1 \rangle} \otimes \underline{e}_{\langle 1 \rangle} + \lambda_2 \underline{e}_{\langle 2 \rangle} \otimes \underline{e}_{\langle 2 \rangle} + \lambda_3 \underline{e}_{\langle 3 \rangle} \otimes \underline{e}_{\langle 3 \rangle};$$

and on comparison with Definition B1.4.5, we see that we are indeed talking about a pure stretch.

By the argument on p. B1.2.4, $\underline{F} = \underline{U}$, and in terms of components

$$[\underline{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

¹Here it is useful to resort to the convention that under-scoring an index suspends the summation convention for that index.

Note that

$$\det \underline{F} = \lambda_1 \lambda_2 \lambda_3 > 0.$$

Since $\underline{B} = \underline{F}\underline{F}^T$,

$$[\underline{B}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, [\underline{B}^{-1}] = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix},$$

and

$$I_1(\underline{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, I_2(\underline{B}) = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, I_3(\underline{B}) = \lambda_1^2 \lambda_2^2 \lambda_3^2.$$

By the constitutive equation (*), the components of the stress tensor are the constants

$$T^{(11)} = \psi_0 + \psi_1 \lambda_1^2 + \psi_{-1} \lambda_1^{-2},$$

$$T^{(22)} = \psi_0 + \psi_1 \lambda_2^2 + \psi_{-1} \lambda_2^{-2},$$

$$T^{(33)} = \psi_0 + \psi_1 \lambda_3^2 + \psi_{-1} \lambda_3^{-2},$$

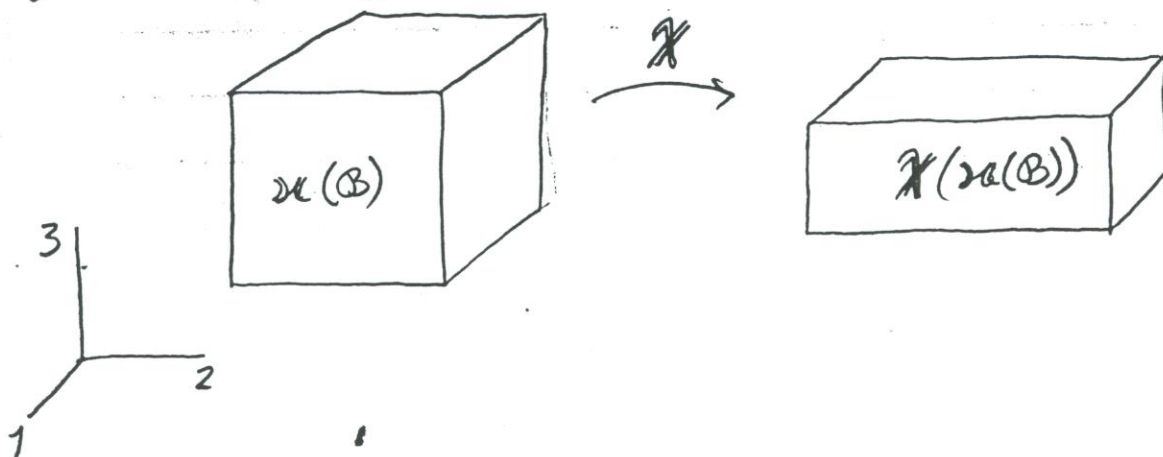
$$T^{(12)} = T^{(23)} = T^{(31)} = 0,$$

where

$$\psi_0 = \psi_0(\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \lambda_1^2 \lambda_2^2 \lambda_3^2), \text{ etc.}$$

Obviously, the equilibrium equation (**) is satisfied independently of the choice of the response functions ψ_0 , ψ_1 , and ψ_{-1} ; but, of course, we knew this in advance from Theorem C2.3.1.

A natural interpretation of this inverse solution is to take the undeformed body to be a rectangular block, and then the deformed body is also a rectangular block (with the same orientation) which is held in equilibrium by normal tractions on its faces,



Uniform Dilatation

Taking $\lambda_1 = \lambda_2 = \lambda_3 = \lambda > 0$ in the pure stretch deformation above, we get the uniform dilatation

$$\mathbf{x} = \mathbf{0} + \lambda (\mathbf{X} - \mathbf{0}).$$

For this deformation, the stress comes out to be

$$\underline{\mathbf{T}} = \psi(\lambda^2) \underline{\mathbf{I}},$$

where

$$\psi(\lambda^2) = \psi_0 + \psi_1 \lambda^2 + \psi_{-1} \lambda^{-2}, \quad \psi_0 = \psi_0(3\lambda^2, 3\lambda^4, \lambda^6), \text{ etc.}$$

Thus, the stress is a hydrostatic pressure, which has a

useful interpretation for any body shape.

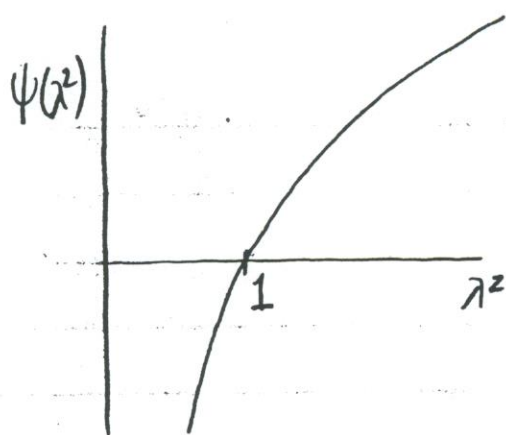
We can use this solution to see that there are some conditions which the response functions Ψ_0 , Ψ_1 , and Ψ_{-1} must satisfy in order that an elastic body behave in a physically reasonable way. If $\lambda > 1$, then $\mathcal{K}(\mathcal{R}(\mathcal{B}))$ has a larger volume than $\mathcal{R}(\mathcal{B})$, and it is physically reasonable to expect that the hydrostatic stress $\underline{\underline{T}} = \Psi(\lambda^2) \underline{\underline{I}}$ be tensile. Similarly, if $\lambda < 1$, then $\mathcal{K}(\mathcal{R}(\mathcal{B}))$ has a smaller volume than $\mathcal{R}(\mathcal{B})$, and $\underline{\underline{T}}$ should be compressive. If $\lambda = 1$, then the deformation is the identity, and $\underline{\underline{T}}$ should vanish if there is no residual stress. Thus, the response functions Ψ_0 , Ψ_1 , Ψ_{-1} should satisfy the conditions implied by

$$\Psi(\lambda^2) > 0 \text{ for } \lambda > 1,$$

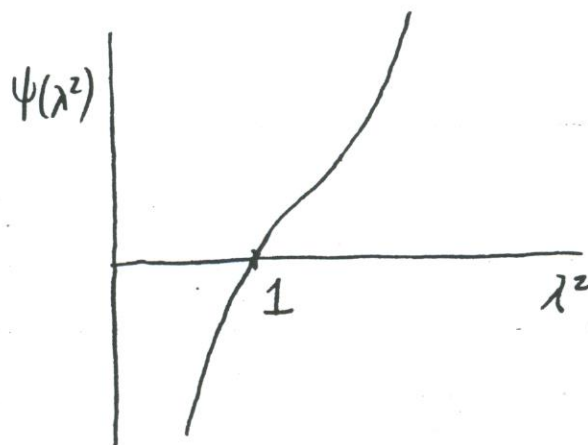
$$\Psi(1) = 0,$$

$$\Psi(\lambda^2) < 0 \text{ for } \lambda < 1.$$

Furthermore, increasing λ (and hence λ^2) increases the volume of the deformed body, and since $\underline{\underline{T}} = \Psi(\lambda^2) \underline{\underline{I}}$, this should call for an increase in $\Psi(\lambda^2)$. Hence, it is physically reasonable to expect that Ψ is a monotonically increasing function. Pictorially, then, we expect



or



The material depicted on the left "softens" in tension, while the one on the right "hardens" in tension. Both are shown as hardening in compression, because it seems reasonable to expect that to compress a body into a smaller and smaller volume would call for a larger and larger pressure. Of course, in reality, in both tension and compression, a body might cease to behave elastically long before one needs to think asymptotically.

To get $\psi(1) = 0$ above, we assumed that the residual stress in the reference configuration was zero. Actually, this is always the case for an elastic body which is isotropic with respect to an unloaded (meaning zero body force and zero surface traction) reference configuration. Traditionally, at least in the mechanics of solids, residual stress always refers to an unloaded reference configuration. If a reference configuration is loaded, then the preferred terminology for the stress in it is initial stress.¹ As usual, we assume that the body is at rest in the reference configuration.

¹cf. the remarks on p. C2.1.3.

Theorem C2.3.2.¹ (unconstrained) If an elastic body is isotropic² w.r.t. an unloaded and stationary reference configuration $\mathcal{R}$, then the residual stress in $\mathcal{R}$ is identically zero.

Proof. Since the body is elastic, it follows from Definition C2.1.1 that its constitutive equation is given at each particle x by

$$\underline{T}_x(\mathbb{X}, t) = \underline{G}_x^x(\underline{F}_x(\mathbb{X}, t)).$$

Since the body is isotropic w.r.t. $\mathcal{R}$, Theorem B4.5.5 requires that for each $\underline{F}_x \in \text{Inv } \mathcal{V}_3$

$$\underline{G}_x^x(\underline{Q} \underline{F}_x \underline{Q}^T) = \underline{Q} \underline{G}_x^x(\underline{F}_x) \underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3.$$

Choosing $\underline{F}_x = \underline{I}$, we get

$$\underline{G}_x^x(\underline{I}) = \underline{Q} \underline{G}_x^x(\underline{I}) \underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3.$$

But

$$\underline{G}_x^x(\underline{I}) = \underline{\dot{T}}_m(x) = \underline{\dot{T}}_x(\mathbb{X})$$

is the residual stress in $\mathcal{R}$. Thus,

¹ A. HÖGER, "On the residual stress possible in an elastic body with material symmetry", Arch. Rational Mech. Anal., BB, 271-289 (1985). See also F. MURNAGHAN, Finite Deformation of an Elastic Solid, Wiley (1951) p. 64 for partial results along this line.

² Not necessarily special isotropic nor solid nor homogeneous.

$$\overset{\circ}{\underline{T}}_{\underline{x}}(\underline{X}) = \underline{Q} \overset{\circ}{\underline{T}}_{\underline{x}}(\underline{X}) \underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3;$$

and by Theorem B4.2.4, $\overset{\circ}{\underline{T}}_{\underline{x}}(\underline{X})$ has the form

$$\overset{\circ}{\underline{T}}_{\underline{x}}(\underline{X}) = \alpha(\underline{X}) \underline{I}.$$

Since the body is in equilibrium in the unloaded reference configuration

$$\text{div } \overset{\circ}{\underline{T}}_{\underline{x}}(\underline{X}) = \nabla \alpha(\underline{X}) = \underline{0} \quad \forall \underline{X} \in \mathcal{M}(\mathcal{B}). \quad)^1$$

$\therefore$ since $\mathcal{M}(\mathcal{B})$ is connected,

$$\alpha(\underline{X}) = \text{const.} = \overset{\circ}{\alpha}, \quad)^2$$

Then the condition that there be zero surface traction on the boundary $\partial \mathcal{M}(\mathcal{B})$, i.e.

$$\overset{\circ}{\underline{T}}_{\underline{x}} \underline{n} = \overset{\circ}{\alpha} \underline{n} = \underline{0} \quad)^3$$

at regular points of $\partial \mathcal{M}(\mathcal{B})$, requires that $\overset{\circ}{\alpha} = 0$. $\square$

¹ See Theorem A6.9.3.

² See Theorem A5.9.3.

³ Cf. Theorem B3.4.2; $\underline{n}$, of course, is the outward unit normal to $\partial \mathcal{M}(\mathcal{B})$.

Application of this result to our constitutive equation (*) yields

Theorem C2.3.3. If the reference configuration is unloaded (and stationary,) then the response functions Ψ_0, Ψ_1 , and Ψ_{-1} , in the constitutive equation (*) for a compressible, special isotropic, homogeneous, elastic solid must meet

$$\Psi_0(3,3,1) + \Psi_1(3,3,1) + \Psi_{-1}(3,3,1) = 0.$$

Biaxial Extension

Let us return momentarily to a general pure stretch. The corresponding homogeneous state of stress is generally triaxial, and this is difficult to control in the laboratory. It would be desirable to adjust the λ 's (in the semi-inverse spirit) to clear one of the faces of traction. E.g., to get $T^{(33)} = 0$, we must have

$$\Psi_0 + \Psi_1 \lambda_3^2 + \Psi_{-1} \lambda_3^{-2} = 0.$$

For given λ_1 and λ_2 , we can easily see what restrictions would need to be placed on Ψ_0, Ψ_1 , and Ψ_{-1} so that the above equation could be solved for λ_3 by the implicit function theorem.

Uniaxial Extension

Referring to the above discussion of biaxial extension, suppose that we wish to have $T^{(22)} = 0$ as well as $T^{(33)} = 0$ so that state of stress is uniaxial¹. Thus, we must have both

$$\psi_0 + \psi_1 \lambda_3^2 + \psi_{-1} \lambda_3^{-2} = 0$$

and

$$\psi_0 + \psi_1 \lambda_2^2 + \psi_{-1} \lambda_2^{-2} = 0.$$

It is natural to try

$$\lambda_2 = \lambda_3 = \alpha \lambda_1,$$

for some $\alpha > 0$. Actually, by a neat argument BATRA² has shown that if the stress is uniaxial, i.e.,

$$[\mathbf{T}] = \begin{bmatrix} T^{(11)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and if the response functions ψ_0 , ψ_1 , and ψ_{-1} , satisfy the

² R.C. Batra, Deformation produced by a simple tensile load in an isotropic elastic body, *J. Elasticity*, 6, 109-111 (1976).

¹ Cf. the discussion p. B4.9.13.

so-called empirical inequalities

$$\Psi_0 \leq 0, \Psi_1 > 0, \Psi_{-1} \leq 0 \text{ on } \mathcal{Q}(\text{Psym } \mathcal{V}_3), \quad)^1$$

then

$$[\tilde{B}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \alpha^2 \lambda_1^2 & 0 \\ 0 & 0 & \alpha^2 \lambda_1^2 \end{bmatrix} \text{ with } \alpha < 1,$$

Of course, BATRA answered a question that is subtly but fundamentally different from the one that we are asking. He asked what deformations could produce a given stress field.² He then found a necessary condition that the corresponding deformations would have to satisfy. We are looking for a sufficient condition, and we are considering only those deformations that are pure stretches.

We are then left with

$$\Psi_0 + \Psi_1 \alpha^2 \lambda_1^2 + \Psi_{-1} \alpha^{-2} \lambda_1^{-1} = 0,$$

where

$$\Psi_0 = \Psi_0((1+2\alpha^2)\lambda_1^2, \alpha^2(2+\alpha^2)\lambda_1^4, \alpha^4\lambda_1^6), \text{ etc.}$$

Again, for given λ_1 , α can be found if the Ψ 's are suitably restricted.

¹For a comprehensive discussion of such inequalities, see §51 of Trawsdel and Noll's NFTM.

²There could conceivably be more than one corresponding deformation.

Simple Shear

Let $\{0; \underline{e}_{\langle i \rangle}\}$ be a reference frame for E_3 and denote the corresponding coordinates of $\mathbb{X}$ and $\underline{x}$ by X^i and x^i , respectively. Consider the homogeneous deformation defined (on all of E^3) by

$$x^1 = X^1 + \gamma X^2, \quad x^2 = X^2, \quad x^3 = X^3 \quad (\gamma = \text{const}).$$

In direct notation this says that

$$\underline{x} = 0 + \underline{\overset{\circ}{F}}(\mathbb{X} - 0) = \mathbb{X}(\mathbb{X}),$$

where

$$\underline{\overset{\circ}{F}} = \underline{\overset{\circ}{I}} + \gamma \underline{e}_{\langle 1 \rangle} \otimes \underline{e}_{\langle 2 \rangle}.$$

In terms of components,

$$[\underline{\overset{\circ}{F}}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which makes it clear that

$$\det \underline{\overset{\circ}{F}} = 1 \neq 0.$$

$\therefore$ by Theorem B1.4.1, the deformation is indeed homogeneous,

By the argument on p. B1.2.4, $\underline{\overset{\circ}{F}} = \underline{\overset{\circ}{F}}$. Since $\det \underline{\overset{\circ}{F}} = 1$, simple shear

is an isochoric deformation; but, of course, the elastic body $\mathcal{B}$ under consideration here is compressible.

Since $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ and $\underline{\underline{B}} = \underline{\underline{F}} \underline{\underline{F}}^T$, we easily calculate that

$$[\underline{\underline{C}}] = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & (1+\gamma^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\underline{\underline{B}}] = \begin{bmatrix} (1+\gamma^2) & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[\underline{\underline{B}}^{-1}] = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & (1+\gamma^2) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\lambda_1(\underline{\underline{B}}) = 3 + \gamma^2, \quad \lambda_2(\underline{\underline{B}}) = 3 + \gamma^2, \quad \lambda_3(\underline{\underline{B}}) = 1.$$

By Theorem B1.4.9, the stretches and shears in the coordinate directions are given by

$$\Lambda(\underline{\underline{e}}_{\langle 1 \rangle}) = 1, \quad \Lambda(\underline{\underline{e}}_{\langle 2 \rangle}) = \sqrt{1 + \gamma^2}, \quad \Lambda(\underline{\underline{e}}_{\langle 3 \rangle}) = 1$$

$$\sin \Gamma(\underline{\underline{e}}_{\langle 1 \rangle}, \underline{\underline{e}}_{\langle 2 \rangle}) = \frac{\gamma}{\sqrt{1 + \gamma^2}},$$

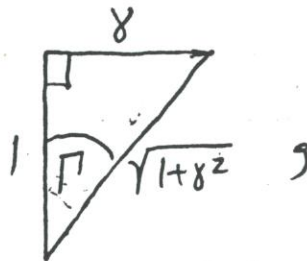
$$\sin \Gamma(\underline{\underline{e}}_{\langle 2 \rangle}, \underline{\underline{e}}_{\langle 3 \rangle}) = \sin \Gamma(\underline{\underline{e}}_{\langle 3 \rangle}, \underline{\underline{e}}_{\langle 1 \rangle}) = 0.$$

Recall from Definition B1.4.3, that the shear $\Gamma(\underline{\underline{e}}_{\langle 1 \rangle}, \underline{\underline{e}}_{\langle 2 \rangle})$ is the angle decrease between material line segments

initially along the 1- and 2- directions. By right triangle trigonometry, the statement

$$\sin \Gamma(\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}) = \frac{\gamma}{\sqrt{1+\gamma^2}}$$

corresponds to the figure



from which we may read off that

$$\gamma = \tan \Gamma(\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}).$$

The components of the stress tensor corresponding to simple shear are given by (*) as

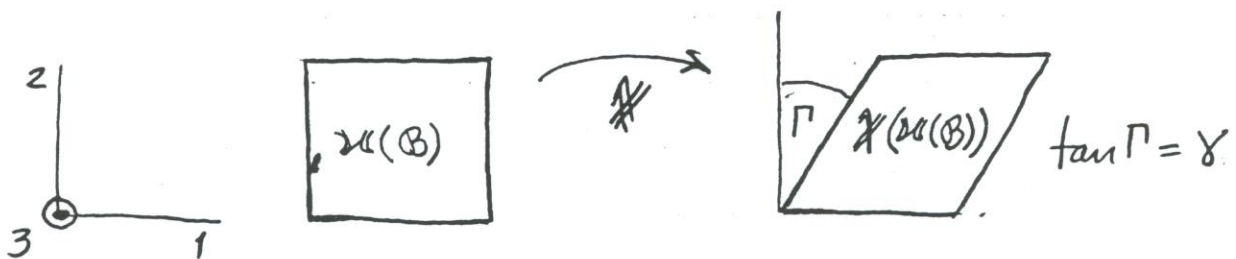
$$\left. \begin{aligned} T^{\langle 11 \rangle} &= \psi_0 + \psi_1(1+\gamma^2) + \psi_{-1}, \\ T^{\langle 22 \rangle} &= \psi_0 + \psi_1 + \psi_{-1}(1+\gamma^2), \\ T^{\langle 33 \rangle} &= \psi_0 + \psi_1 + \psi_{-1}, \\ T^{\langle 12 \rangle} &= (\psi_1 - \psi_{-1})\gamma, \\ T^{\langle 23 \rangle} &= T^{\langle 31 \rangle} = 0, \end{aligned} \right\} \quad (\#)$$

where

$$\psi_0 = \psi_0(3+\gamma^2, 3+\gamma^2, 1), \text{ etc.}$$

In view of Theorem C2.3.1 or by inspection, equilibrium is satisfied for zero body force regardless of the choice of the response functions Ψ_0 , Ψ_1 , and Ψ_{-1} .

A natural interpretation of this inverse solution is to take the undeformed body to be a rectangular block. It is then "sheared" into a parallelepiped and held in equilibrium in this deformed configuration by uniform tractions applied to the faces of the parallelepiped.



It is interesting to consider the linearization in γ of the above solution. By the assumed smoothness of the Ψ 's, we have, e.g., that

$$\begin{aligned} \Psi_0(3+\gamma^2, 3+\gamma^2, 1) &= \Psi_0(3, 3, 1) + \Psi_{0,1}(3, 3, 1)\gamma^2 \\ &\quad + \Psi_{0,2}(3, 3, 1)\gamma^2 + o(\gamma^2) \text{ as } \gamma \rightarrow 0. \end{aligned}$$

Assuming that the reference configuration is (and stationary) unloaded, so that

$$\Psi_0(3, 3, 1) + \Psi_1(3, 3, 1) + \Psi_{-1}(3, 3, 1) = 0$$

by Theorem C2.3.3, we get the estimates

$$T^{<11>} = o(\gamma), T^{<22>} = o(\gamma), T^{<33>} = o(\gamma),$$

$$T^{<12>} = [\psi_+(3,3,1) - \psi_-(3,3,1)]\gamma + o(\gamma),$$

$$T^{<23>} = T^{<31>} = 0,$$

As $\gamma \rightarrow 0$. Of course, when the $o(\gamma)$ terms are ignored, we have exactly the stress field that would be predicted by the linear theory of elasticity with the material constant

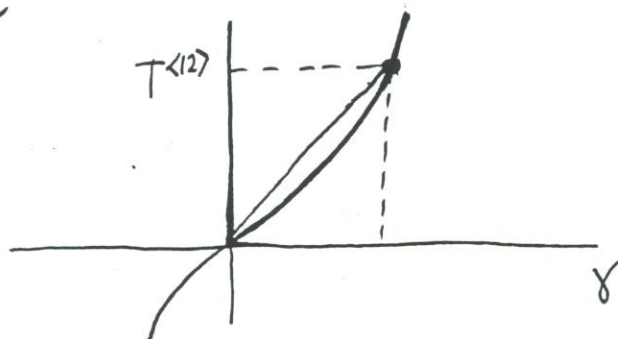
$$\psi_+(3,3,1) - \psi_-(3,3,1) = T^{<12>}/\gamma$$

being the classical shear modulus μ .

In the nonlinear case, the ratio

$$T^{<12>}/\gamma = \psi_+(3+\gamma^2, 3+\gamma^2, 1) - \psi_-(3+\gamma^2, 3+\gamma^2, 1) \equiv \hat{\mu}(\gamma^2) \quad (\#\#)$$

is called the generalized shear modulus or, in view of the graph



the secant shear modulus. Its knowledge would be of some interest for the simple shear problem, but it falls far

short of informing us of the elastic response functions ψ_0, ψ_1 , and ψ_{-1} , which are what we need to fully know the material. (##) makes it clear that the shear stress $T^{<12>}$ is an odd function of the "shear strain" δ , which is what we would expect.

In the linearized theory, the normal stresses $T^{<11>}, T^{<22>}$, and $T^{<33>}$ all vanish; while in the nonlinear case,

$$T^{<11>} = \psi_0 + \psi_1 + \psi_{-1} + \delta^2 \psi_1,$$

$$T^{<22>} = \psi_0 + \psi_1 + \psi_{-1} + \delta^2 \psi_{-1},$$

$$T^{<33>} = \psi_0 + \psi_1 + \psi_{-1}.$$

Thus, for all of ^{the} normal stresses to vanish in the nonlinear case, the material would have to be such that at $(l_1(\underline{B}), l_2(\underline{B}), l_3(\underline{B})) = (3 + \delta^2, 3 + \delta^2, 1)$,

$$\psi_0 = \psi_1 = \psi_{-1} = 0.$$

This is physically unreasonable for then even $T^{<12>} = 0$. We leave it as Exercise C2.3.1 to show that even the milder assumption

$$T^{<11>} = T^{<22>} = 0$$

leads to the same unacceptable conclusion, Such normal

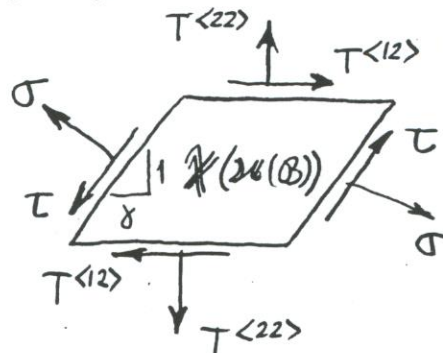
stress effects are a typical distinction between linear and nonlinear theories in continuum mechanics.

The first, second, and fourth equations in (#) $\Rightarrow$

$$T^{<11>} - T^{<22>} = \gamma T^{<12>}.$$

Since this holds independently of what the response functions Ψ_0 , Ψ_1 , and Ψ_2 might be, it is called a universal relation for the simple shear of compressible, special isotropic, homogeneous, elastic solids. Its significance comes from the observation that if a block of material is subjected to simple shear and the required stresses (as measured) do not satisfy the above relation, then we know that the material at hand is not a compressible, special isotropic, homogeneous, elastic solid.

Of course, one could not directly measure the normal stress $T^{<11>}$ directly for it acts on planes normal to e_1 , and such planes are inaccessible in the deformed body. Let us denote the normal and shear stress on the slant face of the deformed body by σ and τ , respectively.



It is easy to show (Exercise C2.3.2) that

$$\tau = \frac{T^{(12)}}{1+\gamma^2} \quad \text{and} \quad \sigma = T^{(22)} - \frac{\gamma T^{(12)}}{1+\gamma^2}.$$

Those equations are universal relations between more accessible quantities. However, the measurement of σ , τ , $T^{(22)}$, and $T^{(12)}$ is still no easy task.

The universal relation $T^{(11)} - T^{(22)} = \gamma T^{(12)}$ makes it clear that $T^{(11)} \neq T^{(22)}$ in the simple shear of compressible, special isotropic, homogeneous, elastic solids. This inequality of normal stresses (which does not occur in the linearized theory where they are both zero) is known as the POYNTING effect since he predicted it (from a special theory of finite elasticity) and observed it experimentally (actually in torsion) in 1905.

That the mean stress, $\frac{1}{3} \text{tr} \underline{T}$, does not vanish in finite simple shear was observed experimentally (again in torsion) by WERTHEIM in 1858 and predicted (again from a special theory) by KELVIN in 1867. Consequently, effects associated with a mean stress (usually volume change) which vanish in the linearized case but not in the nonlinear case are called WERTHEIM-KELVIN effects.

Finally, we turn to the converse of Theorem C2.3.1. The result is originally due to ERICKSEN¹; our proof is based on SHIELD's elegant demonstration².

Theorem C2.3.4. If a deformation is universal for the class of all compressible, special isotropic, homogeneous, elastic solids, then it is homogeneous.

Proof. Suppose that the deformation $\mathcal{K}$ (with deformation gradient $\underline{F}$ and left deformation tensor $\underline{B}$) is universal for the class of all compressible, special isotropic, homogeneous, elastic solids. Then by Definition C2.3.1 and the remarks on pp. C2.3.3-5, $\forall$ choices of the response functions

$$\Psi_i: \mathcal{Q}(\text{Sym } \mathcal{V}_3) \rightarrow \mathbb{R} \quad (i = -1, 0, 1),$$

we must have

$$\text{div } \underline{T}(\underline{x}) = \underline{0}, \quad \underline{x} \in \mathcal{K}(\mathcal{K}(\mathcal{B})), \quad (**)$$

where

$$\underline{T}(\underline{x}) = \underline{T}_m(\mathcal{K}^{-1}(\underline{x})). \quad (***)$$

and

¹ J. L. ERICKSEN, "Deformations possible in every compressible, perfectly elastic material," J. Maths. Phys., 34 (1955) 126-128.

² R. T. SHIELD, "Deformations possible in every compressible, isotropic, perfectly elastic material," J. Elasticity 1 (1971) 91-92.

$$\begin{aligned} \underline{T}_n(\mathbb{X}) = & \psi_0(\ell_1(\underline{B}(\mathbb{X})), \ell_2(\underline{B}(\mathbb{X})), \ell_3(\underline{B}(\mathbb{X}))) \underline{I} \\ & + \psi_1(\cdot) \underline{B}(\mathbb{X}) + \psi_{-1}(\cdot) \underline{B}^{-1}(\mathbb{X}). \end{aligned} \quad (*)$$

In order to ease the computation required in (**), we work with the asymmetric nominal stress tensor

$$\underline{S}(\mathbb{X}) = \det \underline{F}(\mathbb{X}) \underline{T}_n(\mathbb{X}) \underline{F}^{-T}(\mathbb{X}), \quad (d) \quad)'$$

so that by Theorem B3.6.2 (**) $\iff$

$$\operatorname{div} \underline{S}(\mathbb{X}) = \underline{0}, \quad \mathbb{X} \in \mathcal{K}(\mathcal{B}). \quad (dd)$$

We proceed by making special choices of the ψ_i . The first choice is defined by

$$\psi_0 \equiv 0, \quad \psi_1(\ell_1(\underline{B}), \ell_2(\underline{B}), \ell_3(\underline{B})) = \frac{2}{\sqrt{\ell_3(\underline{B})}}, \quad \psi_{-1} \equiv 0.$$

Then by (*) and (d),

$$\underline{S}(\mathbb{X}) = 2 \underline{F}(\mathbb{X})$$

and (dd) $\implies$

$$\operatorname{div} \underline{F} = \operatorname{div}(\underline{0} \mathbb{X}) = \underline{0} \quad \text{on } \mathcal{K}(\mathcal{B}). \quad (-v-)$$

¹See Theorem B3.6.2; here we have used $\det \underline{F} > 0$.

Thus, $\mathbb{X}$ is harmonic on $\mathcal{M}(\mathcal{B})$, and consequently

$$\mathbb{X} \in C^\infty(\mathcal{M}(\mathcal{B}), \mathbb{E}_3).)^1$$

Our second choice for the ψ 's is

$$\psi_0(\underline{\zeta}_1(\underline{B}), \underline{\zeta}_2(\underline{B}), \underline{\zeta}_3(\underline{B})) = 2\sqrt{\underline{\zeta}_3(\underline{B})}, \quad \psi_1 \equiv 0, \quad \psi_{-1} \equiv 0.$$

Then by $(*)$ and (∇) ,

$$\underline{S}(\mathbb{X}) = 2 [\det \underline{F}(\mathbb{X})]^2 \underline{F}^{-T}(\mathbb{X}).$$

We leave it as Exercise C2.3.2 to show that in this case

$$\operatorname{div} \underline{S} = 2(\det \underline{F}) \underline{F}^{-T}(\nabla \det \underline{F}).$$

$\therefore (\nabla \nabla) \Rightarrow$

$$\nabla \det \underline{F} = 0 \quad \text{on } \mathcal{M}(\mathcal{B}), \quad (-\nabla - \nabla)$$

Our final choice for the ψ 's is

$$\psi_0(\underline{\zeta}_1(\underline{B}), \underline{\zeta}_2(\underline{B}), \underline{\zeta}_3(\underline{B})) = \underline{\zeta}_1(\underline{B}), \quad \psi_1(\cdot) = 2, \quad \psi_{-1} \equiv 0.$$

Then by $(*)$ and (∇) ,

¹See, e.g., O.D. KELLOGG, Foundations of Potential Theory, Dover (1953), p. 220.

$$\underline{S}(\underline{X}) = [\det \underline{F}(\underline{X})] \left\{ \text{tr} [\underline{F}(\underline{X}) \underline{F}^T(\underline{X})] \underline{F}^{-T}(\underline{X}) + 2 \underline{F}(\underline{X}) \right\}.$$

In view of $(-V-)$ and $(-V-V-)$,

$$\text{div } \underline{S} = (\det \underline{F}) \underline{F}^{-T} \nabla \text{tr} (\underline{F} \underline{F}^T)$$

in this case, (Work this out as Exercise C2.3.3.). $\therefore (dd) \Rightarrow$

$$\nabla \text{tr} (\underline{F} \underline{F}^T) = \underline{0}. \quad (\star)$$

Using the additional smoothness gained from the knowledge that $\underline{X}$ is harmonic, we can take the divergence of $(\star)$ to get

$$\text{div } \nabla \text{tr} (\underline{F} \underline{F}^T) = 0. \quad (\star\star)$$

$(\star\star)$ is most easily exploited in terms of components w.r.t. a reference frame $\{\underline{0}; \underline{e}_{\langle i \rangle}\}$. From various results in Part A, we have

$$\begin{aligned} \text{div } \nabla \text{tr} (\underline{F} \underline{F}^T) &= \sum_{k=1}^3 \left(\sum_{i,j=1}^3 F^{\langle ij \rangle} F^{\langle ij \rangle} \right)_{,kk} \\ &= 2 \sum_{i,j,k=1}^3 \left(F^{\langle ij \rangle}_{,k} F^{\langle ij \rangle}_{,k} + F^{\langle ij \rangle} F^{\langle ij \rangle}_{,kk} \right). \end{aligned}$$

But since

$$F^{\langle ij \rangle} = \chi^i_{,j},$$

$$\sum_{k=1}^3 F_{,kk}^{<ij>} = \left(\sum_{k=1}^3 \chi_{,kk}^i \right)_{,j} \\ = 0 \quad ((-\nabla)).$$

$$\therefore (**) \Rightarrow \sum_{i,j,k=1}^3 F_{,sk}^{<ij>} F_{,sk}^{<ij>} = 0.$$

Observing that the l.h.s. above is a sum of squares, we have that

$$F_{,sk}^{<ij>} = 0 \quad \text{on } \mathcal{R}(\mathcal{B}).$$

Since $\mathcal{R}(\mathcal{B})$ is connected, we conclude that F is uniform on $\mathcal{R}(\mathcal{B})$. Then by Definition B1.4.1, the corresponding deformation $\mathcal{K}$ is homogeneous. $\square$

Note that in assertions like Theorem C2.3.4, the smaller the class of materials under consideration is, the stronger the result is. E.g., Theorem C2.3.4 has the following corollary.

Theorem C2.3.5. If a deformation is universal for the
class of all unconstrained elastic bodies¹, then it is homogeneous.

¹Not necessarily homogeneous nor isotropic nor solid. Of course, the converse of Theorem C2.3.5 (cf. Theorem C2.3.1) is lost for nonhomogeneous bodies.

Proof. Suppose $\mathcal{K}$ is universal for the class of all unconstrained elastic bodies. Then by Definition C2.3.1, it is controllable $\forall$ unconstrained elastic bodies. In particular, it is controllable $\forall$ compressible, special isotropic, homogeneous, elastic solids. $\therefore$ it is universal for the class of all compressible special isotropic, homogeneous, elastic solids; and Theorem C2.3.4 $\Rightarrow$ that $\mathcal{K}$ is homogeneous. $\square$

In this spirit, we remark that each of the three special materials used in the proof of Theorem C2.3.4 is actually hyperelastic¹, and $\therefore$, as was claimed by both ERICKSEN and SHIELD, Theorem C2.3.4 holds for the smaller class of all compressible, isotropic, homogeneous, hyperelastic solids.

Supplementary Reading

Same as for $\S$ C2.1

¹See pp. C2.1.5-6.

C2.4. Some Equilibrium Solutions for Incompressible, Isotropic, Homogeneous, Elastic Solids

Our objectives in this section are the same as in the previous one, i.e., we seek universal solutions of physical interest by the inverse method. However, because of the indeterminate pressure in the constitutive equation, which is the constraint stress that maintains the constraint of incompressibility, there are many more universal solutions than in the compressible case.

For the constitutive equation, we have from Theorem C2.2.4 that the referential stress is given by

$$\begin{aligned} \underline{T}_x(\mathbb{X}) = & -p_x(\mathbb{X}) \underline{\mathbb{I}} + \bar{\Psi}_1(\underline{C}_1(\underline{B}(\mathbb{X})), \underline{C}_2(\underline{B}(\mathbb{X}))) \underline{B}(\mathbb{X}) \\ & + \bar{\Psi}_{-1}(\cdot) \underline{B}^{-1}(\mathbb{X}), \end{aligned} \quad (*)$$

As in §C2.3, we shall suppress explicit mention of the reference configuration $\mathcal{R}$ to the extent that we can without causing confusion. We express the deformation as

$$x = \mathbb{X}(\mathbb{X}), \quad \mathbb{X} = \mathbb{X}^{-1}(x),$$

and also write

$$\underline{F} = D\mathbb{X}, \quad \underline{B} = \underline{F}\underline{F}^T, \text{ etc.}$$

Since we imagine the body as being in the reference

configuration before it is loaded, we require

$$\det \underline{F} > 0.$$

Actually, because of the incompressibility, we have the constraint

$$\det \underline{F} = 1 \quad \text{or} \quad \underline{\epsilon}(\underline{B}) = 1. \quad (**)$$

The response functions $\bar{\Psi}_1$ and $\bar{\Psi}_{-1}$ do not depend explicitly on $\mathbb{X}$ because of the homogeneity. We assume that they have continuous first partial derivatives. In view of the remarks on p. A8.5.22, we note that the single formula representation (*) does not define the most general class of incompressible, isotropic, homogeneous, elastic solids with smooth response functions. We keep this fact in mind by referring to the material defined by (*) with smooth $\bar{\Psi}_1$ and $\bar{\Psi}_{-1}$ as being special isotropic.

By Theorem B3.4.3, the stress equation of equilibrium in the absence of body force is

$$\operatorname{div} \underline{T}(\underline{x}) = \underline{0}, \quad \underline{x} \in \mathbb{X}(\mathcal{M}(\mathcal{B})); \quad (***)$$

and, of course,

$$\underline{T}(\underline{x}) = \underline{T}_n(\mathbb{X}^{-1}(\underline{x})).$$

Recall from Definition C2.3.1 that a deformation is controllable for a specific material if it can be maintained in equilibrium by surface tractions alone. This means that the stress corresponding to a controllable deformation (as calculated from the constitutive equation that defines the specific material under consideration) satisfies the equation of equilibrium for zero body force. When the material is internally constrained, the deformation does not completely determine the stress, and we must think a little more about the concept of controllability.

According to the Principle of Determinism for constrained simple bodies)¹, the stress in an internally constrained body is the sum of the determinate stress, which is determined from the deformation by a constitutive equation, and a constraint stress. Thus, in the constrained case, a deformation)² is controllable if $\exists$ a constraint stress of the form appropriate to the constraint)³ $\Rightarrow$ the sum of the constraint stress and the determinate stress satisfies the equation of equilibrium. With this interpretation understood, the formal definitions of controllable and universal deformations carry over to the constrained case without change.

¹ See p. B4.8.15.

² The deformation, of course, must satisfy the constraint.

³ Cf. Theorem B4.8.7.

Theorem C2.4.1. For the class of all incompressible,
special isotropic, homogeneous, elastic solids, all isochoric
homogeneous deformations are universal; and the corresponding
spatial pressure fields are uniform.

Proof. Exercise C2.4.1. $\square$

In contrast to the compressible case¹, the converse of the above theorem is not valid; i.e., there are nonhomogeneous universal deformations for incompressible, special isotropic, homogeneous, elastic solids. The open problem of finding all of them is known as ERICKSEN's problem.

Because of the isotropy, the residual stress must vanish; i.e., Theorem C2.3.2 holds also in the incompressible case. More precisely, we have

Theorem C2.4.2. If an incompressible elastic body is
isotropic² w.r.t. an unloaded and stationary reference
configuration κ , then the residual stress in κ is identically zero.

Proof. Exercise C2.4.2. $\square$

For the incompressible, special isotropic, homogeneous,

¹ Cf. Theorems C2.3.1 and C2.3.4.

² Not necessarily special isotropic nor homogeneous nor solid.

elastic solid defined by $(*)$, the stress in the reference configuration ($\underline{E} = \underline{I}$) is

$$\underline{\hat{T}}_x^0(\underline{X}) = -\hat{p}_x^0(\underline{X}) \underline{I} + \hat{\Psi}_1(3,3) \underline{I} + \hat{\Psi}_{-1}(3,3) \underline{I}.$$

Then by Theorem C2.4.2,

$$\hat{p}_x^0(\underline{X}) = \hat{\Psi}_1(3,3) + \hat{\Psi}_{-1}(3,3).$$

Thus, we know what the indeterminate reaction pressure is in the reference configuration. Of course, this does not tell us what it is in any other configuration. The above equation is the analog for the incompressible case of Theorem C2.3.3. it does not appear to be particularly useful.

Now we turn to some examples involving homogeneous deformations.

Pure Stretch

As in SC2.3, let $\{\underline{\underline{0}}; \underline{\underline{e}}_{\langle i \rangle}\}$ be a reference frame for E_3 and consider the deformation

$$\underline{\underline{x}} = \underline{\underline{X}}(\underline{\underline{X}}) = \underline{\underline{0}} + \underline{\underline{U}}(\underline{\underline{X}} - \underline{\underline{0}}),$$

where

$$\underline{\underline{U}} = \sum_{i=1}^3 \lambda_i \underline{\underline{e}}_{\langle i \rangle} \otimes \underline{\underline{e}}_{\langle i \rangle}, \quad \lambda_i > 0.$$

Here, by the argument on p. B1.2.4, $\underline{\underline{F}} = \underline{\underline{U}}$, and in terms of components,

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad [\underline{\underline{B}}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \text{and} \quad [\underline{\underline{B}}^{-1}] = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix}.$$

The incompressibility requires that

$$\det \underline{\underline{F}} = \lambda_1 \lambda_2 \lambda_3 = 1.$$

Also

$$\zeta_1(\underline{\underline{B}}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \text{and} \quad \zeta_2(\underline{\underline{B}}) = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2.$$

By Theorem C2.4.1, we know that this homogeneous deformation will produce a stress field that is automatically in equilibrium with zero body force for all incompressible, special isotropic, homogeneous, elastic solids provided the associated pressure field is uniform. Denoting the uniform pressure by p_0 , we see that the stress components are the constants

$$T^{(11)} = -p_0 + \bar{\psi}_1 \lambda_1^2 + \bar{\psi}_{-1} \lambda_1^{-2},$$

$$T^{(22)} = -p_0 + \bar{\psi}_1 \lambda_2^2 + \bar{\psi}_{-1} \lambda_2^{-2},$$

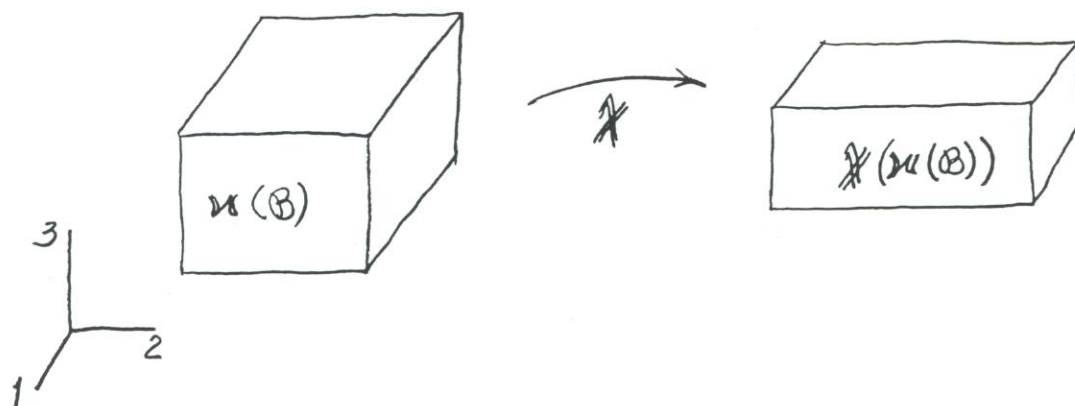
$$T^{(33)} = -p_0 + \bar{\psi}_1 \lambda_3^2 + \bar{\psi}_{-1} \lambda_3^{-2},$$

$$T^{(12)} = T^{(23)} = T^{(31)} = 0,$$

where

$$\bar{\psi}_1 = \bar{\psi}_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2), \text{ etc.}$$

As in the compressible case, a natural interpretation of this inverse solution is to take the undeformed body to be a rectangular block, and then the deformed body is also a rectangular block (with the same orientation), which is held in equilibrium by uniform normal tractions on its faces.



Next we specialize as in §C2.3 by making particular choices of the principal stretches. We do not have complete freedom in this because of the constraint $\lambda_1 \lambda_2 \lambda_3 = 1$. However, we can adjust the pressure p_0 to anything that suits us. We do not formalize uniform dilatation ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda$), because

the constraint requires that $\lambda = 1$, and then the corresponding deformation is the identity map.

Uniaxial Extension

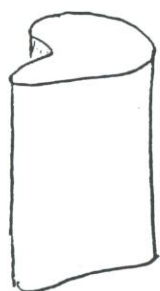
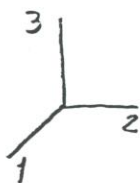
Choosing

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{\lambda_3}},$$

which satisfies the constraint $\lambda_1 \lambda_2 \lambda_3 = 1$, and

$$p_0 = \bar{\psi}_1 \lambda_3^{-1} + \bar{\psi}_{-1} \lambda_3,$$

we get $T^{(11)} = T^{(22)} = 0$; i.e., this choice clears all of the faces of the block except the two with outward unit normals $\underline{e}_{(3)}$ and $-\underline{e}_{(3)}$ of traction. In fact, we need not interpret this inverse solution in terms of a rectangular block. Just as natural (and more useful) is to take the undeformed body to be a right cylinder with generators parallel to $\underline{e}_{(3)}$, and then the deformed body is another right cylinder parallel to $\underline{e}_{(3)}$, which is held in equilibrium by uniform normal traction on its ends with the lateral surface being traction free.



Thus, we have an exact solution which corresponds to the simple tension test. The axial stress is

$$T^{(33)} = \hat{\psi}_1(\lambda_3^2 - \bar{\lambda}_3^{-1}) + \hat{\psi}_{-1}(\bar{\lambda}_3^2 - \lambda_3),$$

where

$$\hat{\psi}_1 = \hat{\psi}_1(\lambda_3^2 + 2\lambda_3^{-1}, \bar{\lambda}_3^2 + 2\lambda_3), \text{ etc.}$$

It is evident that we cannot determine the response functions $\hat{\psi}_1$ and $\hat{\psi}_{-1}$ from the tension test alone.

It is worthwhile to compare the differences in a pure stretch (as interpreted in a cylindrical geometry) in the compressible and incompressible cases. In the compressible case, we could not in general clean the lateral surface of traction so as to realize the tension test. However, in the incompressible case, this was a simple matter because of the presence of the indeterminate reaction pressure. RIVLIN was the first one to see that many finite elasticity problems for incompressible bodies could be solved this way with relative ease, and his work in elasticity in the 1940's provided great impetus to the whole field of modern continuum mechanics.

Biaxial Stretching

Here, we start with a pure homogeneous deformation and choose

$$p_0 = \bar{\psi}_1 \lambda_3^2 + \bar{\psi}_{-1} \lambda_3^{-2}$$

so that

$$T_{\langle 33 \rangle} = 0,$$

Since

$$\lambda_3^2 = (\lambda_1 \lambda_2)^{-2}$$

by the incompressibility condition, the nonzero components of the stress tensor are

$$T_{\langle 11 \rangle} = \bar{\psi}_1 \lambda_1^2 + \bar{\psi}_{-1} \lambda_1^{-2} - \bar{\psi}_1 (\lambda_1 \lambda_2)^{-2} - \bar{\psi}_{-1} (\lambda_1 \lambda_2)^2$$

and

$$T_{\langle 22 \rangle} = \bar{\psi}_1 \lambda_2^2 + \bar{\psi}_{-1} \lambda_2^{-2} - \bar{\psi}_1 (\lambda_1 \lambda_2)^{-2} - \bar{\psi}_{-1} (\lambda_1 \lambda_2)^2,$$

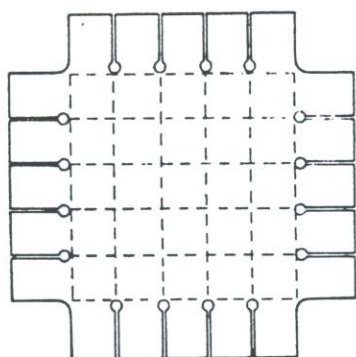
where

$$\bar{\psi}_1 = \bar{\psi}_1 (\lambda_1^2 + \lambda_2^2 + (\lambda_1 \lambda_2)^{-2}, (\lambda_1 \lambda_2)^2 + \lambda_1^{-2} + \lambda_2^{-2}), \text{ etc.}$$

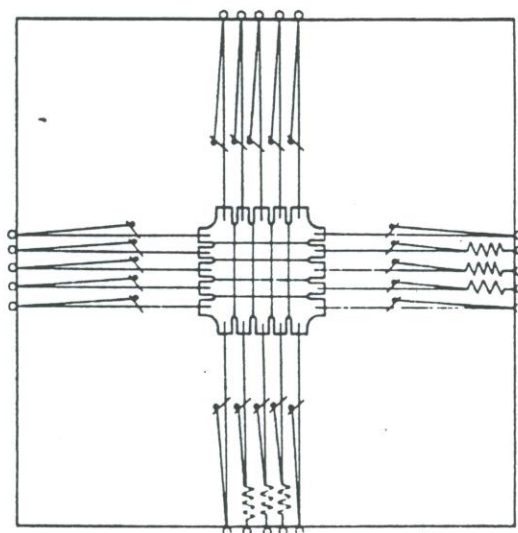
RIVLIN and SAUNDERS¹ used this solution to experimentally determine the response functions $\bar{\psi}_1$ and $\bar{\psi}_{-1}$ for various kinds of rubber. A rectangular grid was drawn on a sheet of rubber and loading lugs were cut into the sides of the sheet as indicated in the sketch. Forces were applied to the lugs and adjusted so as to make the

¹ Phil. Trans. Roy. Soc. London A 243, 1951. Also see the discussion in §3.6 of the text by ATKIN and FOX.

grid rectangular in the deformed configuration — thus ensuring a homogeneous deformation. The stretches λ_1 and λ_2 could be read off from the deformed grid. The stresses $T^{(11)}$ and $T^{(22)}$ were calculated by averaging the measured forces over the deformed areas. When this information is entered into the equations for $T^{(11)}$ and $T^{(22)}$, there results a pair of linear algebraic equations for $\bar{\Psi}_1$ and $\bar{\Psi}_2$ (evaluated at a particular argument). RIVLIN and SAUNDERS were able to cover the range $5 \leq C_1(B) \leq 11$, $5 \leq C_2(B) \leq 30$,



Undeformed



Deformed

You should not fail to notice the important role played by the universal nature of the homogeneous deformation in this experimental program. RIVLIN and SAUNDERS knew that as long as their rubber was an incompressible, special isotropic, homogeneous, elastic solid, then they could produce any isochoric homogeneous deformation by applying the stresses $T^{(11)}$ and $T^{(22)}$ as given by the formulas on p.C2.4.10, and the sheet would be in equilibrium without the aid of body force.

Simple Shear

As in §C2.3, let $\{\mathbf{0}; \underline{\underline{e}}_{\langle i \rangle}\}$ be a reference frame for E_3 and consider the deformation

$$\underline{\underline{x}} = \underline{\underline{X}}(\underline{\underline{X}}) = \mathbf{0} + \underline{\underline{F}}(\underline{\underline{X}} - \mathbf{0}),$$

where

$$\underline{\underline{F}} = \underline{\underline{I}} + \gamma \underline{\underline{e}}_{\langle 1 \rangle} \otimes \underline{\underline{e}}_{\langle 2 \rangle}.$$

In terms of components,

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\underline{\underline{B}}] = \begin{bmatrix} (1+\gamma^2) & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } [\underline{\underline{B}}^{-1}] = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & (1+\gamma^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also

$$\ell_1(\underline{\underline{B}}) = 3 + \gamma^2, \quad \ell_2(\underline{\underline{B}}) = 3 + \gamma^2, \quad \text{and} \quad \ell_3(\underline{\underline{B}}) = 1;$$

and we note that the incompressibility condition is automatically satisfied.

Again by Theorem C2.4.1, we know that this homogeneous deformation will always produce a stress field that is in equilibrium without body force for any incompressible, special isotropic, homogeneous, elastic solid provided that the associated pressure field is uniform. Denoting the uniform pressure by p_0 , we find the components of the stress tensor to be

$$T^{<11>} = -p_0 + \bar{\psi}_1 + \bar{\psi}_{-1} + \bar{\psi}_1 \gamma^2,$$

$$T^{<22>} = -p_0 + \bar{\psi}_1 + \bar{\psi}_{-1} + \bar{\psi}_{-1} \gamma^2,$$

$$T^{<33>} = -p_0 + \bar{\psi}_1 + \bar{\psi}_{-1},$$

$$T^{<12>} = (\bar{\psi}_1 - \bar{\psi}_{-1}) \gamma,$$

$$T^{<23>} = T^{<31>} = 0,$$

where

$$\bar{\psi}_1 = \bar{\psi}_1(3+\gamma^2, 3+\gamma^2), \text{ etc.}$$

As in §C2.3, a natural interpretation of this solution is the shearing of an initially rectangular block.



We can choose the pressure p_0 so as to free any pair of faces of normal traction. E.g., $p_0 = \bar{\psi}_1 + \bar{\psi}_{-1} \Rightarrow T^{<33>} = 0$; and in this case we get the simple relations

$$T^{<11>} = \bar{\psi}_1(3+\gamma^2, 3+\gamma^2) \gamma^2,$$

$$T^{<22>} = \bar{\psi}_{-1}(3+\gamma^2, 3+\gamma^2) \gamma^2,$$

$$T^{<12>} = [\bar{\psi}_1(3+\gamma^2, 3+\gamma^2) - \bar{\psi}_{-1}(3+\gamma^2, 3+\gamma^2)] \gamma,$$

As in the compressible case, the presence of the nonzero $T^{(11)}$ and $T^{(22)}$ is a normal stress effect, we still have the universal relation

$$T^{(11)} - T^{(22)} = \gamma T^{(12)},$$

and $\bar{\Psi}_1(3+\gamma^2, 3+\gamma^2) - \bar{\Psi}_{-1}(3+\gamma^2, 3+\gamma^2)$ can be thought of as the generalized shear modulus,

Next we consider an example of a nonhomogeneous universal deformation.

Confined Torsion and Extension of a Solid Circular Cylinder

Here, following the notational scheme of §B1.15, we refer $\mathbb{X}$ and $\mathbb{X}$ to the same circular cylindrical coordinate system. Consider the nonhomogeneous deformation defined by

$$r = \alpha R, \quad \theta = \Theta + \beta Z, \quad z = \gamma Z,$$

$$(x^1 = \alpha X^1, \quad x^2 = X^2 + \beta X^3, \quad x^3 = \gamma X^3),$$

where α, β , and γ are constants with

$$\alpha, \gamma > 0,$$

Of course, (R, Θ, Z) are the circular cylindrical coordinates of $\mathbb{X}$,

while (r, θ, z) are the circular cylindrical coordinates of $\underline{x}$,

By Theorem B1.15.6, the double components of the deformation gradient $\underline{F}$ are given by

$$[F^i_{\cdot A}] = \left[\frac{\partial x^i}{\partial X^A} \right] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{bmatrix}.$$

Exercise C2.4.3. Show that this deformation is, indeed, nonhomogeneous as claimed, (Hint: Determine rectangular Cartesian components of $\underline{F}$.)

It is left as Exercise C2.4.4 to show that the components (relative to the coordinate system at $\underline{x}$) of $\underline{B}$ and $\underline{B}^{-1}$ and the invariants of $\underline{B}$ are given by

$$[B^{ij}] = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & (r^{-2} + \beta^2) & \beta\gamma \\ 0 & \beta\gamma & \gamma^2 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & (\alpha^2 r^{-2} + \beta^2) & \beta\gamma \\ 0 & \beta\gamma & \gamma^2 \end{bmatrix},$$

$$[B^{<ij>}] = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & (\alpha^2 + \beta^2 r^2) & \beta\gamma r \\ 0 & \beta\gamma r & \gamma^2 \end{bmatrix},$$

$$[(\underline{B}^{-1})^{<ij>}] = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha^{-2} & -\alpha^{-2} \beta \gamma^{-1} r \\ 0 & -\alpha^{-2} \beta \gamma^{-1} r & \alpha^{-2} \gamma^{-2} (\alpha^2 + \beta^2 r^2) \end{bmatrix},$$

and

$$\zeta_1(\underline{B}) = 2\alpha^2 + \gamma^2 + \beta^2 r^2$$

$$\zeta_2(\underline{B}) = \alpha^4 + 2\alpha^2 \gamma^2 + \alpha^2 \beta^2 r^2,$$

$$\zeta_3(\underline{B}) = \alpha^4 \gamma^2.$$

The incompressibility condition (**) requires $\alpha^4 \gamma^2 = 1$,
or since $\alpha, \gamma > 0$,

$$\gamma = \alpha^{-2}.$$

Then we have the simplifications

$$[B^{<i>j>}] = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & (\alpha^2 + \beta^2 r^2) & \alpha^{-2} \beta r \\ 0 & \alpha^{-2} \beta r & \alpha^{-4} \end{bmatrix},$$

$$[(\underline{B}^{-1})^{<i>j>}] = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha^{-2} & -\beta r \\ 0 & -\beta r & \alpha^2 (\alpha^2 + \beta^2 r^2) \end{bmatrix},$$

and

$$\zeta_1(\underline{B}) = 2\alpha^2 + \alpha^{-4} + \beta^2 r^2,$$

$$\zeta_2(\underline{B}) = \alpha^4 + 2\alpha^{-2} + \alpha^2 \beta^2 r^2.$$

By the constitutive equation (*),

$$\underline{T} = -p \underline{I} + \bar{\psi}_1 \underline{B} + \bar{\psi}_{-1} \underline{B}^{-1},$$

and by the above results, the physical components (again relative to the natural basis at x) of the stress tensor are

$$T^{(rr)} = -p + \alpha^2 \bar{\psi}_1 + \alpha^{-2} \bar{\psi}_{-1},$$

$$T^{(\theta\theta)} = -p + (\alpha^2 + \beta^2 r^2) \bar{\psi}_1 + \alpha^{-2} \bar{\psi}_{-1},$$

$$T^{(zz)} = -p + \alpha^{-4} \bar{\psi}_1 + \alpha^2 (\alpha^2 + \beta^2 r^2) \bar{\psi}_{-1},$$

$$T^{(z\theta)} = \alpha^{-2} \beta r \bar{\psi}_1 - \beta r \bar{\psi}_{-1},$$

$$T^{(r\theta)} = T^{(rz)} = 0,$$

where $\bar{\psi}_1$ and $\bar{\psi}_{-1}$ are functions of $\zeta_1(\underline{B})$ and $\zeta_2(\underline{B})$.

Since all of the stress components above depend only on r (apart from the pressure p) and since $T^{(r\theta)} = T^{(rz)} = 0$, the equilibrium equation (**), $\text{div } \underline{T} = \underline{0}$, reduces to)¹

$$\frac{\partial T^{(rr)}}{\partial r} + \frac{1}{r} (T^{(rr)} - T^{(\theta\theta)}) = 0,$$

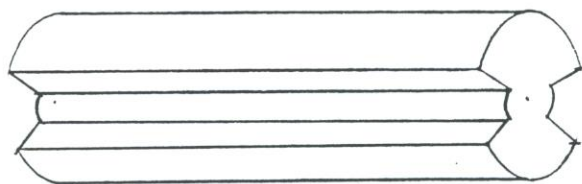
$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0.$$

¹ The physical components of $\text{div } \underline{T}$ relative to circular cylindrical coordinates are given on p. A6.16.3.

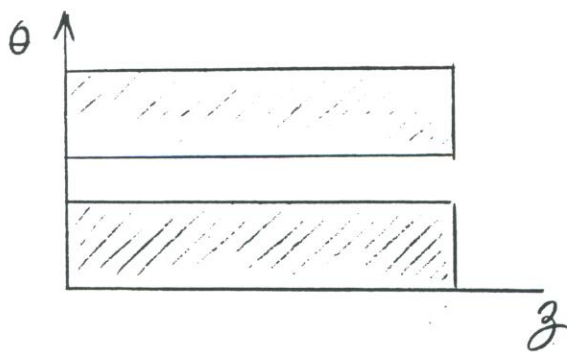
In integrating these simple equations, we must not be too cavalier. What we can do depends on the domain of the independent variables (r, θ, z) , and this in turn depends on the topology of the deformed body. Suppose that for each value of r the domain of (θ, z) is connected, then obviously

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \Rightarrow p = p(r). \quad)^1$$

However, if the deformed body is a cylinder of the sort depicted below,



then for some values of r the corresponding (θ, z) domain will look like the disconnected region shown below.



On the other hand, we could adopt the attitude that at this stage in a semi-inverse procedure, we can set the pressure to be anything that we want provided that equilibrium is still satisfied. The choice $p = p(r)$

¹ Note that the two-dimensional gradient vanishes, and argue as in the proof of Theorem A5.9.3.

ensures that

$$(\operatorname{div} \underline{T})^{<\theta>} = (\operatorname{div} \underline{T})^{<\varphi>} = 0,$$

and leaves us with

$$T^{<rr>} = T^{<rr>}(r), \quad T^{<\theta\theta>} = T^{<\theta\theta>}(r), \quad T^{<\varphi\varphi>} = T^{<\varphi\varphi>}(r).$$

$$\text{Then since } T^{<rr>} - T^{<\theta\theta>} = -\beta^2 r^2 \dot{\psi}_1,$$

$$(\operatorname{div} \underline{T})^{<r>} = 0$$

becomes

$$\frac{d}{dr} T^{<rr>} = \beta^2 r \dot{\psi}_1.$$

Here, since both sides of the equation depends only on r , the shape of the deformed body will not cause us any problem; and we can integrate from an arbitrarily chosen base point ($r=a$) to get

$$T^{<rr>} = T^{<rr>}(r) = T^{<rr>}(a) + \beta^2 \int_a^r r \dot{\psi}_1 dr.$$

But from the constitutive equation,

$$T^{<rr>} = -p + \alpha^2 \bar{\psi}_1 + \alpha^{-2} \dot{\psi}_{-1};$$

and $\therefore$

$$p = p(r) = \alpha^2 \bar{\psi}_1 + \alpha^{-2} \dot{\psi}_{-1} - \beta^2 \int_a^r r \dot{\psi}_1 dr - T^{<rr>}(a)$$

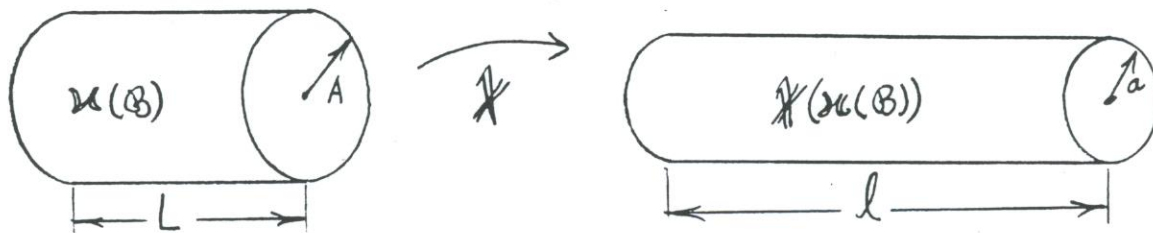
Thus, with this choice of the pressure, the stress

field on p.C2.4.17 satisfies the stress equation of equilibrium for zero body force for any choice of the response functions $\bar{\Psi}_1$ and $\bar{\Psi}_2$. Consequently, the deformation

$$r = \alpha R, \theta = \Theta + \beta Z, z = \alpha^{-2} Z$$

is universal for the class of all incompressible, special isotropic, homogeneous, elastic solids.

As a natural interpretation of this semi-inverse solution, we take the undeformed body to be a right circular cylinder of radius A and length L . Then the deformed body is also a right circular cylinder of radius $a = \alpha A$ and length $l = \alpha^{-2} L$.



We can clear lateral surface of traction by taking

$$T^{(rr)}(a) = 0.$$

Then the deformed cylinder is held in equilibrium by traction on its end faces. The net axial load is

$$N = \int_0^a T^{(zz)} 2\pi r dr ;$$

the net twisting moment is

$$T = \int_0^a r T_{\langle \theta\theta \rangle} 2\pi r dr ;$$

and because of the symmetry of the stress distribution and the cross section, the net bending moments and shear loads are zero. We leave it as Exercise C2.4.5 to show that

$$N = \pi(\alpha\beta)^2 \int_0^a r^3 (2\dot{\psi}_{-1} - \bar{\alpha}^2 \dot{\psi}_1) dr + 2\pi(\alpha^4 - \bar{\alpha}^2) \int_0^a r (\dot{\psi}_{-1} - \bar{\alpha}^2 \dot{\psi}_1) dr$$

and

$$T = 2\pi\beta \int_0^a r^3 (\alpha^{-2} \dot{\psi}_1 - \dot{\psi}_{-1}) dr.$$

(Hint: The calculation of T is straight forward. To get N in the above form start with

$$N = \int_0^a 2\pi r (T_{\langle \theta\theta \rangle} - T_{\langle rr \rangle}) dr + \int_0^a 2\pi r T_{\langle rr \rangle} dr$$

and integrate the last integral by parts using the equilibrium equation

$$\frac{d}{dr} T_{\langle rr \rangle} = \frac{1}{r} (T_{\langle \theta\theta \rangle} - T_{\langle rr \rangle}).$$

If $\delta=1$, then incompressibility requires $\alpha=1$; and the deformation under study here becomes one of pure torsion:

$$r = R, \theta = \Theta + \beta Z, z = Z.$$

In this case,

$$N = \pi \beta^2 \int_0^a r^3 (2\dot{\Psi}_{-1} - \dot{\Psi}_1) dr,$$

where

$$\dot{\Psi}_1 = \dot{\Psi}_1(3 + \beta^2 r^2, 3 + \beta^2 r^2), \text{ etc.}$$

We observe that we cannot have pure torsion without supplying some axial load — in contrast to the situation in the linear theory. Experimental data suggest

$$\dot{\Psi}_1 > 0 \text{ and } \dot{\Psi}_{-1} \leq 0.$$

If these inequalities are valid, then $N < 0$; and we conclude that one must supply a compressive axial load in order to effect a pure torsion. This was theoretically predicted and experimentally observed by RIVLIN¹ in 1947.

A quite readable account of experimental work based on the above universal solution can be found in the text ATKIN and FOX.

¹R.S. RIVLIN, "Torsion of a rubber cylinder," J. Appl. Phys., 18 (1947) 444-449.