Crack Bridging. Lecture 1

Following Griffith (1921), we distinguish two processes: deformation in the body and separation of the body. Up to this point, the process of deformation has been described by field theories of various kinds, such as

- linear elastic theory (infinitesimal deformation, linear elastic material)
- nonlinear elastic theory (finite deformation, nonlinear elastic material)
- deformation theory of plasticity (infinitesimal deformation, fictitious nonlinear elastic material)

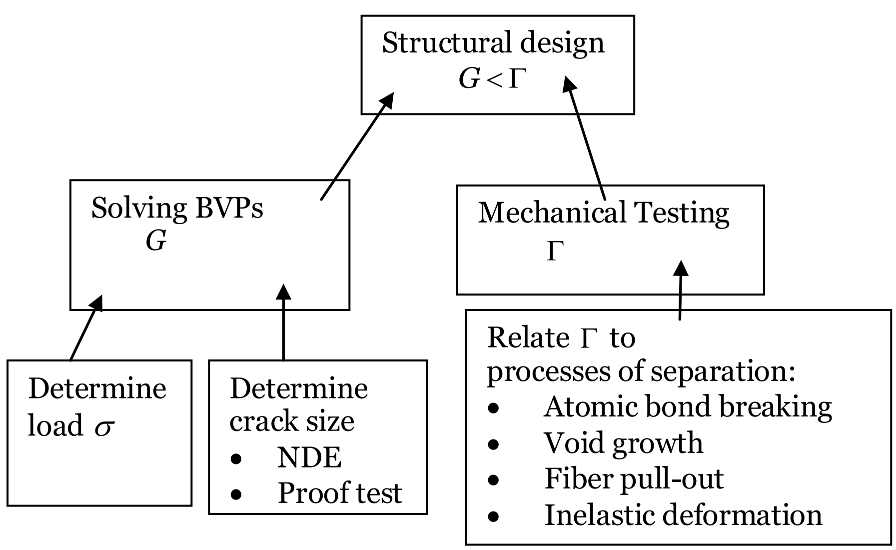
By contrast, the process of separation has been described, if it is described at all, by micrographs, cartoons, and words. A picture is worth a thousand words, but an equation is worth a thousand pictures. So far we have not used a single equation to describe any process of separation.

This negligence in describing the process of separation seems odd, particularly in a subject called fracture mechanics. Without specifying a process of separation, the artificial singularity would remain the black hole in our subject.

Barrenblatt (1959) modeled the process of separation by an array of nonlinear springs. By now this idea has permeated into fracture mechanics in many ways.

Griffith approach. Before critiquing the Griffith approach, we should take a moment to appreciate its accomplishments. Griffith (1921) initiated the approach to study fracture of glass. The approach was extended by Irwin and Orowan (1950s) to study metals, and by Rivlin and Thomas (1953) to study elastomers. Several perspectives have emerged:

- Structure vs. material
- Calculation vs. testing



The Griffith approach has taken us far. The Rivlin-Thomas (1953) experiment and the Begley-Landes (1972) experiment have shown that fracture energy can be determined without specifying either a theory of deformation or a theory of separation. Once a field theory of deformation is specified, however, we can do at least two things:

1. Relate the energy release rate to the external boundary conditions by solving boundary-value problems.

2. Identify conditions under which the energy release rate is the only parameter that represents the external boundary conditions.

The trouble with the Griffith approach. The accomplishments of the Griffith approach should not obscure an obvious fact: the approach lacks a description of the process of separation. The approach is hinged around a single quantity: the energy release rate G. Obvious limitations of the approach include:

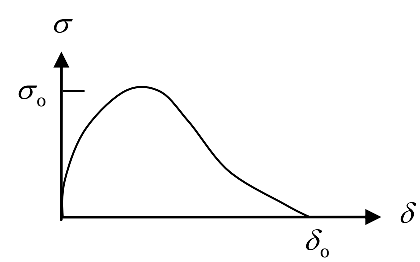
- When G is the only parameter that represents the external boundary conditions, we may measure the fracture energy, but we do not really understand the value.
- When G is not the only parameter that represents the external boundary conditions, we try to extend the approach, but with limited success.

Model the process of separation by a traction-separation curve. Barrenblatt (1959) modeled the process of separation by using an array of nonlinear springs. The springs are characterized by a traction-separation curve:

$$\sigma = \sigma(\delta).$$

The curve should have the following trends: going up and then going down.

The traction-separation curve introduces a scale of stress and a scale of length. Let σ_0 be the scale of traction, e.g., the maximum traction. Let δ_0 be the scale of separation, e.g., the separation beyond which the traction becomes negligible.



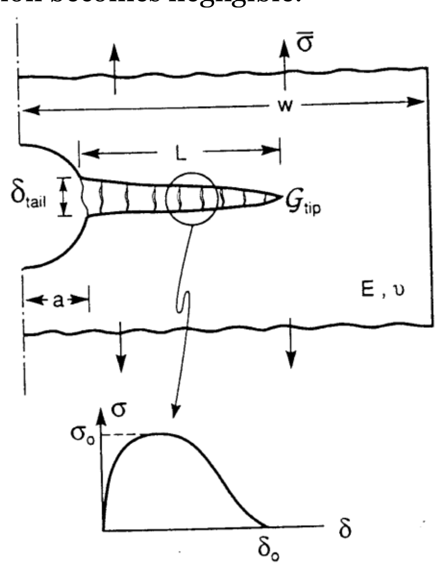
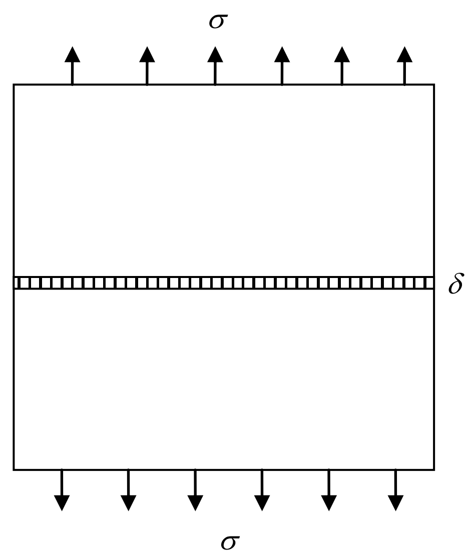


Fig. 1 An inelastic band is viewed as either a bridged crack or an array of continuously distributed dislocations

Crack-bridging model (cohesive-zone model). In this model, an array of nonlinear springs joins two halves of a body. The separation of the body is modeled by the traction-separation curve. The deformation in the body can be modeled by any field theory. For the time being, we adopt the linear elastic theory. The combination of the nonlinear springs the linear elastic body and defines a boundary-value problem. Separation and deformation are modeled in a single boundary-value problem.

Fracture energy. To explore consequences of the model, we begin with a simple boundary-value problem of this kind. Two halves the body are joined by an array of nonlinear springs. Subject to a remote stress σ, each half of the body is in a state of homogenous deformation. The springs are also in a state of homogenous deformation. The stress in the springs is the same as the applied stress.



As the separation δ increases, the stress σ goes up and then does down, following the traction-separation curve. The work done by the applied stress to separate the two halves is the area under the traction-separation curve. By definition, this work equals the fracture energy Γ, namely,

$$\Gamma = \int_0^\infty \sigma(\delta) d\delta .$$

Incidentally, because the traction-separation curve is not monotonic, the trivial solution described above is not the only solution to the boundary-value problem. Other solutions are possible. Springs in some region may deform more than those in other regions, and the deformation in the body can be inhomogeneous. For example, see Z. Suo, M. Ortiz and A. Needleman, Stability of solids with interfaces. *J. Mech. Phys. Solids*. **40**, 613-640 (1992). http://www.seas.harvard.edu/suo/papers/019.pdf

Square-root singular field. Recall the elastic field in the body without bridging zone. Let *r* be the distance from the tip of the crack. When *r* is small compared to all lengths specifying the geometry of the body, the external boundary conditions affect the field at *r* through the energy release rate alone. In addition to *r*, the problem has only one other length, G/E. The two lengths form a dimensionless ratio $G/(Er)$. The stress field is linear in the applied load, but the energy release rate is quadratic in the applied load. Consequently, the crack-tip field takes the form:

$$\sigma_{ij}(r,\theta) = E\sqrt{\frac{G}{Er}} f_{ij}(\theta).$$

Note that the square-root singularity results from the above qualitative considerations.

Small-scale bridging. We next place an array of springs ahead the tip of the crack. The deformation in the springs is inhomogeneous. The springs right at the tip of the crack deform more, but the springs far ahead the tip of the crack deform negligibly. Thus, we may speak of a bridging zone of length *L*. When the length of the bridging zone is small compared to the length *a* characteristic of the external boundary conditions, the singular solution predominates in the annulus

$$L < r < a.$$

We call this annulus the G-annulus. This lecture assumes the small-scale bridging condition.

Calculate the fracture energy by using the J integral (Rice, 1968). Under the small-scale bridging condition, we can apply the J integral as follows. For the J integral over a contour in the G-annulus,

$$J = G.$$

We then select a contour just includes the deformed springs. Recall the definition of the J integral:

$$J = \int \left(W n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds.$$

Over this small contour, the integral gives

$$J = \int_0^{\delta_{tail}} \sigma(\delta) d\delta,$$

where δ_{tail} is the separation at the tail of the bridging zone. Because the J integral is path-independent, we find that

$$G = \int_0^{\delta_{tail}} \sigma(\delta) d\delta.$$

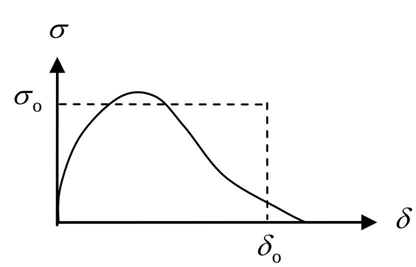
As the energy release rate G increases, the crack opens more, and δ_{tail} increases. In the steady state, the bridging zone translates in the body, stretching the springs in the front, and breaking the springs in the tail. We recover the relation

$$\Gamma = \int_0^{\infty} \sigma(\delta) d\delta.$$

Size of bridging zone. In obtaining the square-root singular field, the material is assumed to be linearly elastic. Once the nonlinear spring is introduced into the model, the stress cannot be singular, but is bounded by the strength of the springs, σ_0. In the steady state, the size of the bridging zone scales as

$$L_{ss} \sim \frac{\Gamma E}{\sigma_0^2}.$$

An idealized traction-separation curve. Because the traction-separation curve is nonlinear, the resulting boundary-value problem is also nonlinear. It appears that much insight can be gained by an idealized traction-separation curve, as shown in the figure.



Orders of magnitude. The scale of the traction-separation curve is set by σ_0 and δ_0. The fracture energy scales as

$$\Gamma \sim \sigma_0 \delta_0.$$

The size of the bridging zone scales as

$$L_{ss} \sim \frac{E}{\sigma_o}\delta_o .$$

For orders of magnitude, see the following table taken from Bao and Suo (1992).

TABLE 1 Illustrative properties for bridging mechanisms

	atomic bond	fiber pull-out	fiber cross-over
σ_0 (N/m^2)	10^{10}	10^9	10^7
δ_0 (m)	10^{-10}	10^{-5}	10^{-4}
$\sigma_0\delta_0$(J/m^2)	1	10^4	10^3
$\delta_0 E/\sigma_0$ (m)	10^{-9}	10^{-3}	1

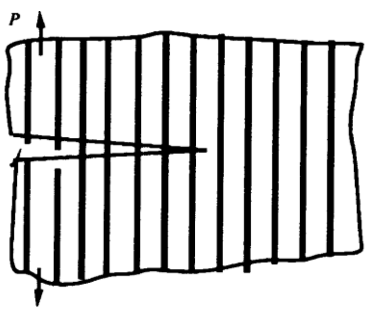


FIG. 2 Fiber pull-out.

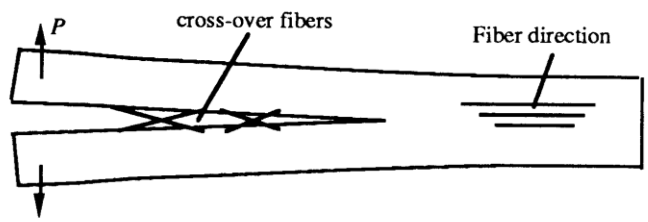


FIG. 3 Fiber cross-over.

Recap of what we have done. A specimen contains a crack and is subject to external loads. Represent the fracture process at the crack tip by the bridging law. Except for the bridging zone, the specimen is linear elastic. When the bridging zone length is much smaller than the crack length (SSB), an annulus exists, within which the stress field is described by the square-root singular field. Consequently, G is the only messenger between the crack tip process and the external load. We do two things:

(1) Relate G to the external loads.

(2) Relate G to the bridge law.

We have been doing (1) all along. Doing (2) is new to us. Previously, we have not specified any process of separation. Now we do, by prescribing a traction-separation curve. Once we have done (1) and (2), the external loads are linked to the crack tip process, through G.

(1) Relate G to the external loads

Determine the stress intensity by solving the elasticity boundary value problem, treating the crack tip as a mathematical point. The solutions to many specimen geometries are contained in handbooks. For example, for the Griffith crack, the stress intensity factor is

$$G = \pi\frac{a\sigma^2}{E}.$$

If you are given an odd specimen, you can always determine K by using the finite element method. Remember, in this step, the specimen is elastic all the way to the crack tip. You simply neglect the bridging at the crack tip.

(2) Relate G to process of separation

Let $\sigma(\delta)$ be the bridging law. The fracture energy is given by

$$\Gamma = \int_0^\infty \sigma(\delta) d\delta$$

The bridging model relates the fracture toughness to the bridging law, $\sigma(\delta)$. Often we have some idea of the bridging law in terms of the physical processes of the fracture, such as atomic decohesion, or fiber breaking. Consequently, the bridging model allows us to relate the fracture toughness to the physical process of fracture. Previously, we have emphasized that fracture toughness is measured experimentally. Now we have made an attempt to understand what we measure. The bridging model can tell us more. Read on.

Resistance curve. A previous lecture introduced the *R*-curve as determined by experimental measurement (http://imechanica.org/node/7674). The *R*-curve can now be calculated once we specify a process of separation. As G increases, the size of the bridging zone L increases. The $G-L$ function represents the *R*-curve.

To calculate the *R*-curve, we have to solve the boundary value problem. A semi-infinite crack lies in an infinite elastic body. Near the crack tip, the body is bridged, as described by a traction-separation curve. Remote from the crack tip, the body falls into the *G*-annulus, in which the stress field is characterized by the energy release rate G, namely,

$$\sigma_{ij}(r,\theta) = E\sqrt{\frac{G}{Er}} f_{ij}(\theta).$$

The process of separation is modeled by the ideal traction-separation curve. The stress intensity factor at the crack tip due to a pair of forces on the crack faces is

$$K_{\text{tip}} = \sqrt{\frac{2}{\pi}} \frac{P}{\sqrt{\xi}}.$$

This is a basic solution found in handbooks. Here P is force per unit thickness. The form of the solution is anticipated by dimensional considerations.

The remote K tends to open the crack, and the bridging traction tends to close the crack. The stress intensity factor at the tip of the crack, K_{tip}, is a linear superposition of the remote K and that caused by the bridging traction.

$$K_{\text{tip}} = K - \sqrt{\frac{2}{\pi}} \int_0^L \frac{\sigma_0 d\xi}{\sqrt{\xi}}$$

$$= K - \sqrt{\frac{2}{\pi}} 2\sigma_0 \sqrt{L}$$

The form of this result can be readily understood in terms of a dimensional consideration.

Assume that the stress is bounded everywhere, the stress intensity factor at the tip of the crack must vanish,

$$K_{\text{tip}} = 0.$$

Consequently, we obtain that

$$K = \sqrt{\frac{2}{\pi}} 2\sigma_0 \sqrt{L} .$$

Recall Irwin's relation $G = K^2 / E$. We can express this result as

$$G = \frac{8\sigma_0^2}{\pi E} L .$$

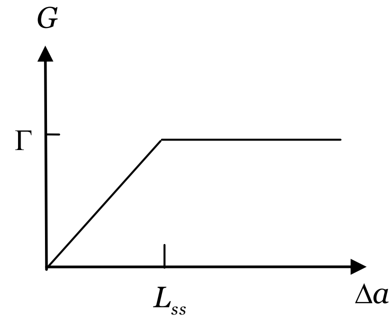
The steady state is reached when

$$G = \sigma_0 \delta_0 .$$

Thus, the steady-state bridging length is

$$L_{ss} = \frac{\pi E \delta_0}{8\sigma_0} .$$

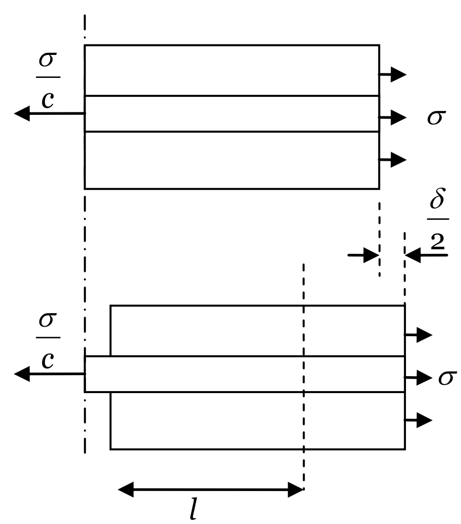
The calculated R-curve is sketched in the figure.



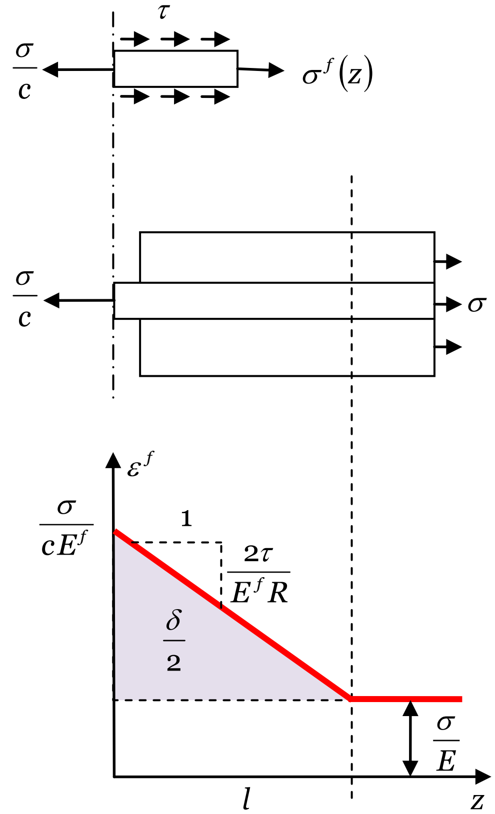
Ceramic-matrix Composites: How can a brittle solid toughen another brittle solid? Consider a crack in a plane normal to the fiber direction. The crack breaks the matrix, but leaves fibers intact. Three main requirements:

- *Week interface*. The fiber-matrix interface deflects the crack.
- *Strong fibers*. Fibers have small diameters (say 10 μm), and therefore have small natural flaws. The fibers bridge the crack. The strength of fibers is typically on the order of GPa.
- *Frictional sliding*. The fibers slide against the matrix.

Traction-separation curve derived from a micromechanical model. A unit cell consists of a single fiber and its surrounding matrix. Due to the symmetry, only half of the unit cell is illustrated. The stress applied on the composite σ. The matrix is cracked, but the fiber is unbroken. Illustrated are two configurations: When the fiber and the matrix are bonded, the composite deforms elastically. When the fiber and the matrix are debonded, the fiber slides relative to the matrix. The length of the sliding zone is denoted as l. Relative to the first configuration, the second configuration has the excess displacement δ. This excess displacement is due to sliding, and is taken to be the separation in the traction-separation curve. We would like to use this micromechanical model to obtain the relation between σ and δ.



Let E^{f} and E^{m} be Young's modulus of the fiber and the matrix. For the composite without sliding, the axial strain is the same in the fiber and the matrix, so that Young's modulus of the composite E is given by the rule of mixture:



$$E = cE^{\mathrm{f}} + (1-c)E^{\mathrm{m}},$$

where c is the volume fraction of the fiber.

In the second configuration, because the fiber slides relative to the matrix, the stress in the fiber varies along the z-axis, $\sigma^{\mathrm{f}}(z)$. At the crack plane, $z = 0$, the fiber carries all the load, so that

$$\sigma^{\mathrm{f}}(0) = \frac{\sigma}{c}.$$

The sliding between the fiber and the matrix is modeled by a constant frictional stress, τ. This model is known as the shear-lag model. Draw the free-body diagram of the fiber between the crack plane and z. Balancing the force of the free body, we obtain that

$$\sigma^{\mathrm{f}}(z) = \frac{\sigma}{c} - \frac{2\tau}{R}z.$$

where R is the fiber radius. The strain in the fiber also varies with the position:

$$\varepsilon^{\mathrm{f}}(z) = \frac{\sigma}{E^{\mathrm{f}}c} - \frac{2\tau}{E^{\mathrm{f}}R}z.$$

Beyond the sliding zone, $z > l$, the strain is uniform in the fiber and the matrix, so that

$$\varepsilon^{\mathrm{f}}(l) = \frac{\sigma}{E}.$$

A combination of the above two expressions gives the length of the sliding zone:

$$l = R\frac{\sigma}{\tau}\frac{(1-c)}{2c}\frac{E^{\mathrm{m}}}{E}.$$

The length of the sliding zone is inversely proportional to the frictional stress. Sketch the function $\varepsilon^{\mathrm{f}}(z)$ in the interval $0 < z < l$.

The displacement is the integration of the strain along the axis, and is given by the area of the triangle illustrated in the figure, so that

$$\delta = \frac{R}{2}\left(\frac{1-c}{c}\right)^2\left(\frac{E^{\mathrm{m}}}{E}\right)^2\frac{\sigma^2}{\tau E^{\mathrm{f}}}.$$

This expression gives the traction-displacement curve. The curve takes the form

$$\delta = \lambda\sigma^2.$$

The fracture energy due to fiber-matrix sliding. Let the fiber breaking strength be S. The scale of the traction scales with the strength of the fiber:

$$\sigma_0 \sim S.$$

Examining the traction-separation curve derived above, we find that the scale of the separation scales as

$$\delta_0 \sim \frac{RS^2}{\tau E}.$$

In the above estimates, I have dropped all the dimensionless numbers of order unity. The fracture energy scales as

$$\Gamma \sim \frac{S^3 R}{E\tau}.$$

Now we see the options in making a tough composite.

Reviews

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