Crack-Tip Field

We have modeled a body by using the linear theory of elasticity, the faces of a crack by two flat planes, and the front of the crack by a straight line. This model predicts that the field near the front of the crack is singular. The singular field is clearly an artifact of the idealized model, but Irwin and others made the singular field a centerpiece of fracture mechanics.

The mathematics of this singular field had been known long before Irwin entered the field. We will focus on the mathematics in this lecture, and will describe Irwin's way of using the singular field in the following lecture.

Modes of fracture. Depending on the symmetry of the field near the tip of a crack, fracture is classified into three modes.

- Mode I: tensile mode, or opening mode.
- Mode II: in-plane shear mode, or sliding mode.
- Mode III: anti-plane shear mode, or tearing mode.

The modes describe the local field near a point on the front. For example, consider a penny-shaped crack in an infinite body. The front of the crack is a circle. When a load pulls the body in the direction perpendicular to the plane of the crack, every point along the front of the crack is under the mode I condition. When a load shears the body in the direction parallel to the plane of the crack, every point along the front of the crack is under a mixed mode II and mode III condition. Only a few special points on the front are under either a pure mode II condition, or a pure mode III condition.

The concept of the modes appeals to our intuition, but this concept is absent in the energy-based approach. Energy release rate by itself does not differentiate the modes of fracture. Energy release rate characterizes the amplitude of the applied load, but not the mode of the applied load. To describe the mode of the load will require us to talk about the field near the tip of the crack.

Governing equations of the linear theory of elasticity. The state of a body is characterized by three fields: displacement $u_i(x_1,x_2,x_3)$, strain $\varepsilon_{ij}(x_1,x_2,x_3)$, and stress $\sigma_{ij}(x_1,x_2,x_3)$. In the body, the three fields satisfy the governing equations:

- Strain-displacement relations $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$
- Balance of forces $\sigma_{ij,j} = 0$
- Stress-strain relations $\sigma_{ij} = C_{ijpq}\varepsilon_{pq}$

At every point on the surface of the body, in each direction, we prescribe either the displacement, or the traction $\sigma_{ij}n_j$. If these equations look vague to you, please review the notes on the elements of linear elasticity (http://imechanica.org/node/205). Note two salient features of the linear theory of elasticity:

1. The theory is linear.
2. The theory has no intrinsic length scale.

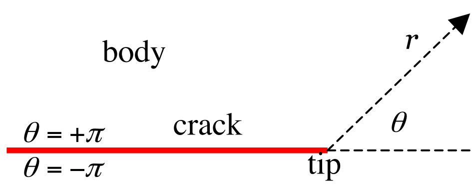
These two features greatly simplifies the analysis.

Energy release rate for a crack in a linear elastic material. Consider a crack of length $2a$ in a sheet subject to applied stress σ_{appl}. The body is made of a homogeneous, isotropic, linearly elastic material, with Young's modulus E and Poisson's ratio ν. Linearity of the boundary-value problem and dimensional consideration dictate that the energy release rate should take the form

$$G = \beta \frac{\left(\sigma_{\mathrm{appl}}\right)^2 a}{E}.$$

The dimensionless number β must be determined by solving the boundary-value problem.

Linear elastic field around the tip of a crack. Focus on the field near one tip of the crack. We model the tip of the crack as a point, and the two faces of the crack as straight lines. Let (r,θ) be the polar coordinates centered at the tip of the crack. The two faces of the crack coincide with the lines $\theta = +\pi$ and $\theta = -\pi$. A material particle in the body is at a distance r from the tip of the crack. The material particle is said to be near the tip of the crack if r is much smaller than the length characteristic of the boundary-value problem; for example, $r << a$, where a is the length of the crack.



We want to determine the fields near the tip of the crack, including the stress $\sigma_{ij}(r,\theta)$, strain $\varepsilon_{ij}(r,\theta)$ and displacement $u_i(r,\theta)$. To this end, we regard the body as infinite, and the crack as semi-infinite. The faces of the crack are traction-free, and the load is applied remotely from the tip of the crack. We represent the applied load by a single parameter: the energy release rate G. In doing so we change our perspective. We have regarded G as part of the solution of a boundary-value problem. We now regard G as the applied load in a boundary-value problem.

This mathematical model differs from a real crack in a real material in many aspects, but for the time being we stick to the model itself.

The crack-tip field is square-root singular. In teaching fracture mechanics in the spring of 2010, I found a way to reach the square-root singularity without solving the boundary-value problem.

The energy release rate G is the loading parameter of the boundary-value problem. As G increases, the field increases. In the linear theory of elasticity, the stress, strain, and displacement are linearly proportional to the applied stress, but G is quadratic in the applied stress. Thus, the stress field scales with G as

$$\sigma_{ij}(r,\theta) \propto \sqrt{G} .$$

The infinite body and the semi-infinite crack provide no length scales—that is, the geometry of the boundary-value problem is scale-free. The three quantities G, E and r form a single dimensionless group:

$$\frac{G}{Er} .$$

A combination of linearity and dimensional consideration dictates that the field of stress near the tip of the crack should take the form

$$\sigma_{ij} = \sqrt{\frac{EG}{r}} f_{ij}(\theta) ,$$

where $f_{ij}(\theta)$ are dimensionless functions of θ, and possibly also depend on Poisson's ratio ν. Thus, from these elementary considerations, we find out that the stress field near the tip of the crack is square-root singular in r. The energy release rate represents the applied load and sets the amplitude of the crack-tip field.

Similar considerations show that the fields of strain and displacement near the tip of the crack take the following forms:

$$\varepsilon_{ij}(r,\theta) = \sqrt{\frac{G}{Er}} g_{ij}(\theta) ,$$

$$u_i(r,\theta) = \sqrt{\frac{Gr}{E}} h_i(\theta) .$$

These angular functions can be determined by solving the boundary-value problem, as described below.

The singular field around the front of a crack. While a body is in a three-dimensional space, the singular field around the front of a crack is nearly two-dimensional. Let us clarify this reduction in dimension. In the three-dimensional space, we model a body by a volume, a crack by a smooth surface, and the front of the crack by a smooth curve. For any point on the front, we use the point as the origin to set up local coordinates, with the axis x pointing in the direction of propagation of the front, y normal to the plane of the crack, and z tangent to the front.

Because the front is assumed to be a smooth curve, the field around the front is singular in x and y, but smooth in z. That is, for any component of the field, $f(x,y,z)$, the derivative $\partial f / \partial z$ is small compared to the derivatives $\partial f / \partial x$ and $\partial f / \partial y$. Consequently, we may drop all partial derivatives with respect to z in the governing equations. Consequently, the singular field around the front is locally characterized by a field of the form

$$u(x,y), v(x,y), w(x,y).$$

We further assume that the elastic behavior of the material is isotropic. The field decouples into two types:

- Plane-strain deformation: $u(x,y) \neq 0, v(x,y) \neq 0$, but $w(x,y) = 0$.
- Anti-plane deformation: $u(x,y) = 0, v(x,y) = 0$, but $w(x,y) \neq 0$.

The plane-strain deformation describes mode I and mode II cracks, and the anti-plane deformation describes mode III cracks. In class we will mostly talk about mode I cracks.

Plane-strain conditions. To analyze this locally plane-strain field, we focus on an actual plane-strain problem:

- The body is under the plane-strain conditions.
- The crack is a flat plane.
- The front of the crack is a straight line.
- The field is elastic all the way to the front.

The fields reduced to

- Displacement field: $u(x,y)$ and $v(x,y)$. Note that $w = 0$.
- Strain field: $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$ are functions of x and y. Other strain components vanish.
- Stress field: $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{zz}$ are functions of x and y. Other stress components vanish.

Strain-displacement relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Balance of forces:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

Hooke's law:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right),$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right),$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\right),$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy},$$

Under the plane strain conditions, $\varepsilon_{zz} = 0$, so that

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right).$$

Substituting this relation to Hooke's law, we obtain that

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right),$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right),$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy}.$$

The quantity

$$\bar{E} = \frac{E}{1-\nu^2},$$

is known as the plane-strain modulus.

Under the plane strain conditions, the elastic field is represented by 2 displacements, 3 strains, and 4 stresses. The 9 functions are governed by 9 field equations (3 strain-displacement relations, 3 stress-strain relations, 2 equations to balance forces, and 1 relation between σ_z, σ_x, σ_y. If you have not studied plane-strain problems before, please review the notes http://imechanica.org/node/319.

Airy's function. We now have 9 equations for 9 functions. As usual we can eliminate some functions by combining equations. Many approaches of elimination have been devised. Here we will follow the approach due to Airy (1863), who reduced the system of equations to one equation for one function.

Recall a theorem in calculus. If functions $f(x,y)$ and $g(x,y)$ satisfy the following relation

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

then a function $\alpha(x,y)$ exists, such that

$$f = \frac{\partial \alpha}{\partial y}, \quad g = \frac{\partial \alpha}{\partial x}.$$

According to this theorem, one equation of force balance,

$$\frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial \sigma_{xy}}{\partial y},$$

implies that a function $\alpha(x,y)$ exists, such that

$$\sigma_{xx} = \frac{\partial \alpha}{\partial y}, \sigma_{xy} = -\frac{\partial \alpha}{\partial x}.$$

The other equation of force balance,

$$\frac{\partial \sigma_{xy}}{\partial x} = -\frac{\partial \sigma_{yy}}{\partial y},$$

implies that a function $\beta(x,y)$ exists, such that

$$\sigma_{yy} = \frac{\partial \beta}{\partial x}, \sigma_{xy} = -\frac{\partial \beta}{\partial y}.$$

In the above, we have expressed the shear stress σ_{xy} in two ways. Equating the two expressions, we obtain that

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}.$$

According to the theorem in calculus, this equation implies that a function $\phi(x,y)$ exists, such that

$$\alpha = \frac{\partial \phi}{\partial y}, \beta = \frac{\partial \phi}{\partial x}.$$

Summing up, we can express the three stresses in terms of one function:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \ \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \ \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

The function $\phi(x,y)$ is known as Airy's function. Stresses expressed by Airy's function satisfy the equations of force balance.

Strains in terms of Airy's function. Using Hooke's law, we express the strains in terms of Airy's function:

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu}{1-\nu}\frac{\partial^2 \phi}{\partial x^2}\right),$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\nu}{1-\nu}\frac{\partial^2 \phi}{\partial y^2}\right),$$

$$\varepsilon_{xy} = -\frac{(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}.$$

The equation of compatibility. Eliminate the displacements from the strain-displacement relations, and we obtain that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial \varepsilon_{xy}}{\partial x \partial y}.$$

This equation is known as the equation of compatibility.

The compatibility equation can be expressed in terms of Airy's function:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

This equation is known as the *bi-harmonic equation*. According to Meleshko, this equation was first derived by Maxwell when asked by Stokes to review the paper by Airy. The plane-strain problem is governed by the bi-harmonic equation and the boundary conditions. Once ϕ is solved, one can determine the stresses, strains, and displacements.

The biharmonic equation is often written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 0 .$$

Polar coordinates. When Airy's function is expressed as a function of the polar coordinates, $\phi(r,\theta)$, the polar components of the stress are expressed as

$$\sigma_{rr}=\frac{\partial^2\phi}{r^2\partial\theta^2}+\frac{\partial\phi}{r\partial r},\quad \sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2},\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right).$$

The bi-harmonic equation is

$$\left(\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}\right)\left(\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}\right)\phi = 0 .$$

The Williams expansion. Within the linear elastic theory, the field in a body is determined by a boundary-value problem. The field depends on the boundary conditions, namely, the size and the shape of the crack and the body, as well as the magnitude and the distribution of the load. Some such boundary-value problems had been solved before Williams entered the field. Williams took a different approach. Instead of solving individual boundary-value problems, he focused on the singular field around the tip of the crack, in a zone so small that the crack can be assumed to be semi-infinite, and the boundary of the body infinitely far away. He discovered that the form of the singular field is universal, independent of the shape of the body and the crack.

Let (r,θ) be the polar coordinates, centered at a particular point on the front of the crack. The crack propagates in the direction $\theta = 0$, and the two faces of the crack coincide with $\theta = \pm\pi$. The two faces of the crack are traction-free.

We solve the biharmonic equation using the method of separation of variables. Each term in the bi-harmonic equation has the same dimension in *r*. For such an equi-dimensional equation, the solution is *r* to some power. Write the solution in the form

$$\phi(r,\theta)=r^{\lambda+1}f(\theta),$$

where the constant λ and the function $f(\theta)$ are to be determined. Insert this form into the biharmonic equation, and we obtain an ordinary differential equation (ODE):

$$\left[\frac{d^2}{d\theta^2}+(\lambda-1)^2\right]\left[\frac{d^2}{d\theta^2}+(\lambda+1)^2\right]f(\theta)=0.$$

This is an ODE with constant coefficients. The solution is of the form

$$f(\theta)=\exp(b\theta),$$

where *b* is to be determined. Inserting this form into the ODE, we find that

$$\left(b^2+(\lambda-1)^2\right)\left(b^2+(\lambda+1)^2\right)=0.$$

This is a fourth-order algebraic equation for b. The four roots are

$$b=\pm(\lambda-1)i, \quad \pm(\lambda+1)i,$$

where $i=\sqrt{-1}$. The general solution to the ODE is

$$f(\theta)=A\cos(\lambda+1)\theta+B\cos(\lambda-1)\theta+C\sin(\lambda+1)\theta+D\sin(\lambda-1)\theta,$$

where A, B, C and D are constants. For the mode I crack, the field is symmetric with respect to the x-axis, so that $C=D=0$.

Summarizing, we find that the solution for a mode I crack is

$$\phi(r,\theta)=r^{\lambda+1}\left[A\cos(\lambda+1)\theta+B\cos(\lambda-1)\theta\right],$$

where λ, A and B are constants to be determined by using the boundary conditions.

The stresses are

$$\sigma_{rr}=-\lambda r^{\lambda-1}\left[A(\lambda+1)\cos(\lambda+1)\theta+B(\lambda-3)\cos(\lambda-1)\theta\right]$$

$$\sigma_{\theta\theta}=-(\lambda+1)\lambda r^{\lambda-1}\left[A\cos(\lambda+1)\theta+B\cos(\lambda-1)\theta\right]$$

$$\sigma_{r\theta}=(\lambda-1)r^{\lambda-1}\left[A(\lambda+1)\sin(\lambda+1)\theta+B(\lambda-1)\sin(\lambda-1)\theta\right].$$

At $\theta=\pi$, both components of the traction vanish, $\sigma_{\theta\theta}=\sigma_{r\theta}=0$, namely,

$$\begin{bmatrix}(\lambda+1)\lambda\cos\lambda\pi & (\lambda+1)\lambda\cos\lambda\pi \\ (\lambda^2-1)\sin\lambda\pi & (\lambda-1)^2\sin\lambda\pi\end{bmatrix}\begin{bmatrix}A \\ B\end{bmatrix}=\begin{bmatrix}0 \\ 0\end{bmatrix}$$

This pair of linear algebraic equations for A and B form an eigenvalue problem. To have a solution such that A and B are not both zero, the determinant must vanish, namely,

$$\lambda(\lambda^2-1)\sin\lambda\pi\cos\pi\lambda=0.$$

The solutions are

$$\lambda=\ldots-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},\ldots$$

Consequently, the field of stress takes the form of an expansion:

$$\sigma_{ij}=\sum_{m=-\infty}^{+\infty}a_m r^{m/2}f_{ij}^{(m)}(\theta).$$

The functions $f_{ij}^{(m)}(\theta)$ are determined by the eigenvalue problem. The amplitudes a_m, however, are not determined by the eigenvalue problem, and should be determined by the full boundary-value problem.

The square-root singularity. Each term in the Williams expansion corresponds to a solution to the biharmonic equation. Which term should we choose? Note that

$$\sigma\sim r^{\lambda-1},\ \varepsilon\sim r^{\lambda-1},\ u\sim r^{\lambda}.$$

Recall the stress concentration for the ellipse. Require that the displacement to be bounded, so that $\lambda>0$. Require the stress to be singular, so that $\lambda<1$. The two requirements force us to choose

$$\lambda = +\frac{1}{2}.$$

We may justify the choice of the square-root singularity on the basis of the energy release rate. As we noted in the beginning of the lecture, the square-root singular field yields the finite energy release rate. All the more singular terms yield infinite energy release rate, and all the non-singular terms yield zero energy release rate.

The above justifications for the choice of the square-root singularity are flimsy. The significance of this choice has to be understood later, when we see how this singularity is used in practice. For further discussion, see C.Y. Hui and Andy Ruina, Why K? High order singularities and small scale yielding, International Journal of Fracture 72, 97-120 (1995).

Substituting $\lambda = 1/2$ into the algebraic equations, we find that $B = 3A$. The eigenvalue problem does not specify any applied load: both the field equations and the boundary conditions are homogeneous. Like any eigenvalue problem, this eigenvalue problem leaves the amplitude undetermined. In this case, all the field is determined up to the constant *B*. Following a convention, we write amplitude B as $B = -K/\sqrt{2\pi}$.

Substituting $\lambda = 1/2$, $B = 3A$ and $B = -K/\sqrt{2\pi}$ into the expressions for the stresses, we obtain the square-root singular field around the tip of the crack:

$$\sigma_{rr} = \frac{K}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left(1+\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\sigma_{\theta\theta} = \frac{K}{\sqrt{2\pi r}}\cos^3\left(\frac{\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K}{\sqrt{2\pi r}}\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

In particular, the stress at a distance *r* directly ahead the crack tip is

$$\sigma_{\theta\theta}(r,0) = \frac{K}{\sqrt{2\pi r}}.$$

One can also determine the displacement components. In particular, the crack opening displacement a distance *r* behind the crack tip is

$$\delta = \frac{8K}{\overline{E}}\sqrt{\frac{r}{2\pi}},$$

where $\overline{E} = E/(1-\nu^2)$ under the plane strain conditions, and $\overline{E} = E$ under the plane stress conditions.

Notes

- *K* is the amplitude of the near tip field.
- The magnitude of *K* is undetermined in the eigenvalue problem.

- The numerical factor $(2\pi)^{-1/2}$ is set by convention. A different numerical factor will rescale the amplitude K. The particular numerical factor $(2\pi)^{-1/2}$ is selected to make the relation between K and G look simple. See below.
- K is called the stress intensity factor.
- The r and θ dependence are fully determined by the eigenvalue problem, and are independent of the external boundary conditions.

Energy release rate determines the amplitude of the crack-tip field. The above eigenvalue problem invokes the traction-free boundary conditions of the two faces of the crack, but not the load applied remotely from the tip of the crack. We have talked about representing the remote applied load by using the energy release rate G, but have so far not used G in solving the boundary-value problem. We expect that G determines the amplitude of the crack-tip field K. A dimensional consideration gives that

$$G \propto \frac{K^2}{\bar{E}}.$$

Of course, the dimensional analysis will not fix the numerical coefficient in the above expression. To calculate the numerical coefficient, we the follow Irwin's approach.

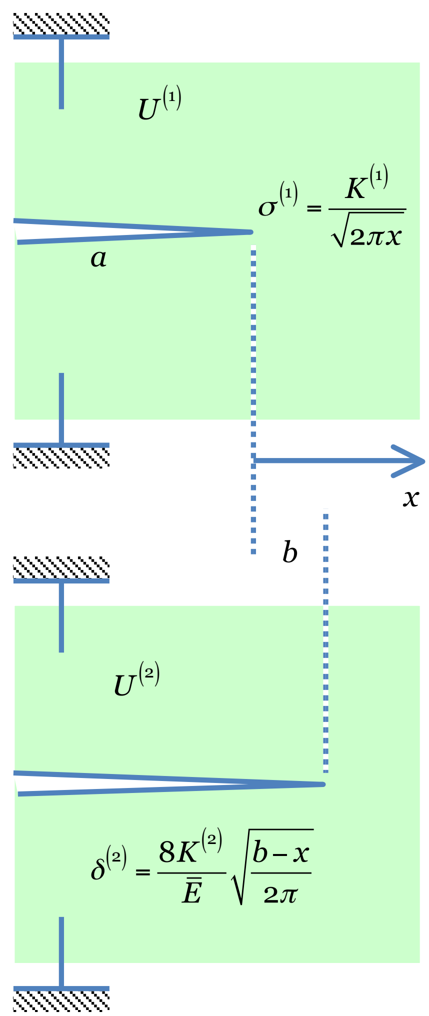
Consider two bodies, 1 and 2, of the same configuration: a sheet of unit thickness containing a crack. The crack in Body 2 is longer than that in Body 1 by a length b. Let $K^{(1)}$ be the stress intensity factor of the crack in Body 1. The stress at distance x ahead the tip of the crack in Body 1 is

$$\sigma^{(1)} = \frac{K^{(1)}}{\sqrt{2\pi x}}.$$

Let $K^{(2)}$ be the stress intensity factor of the crack in Body 2. The opening displacement at distance $b-x$ behind the tip of the crack in Body 2 is

$$\delta^{(2)} = \frac{8K^{(2)}}{\bar{E}}\sqrt{\frac{b-x}{2\pi}}.$$

Let $U^{(1)}$ and $U^{(2)}$ be the strain energy stored in the two bodies, respectively. The displacement of the applied load is fixed. The difference in the energy in the two bodied is due to the work done by the closing traction:



$$U^{(1)} - U^{(2)} = \frac{1}{2}\int_0^b \sigma^{(1)}\delta^{(2)} dx .$$

Inserting the expressions for the stress and crack opening displacement, we obtain that

$$U^{(1)} - U^{(2)} = \frac{1}{2}\int_0^b \frac{K^{(1)}}{\sqrt{2\pi x}} \frac{8K^{(2)}}{\bar{E}} \sqrt{\frac{b-x}{2\pi}} dx = \frac{K^{(1)}K^{(2)}b}{\bar{E}} .$$

The integral is evaluated by a change of variable, $x = b\sin^2\alpha$.

By definition, the energy release rate is

$$G = \frac{U^{(1)} - U^{(2)}}{b} .$$

In this definition, the change in the length of the crack b is small compared to the size of body or the total length of the crack a. As $b/a \to 0$, the stress intensity factors of the cracks in the two bodies approach each other, $K^{(1)} = K^{(2)} = K$. Thus, we reach Irwin's *G*-*K* relation:

$$G = \frac{K^2}{\bar{E}} .$$

Given that K and G are related by a simple formula, why should we elevate K to such a high stature in fracture mechanics? Perhaps this question is moot. By now K is all over the literature, and you will just have to deal with it. But you can always replace K by $\sqrt{\bar{E}G}$ without missing anything.

More broadly, however, we often allow redundancy in developing any theory. Different expressions of one idea may suggest different extensions. In linearly elastic solids, K and G both represent the amplitude of the crack-tip field. However, K is readily extended to situations where G does not apply, such as singular fields around corners. On the other hand, G still applies in situations where the crack-tip field is difficult to determine, such as in nonlinear elastic solids like rubber. We will make various extensions later in the course.

Mode II. The stress a distance r ahead the crack tip is

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}$$

Behind the crack tip, the sliding displacement between the two crack faces is

$$\delta_{II} = \frac{8K_{II}}{\bar{E}}\sqrt{\frac{r}{2\pi}} .$$

The energy release rate relates to the stress intensity factor as

$$G = \frac{K_{II}^2}{\bar{E}} .$$

Mode III. The stress a distance r ahead the crack tip is

$$\sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}}$$

Behind the crack tip, the tearing displacement between the two crack faces is

$$\delta_{III} = \frac{4K_{III}}{\mu}\sqrt{\frac{r}{2\pi}},\ \mu = \frac{E}{2(1+\nu)}.$$

The energy release rate relates to the stress intensity factor as

$$G = \frac{K_{III}^2}{2\mu}.$$

When all three modes are present, the energy release rate is the sum:

$$G = \frac{K_I^2}{\bar{E}} + \frac{K_{II}^2}{\bar{E}} + \frac{K_{III}^2}{2\mu}.$$

Historical Notes

For a historical account involving Airy, Stokes and Maxwell, see V.V. Meleshko, Selected topics in the history of the two-dimensional biharmonic problem, Applied Mechanics Review 56, 33-85 (2003).

The square-root singularity near the tip of a crack was known before Irwin entered the field. Here are two papers that clearly display the square-root singularity.

- H.M. Westergaard, Bearing pressures and cracks. Journal of Applied Mechanics 6, A49-A53 (1939). Westergaard was a professor at Harvard.
- I.N. Sneddon, The distribution of stress in the neighborhood of a crack in an elastic solid. Proceedings of the Royal Society of London A 187, 229-260 (1946).

M.L. Williams, On the stress distribution at the base of a stationary crack. Journal of Applied Mechanics 24, 109-115 (1957).

G.R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate, Journal of Applied Mechanics 24, 361-364 (1957).

The legend has it that Irwin chose letter *K* after J.A. Kies, one of his co-workers.