The Darboux Classification of Curl Forces*

Arash Yavari[1] and Alain Goriely[†2]

[1] *School of Civil and Environmental Engineering and The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA,*
[2] *Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK*

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Abstract

We study particle dynamics under curl forces. These forces are a class of non-conservative, non-dissipative, position-dependent forces that cannot be expressed as gradient of a potential function. We show that the fundamental quantity of particle dynamics under curl forces is a *work* 1*-form*. By using the Darboux classification of differential 1-forms on $\mathbb{R}^2$ and $\mathbb{R}^3$, we establish that any curl force in two dimensions has at most two *generalized potentials*, while in three dimensions, it has at most three. These potentials generalize the single potential of conservative systems. For any curl force field, we introduce a corresponding conservative force field—*the conservative auxiliary force*. The Hamiltonian of this conservative force is a conserved quantity of motion for the dynamics of a particle under the curl force, although it is not the physical energy.

Keywords: Nonconservative force, curl force, Darboux classification, generalized potential energies.

1 Introduction

In classical mechanics, a conservative force field is one that can be derived from a potential energy. Typical examples of non-conservative forces, such as friction and viscous forces, are dissipative. However, a non-conservative force—one that does not have an associated potential—is not necessarily dissipative. An important class of such non-conservative, non-dissipative force fields are forces of the form $\mathbf{F} = \mathbf{F}(\mathbf{x})$, depend on position $\mathbf{x} \in \mathbb{R}^n$ $(n = 2, 3)$ but not on velocity, and for which curl $\mathbf{F}(\mathbf{x}) \neq \mathbf{0}$. Following Berry and Shukla [2012], we call these force fields *curl forces*. Non-conservative forces that depend on position are also referred to as *positional forces*, *pseudo-gyroscopic forces*, and *circulatory forces* [Ziegler, 1977, Kirillov, 2021]. These forces are non-conservative yet non-dissipative, meaning that they preserve the phase space volume. Another characterization of non-conservative, non-dissipative forces is provided in §2.3. Particle dynamics under such force fields differs fundamentally from that under conservative forces. For instance, Noether's theorem, which connects conservation laws with symmetries, cannot be used in this context [Berry and Shukla, 2012, 2013]. Curl forces in both discrete and continuous mechanical systems such as Cauchy elasticity have a rich history. For a recent detailed literature review, see [Yavari and Goriely, 2024].

For a conservative force field $\mathbf{F} = \mathbf{F}(\mathbf{x})$, there exists a potential energy $U = U(\mathbf{x})$ such that

$$\mathbf{F}(\mathbf{x}) = -\nabla U = -\frac{\partial U}{\partial \mathbf{x}}\,. \tag{1.1}$$

For a curl force, no such potential energy exists. However, we will show in the following that any curl force has at most two and three *generalized potential energies* in $2D$ and $3D$, respectively.

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†Corresponding author, e-mail: alain.goriely@maths.ox.ac.uk

Berry and Shukla [2015] showed that a subset of curl forces admits Hamiltonians of the form of an anisotropic kinetic energy + a scalar potential. In this paper, we demonstrate that any curl force can be written in terms of three generalized potentials and admits an auxiliary Hamiltonian which, although not representing the physical energy, is nevertheless a conserved quantity of motion (albeit an implicit one that does not divide the phase-space into invariant regions).

This paper is organized as follows. In §2, we present the Darboux classification of curl forces in two and three dimensions, establishing the fundamental structure of the work 1-form. In §2.1 and §2.2, we calculate the generalized potentials associated with two- and three-dimensional curl forces, respectively. In §2.3, we study the work performed by curl forces in closed and cyclic motions. In §2.4, we investigate the change in the kinetic energy of a particle under curl forces and interpret it in the context of Carathéodory's formulation of thermodynamics. The local accessibility property of curl forces is discussed in §2.5. Finally, §2.6 introduces a generalized Hamiltonian formulation for curl forces, where we construct an auxiliary conservative force field and derive the corresponding Hamiltonian. We first analyze the two-dimensional case in §2.6.1, including the calculation of generalized potentials, before extending the formulation to three-dimensional curl forces in §2.6.2. Some concluding remarks are given in §3.

2 Canonical forms of curl forces in two and three dimensions

Let $\boldsymbol{\alpha}$ be a differential form on an n-manifold $\mathcal{M}$. We define

$$(\boldsymbol{\alpha})^k = \overbrace{\boldsymbol{\alpha} \wedge \ldots \wedge \boldsymbol{\alpha}}^{k-\text{factors}} . \tag{2.1}$$

Suppose $\boldsymbol{\Omega}$ is a 1-form on $\mathcal{M}$. k is called the rank of $\boldsymbol{\Omega}$ if $(d\boldsymbol{\Omega})^k \neq 0$ and $(d\boldsymbol{\Omega})^{k+1} = 0$. Note that $(d\boldsymbol{\Omega})^k$ is a $2k$-form and the condition $(d\boldsymbol{\Omega})^k \neq 0$ implies that $n > 2k$ or $n - k > k$.

Theorem 2.1 (Darboux [Darboux, 1882, Slebodzinski, 1970, Sternberg, 1999, Bryant et al., 2013, Suhubi, 2013])**.** *Let $\boldsymbol{\Omega}$ be a 1-form of rank k on an n-manifold $\mathcal{M}$. Suppose $\boldsymbol{\Omega} \wedge (d\boldsymbol{\Omega})^k = 0$ everywhere on $\mathcal{M}$. Then in a neighbourhood of any point, there exist coordinates $\{y^1, \ldots, y^{n-k}, z^1, \ldots, z^k\}$ such that*

$$\boldsymbol{\Omega} = y^1 dz^1 + \ldots + y^k dz^k . \tag{2.2}$$

If $\boldsymbol{\Omega} \wedge (d\boldsymbol{\Omega})^k \neq 0$ everywhere on $\mathcal{M}$, then in a neighbourhood of every point there exist coordinates $\{y^1, \ldots, y^{n-k}, z^1, \ldots, z^k\}$ such that

$$\boldsymbol{\Omega} = y^1 dz^1 + \ldots + y^k dz^k + dy^{k+1} . \tag{2.3}$$

Eqs. (2.2) and (2.3) are referred to as the canonical (or normal) forms of $\boldsymbol{\Omega}$.

Definition 2.2 (Work 1-form)**.** Given a force field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ in $\mathbb{R}^n$ ($n = 2$ or 3), the work 1-form is defined as $\boldsymbol{\Omega} = \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}$. Explicitly, in a (curvilinear) coordinate chart $\{x^a\}$ with metric components $g_{ab} = g_{ba}$, the work 1-form has the representation $\boldsymbol{\Omega} = F^b g_{ba}\, dx^a = F_a\, dx^a$. The work 1-form can also be written in a coordinate-free representation as $\boldsymbol{\Omega} = \mathbf{F}^\flat$, where the flat operator $\flat$ is an isomorphism between vectors and 1-forms.

We note that the analog of the work 1-form in Cauchy elasticity is the stress work 1-form [Yavari and Goriely, 2024].

A force field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ in $\mathbb{R}^n$ is a curl force field if and only if $d\boldsymbol{\Omega} \neq \mathbf{0}$, where d is the exterior derivative and $d\boldsymbol{\Omega}$ is a 2-form. With respect to a coordinate chart $\{x^a\}$, $d\boldsymbol{\Omega} = F_{a,b}\, dx^b \wedge dx^a$, where $\wedge$ is the wedge product of differential forms. In Euclidean space, the curl of a vector field $\mathbf{F} = F^a \frac{\partial}{\partial x^a}$ has the following relation with exterior derivative [Abraham et al., 2012]

$$\operatorname{curl} \mathbf{F} = \left[\star(d\mathbf{F}^\flat)\right]^\sharp , \tag{2.4}$$

where $\sharp$ is the sharp operator, which is an isomorphism between 1-forms and vectors (the inverse of the $\flat$ operator), and $\star$ is the Hodge star operator, which is an isomorphism between k-forms and $(n-k)$-forms $(n = 2, 3)$ and for any 1-form $\boldsymbol{\alpha}$, $\star\star\boldsymbol{\alpha} = (-1)^{n-1}\boldsymbol{\alpha}$. Eq. (2.4) can be rewritten as

$$d\mathbf{F}^\flat = d\boldsymbol{\Omega} = (-1)^{n-1} \star \left[(\operatorname{curl}\mathbf{F})^\flat\right] . \tag{2.5}$$

Therefore, $d\boldsymbol{\Omega} \neq \mathbf{0}$ if and only if $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$.

Curl force fields are classified for $n = 2$ and $n = 3$ using Darboux's theorem as follows:

Proposition 2.3. *In $\mathbb{R}^2$, any force field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ has the following representation:*

$$\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\,\nabla U(\mathbf{x})\,, \tag{2.6}$$

where $U(\mathbf{x})$ and $V(\mathbf{x})$ are the generalized potentials, with $\nabla V(\mathbf{x}) = \mathbf{0}$ if an only if $\operatorname{curl}\mathbf{F} = \mathbf{0}$. More generally, $\operatorname{curl}\mathbf{F} = \mathbf{0}$ if an only if $V = f(U)$ for some function f.

Proof. If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ can be written as $\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x})$ which implies that $V = 1$ and $\nabla V = \mathbf{0}$. If $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$, then in 2D, $d\boldsymbol{\Omega} \neq \mathbf{0}$ but $d\boldsymbol{\Omega} \wedge d\boldsymbol{\Omega}$ is a 4-form, which identically vanishes on $\mathbb{R}^2$. Thus, $\boldsymbol{\Omega}$ has rank $k = 1$. Also, note that $\boldsymbol{\Omega} \wedge d\boldsymbol{\Omega}$ is a 3-form, which vanishes on $\mathbb{R}^2$. Therefore, $\boldsymbol{\Omega}$ has the canonical form $\boldsymbol{\Omega}(\mathbf{x}) = \phi(\mathbf{x})\,d\psi(\mathbf{x})$. Thus, $\mathbf{F}^\flat = \phi(\mathbf{x})\,d\psi(\mathbf{x})$, and hence $\mathbf{F} = \phi(\mathbf{x})\,[d\psi(\mathbf{x})]^\sharp$. Recalling that $[d\psi(\mathbf{x})]^\sharp = \nabla\psi(\mathbf{x})$, one concludes that any curl force field in $\mathbb{R}^2$ has the representation (2.6), where $V(\mathbf{x}) = -\phi(\mathbf{x})$ and $U(\mathbf{x}) = \psi(\mathbf{x})$. Note that[1]

$$\nabla V \times \nabla U = \left[\star\left((\nabla V)^\flat \wedge (\nabla U)^\flat\right)\right]^\sharp = \left[\star\left(-d\phi \wedge d\psi\right)\right]^\sharp = -\left[\star\left(d\boldsymbol{\Omega}\right)\right]^\sharp . \tag{2.7}$$

This implies that $d\boldsymbol{\Omega} = d\phi \wedge d\psi \neq \mathbf{0}$ if and only if $\nabla V \times \nabla U \neq \mathbf{0}$. It is clear that if $\nabla V(\mathbf{x}) = \mathbf{0}$, then $d\boldsymbol{\Omega} = \mathbf{0}$, and hence $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Note that $d\phi \wedge d\psi = \mathbf{0}$ if and only if $\phi = \phi(\psi)$, i.e., ϕ and ψ are functionally dependent. This, in turn, implies that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ if and only if $V = f(U)$ for some function f. In other words, (2.6) represents a curl force if and only if the two potentials are functionally independent. □

Proposition 2.4. *In $\mathbb{R}^3$, any force field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ has the following representation:*

$$\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\,\nabla U(\mathbf{x}) - \nabla W(\mathbf{x})\,, \tag{2.8}$$

where $U(\mathbf{x})$, $V(\mathbf{x})$ and $W(\mathbf{x})$, provided that $\nabla V \times \nabla U \neq \mathbf{0}$, are called the generalized potentials. In particular, $\nabla V(\mathbf{x}) = \mathbf{0}$ if and only if $\operatorname{curl}\mathbf{F} = \mathbf{0}$ (in which case we can take $W = 0$ without loss of generality), and $\nabla W(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{F} \cdot \operatorname{curl}\mathbf{F} = 0$.

Proof. If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ can be written as $\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x})$ which implies that $V = 1$ and $W = 0$. For a 3D curl force, $d\boldsymbol{\Omega} \neq \mathbf{0}$ but $d\boldsymbol{\Omega} \wedge d\boldsymbol{\Omega}$ is a 4-form, which vanishes on $\mathbb{R}^3$. Thus, again $\boldsymbol{\Omega}$ has rank $k = 1$. We have the following two possibilities

$$\begin{cases} \boldsymbol{\Omega} \wedge d\boldsymbol{\Omega} = 0 \;\Rightarrow\; \boldsymbol{\Omega} = \phi\,d\psi\,, \\ \boldsymbol{\Omega} \wedge d\boldsymbol{\Omega} \neq 0 \;\Rightarrow\; \boldsymbol{\Omega} = \phi\,d\psi + d\zeta\,. \end{cases} \tag{2.9}$$

Therefore, the force field has the representation (2.8), where $V = -\phi$, $U = \psi$, and $W = -\zeta$. Note that[2]

$$(\mathbf{F} \cdot \operatorname{curl}\mathbf{F})\,\boldsymbol{\mu} = \mathbf{F}^\flat \wedge \star\,(\operatorname{curl}\mathbf{F})^\flat = \boldsymbol{\Omega} \wedge \star\left[\star(d\mathbf{F}^\flat)\right] = (-1)^{n-1}\boldsymbol{\Omega} \wedge d\boldsymbol{\Omega}\,. \tag{2.10}$$

This implies that $\boldsymbol{\Omega} \wedge d\boldsymbol{\Omega} = 0$ is equivalent to $\mathbf{F} \cdot \operatorname{curl}\mathbf{F} = 0$. □

Remark 2.5. We note that the quantity $\chi(\mathbf{F}) = \mathbf{F} \cdot \operatorname{curl}\mathbf{F}$ that determines whether or not a force derives from one $(V = 1, W = 0)$, two $(W = 0)$, or three potentials, is referred to as the helicity or chirality of the field $\mathbf{F}$ and plays a central role in fluid mechanics [Moffatt, 1969, Moffatt and Tsinober, 1992, Arnold and Khesin, 1998] and electromagnetism [Trueba and nada, 1996]. If the force field is chiral $\chi(\mathbf{F}) \neq 0$, then three potentials are needed to represent it. These three functions can be thought of as an alternative representation of the three components of $\mathbf{F}$ that capture the work performed during motion.

[1] It is known that for arbitrary vectors $\mathbf{u}$ and $\mathbf{w}$ one has $\mathbf{u} \times \mathbf{w} = [\star(\mathbf{u}^\flat \wedge \mathbf{w}^\flat)]^\sharp$ [Abraham et al., 2012].

[2] One can show that for arbitrary vectors $\mathbf{u}$ and $\mathbf{w}$ one has $(\mathbf{u} \cdot \mathbf{w})\boldsymbol{\mu} = \mathbf{u}^\flat \wedge \star(\mathbf{w}^\flat)$, where $\boldsymbol{\mu}$ is the volume element of the Euclidean space ($\boldsymbol{\mu} = dx \wedge dy$ and $\boldsymbol{\mu} = dx \wedge dy \wedge dz$ in 2D and $3D$, respectively) [Abraham et al., 2012].

2.1 Calculation of the generalized potentials for 2D curl forces

From $\mathbf{F} = -V\,\nabla U$, we have $\nabla U = -\frac{\mathbf{F}}{V}$. For this PDE to have a solution the following integrability (compatibility) conditions must be satisfied:[3]

$$\nabla \times \left(\frac{\mathbf{F}}{V}\right) = \frac{1}{V}(\nabla \times \mathbf{F}) - \frac{\nabla V}{V^2} \times \mathbf{F} = \mathbf{0}\,. \tag{2.11}$$

Therefore, the generalized potential V must satisfy the following first-order PDE:

$$\nabla V \times \mathbf{F} - V\,\nabla \times \mathbf{F} = \mathbf{0}\,. \tag{2.12}$$

Solving the above PDE gives V, after which $\nabla U = -\frac{\mathbf{F}}{V}$.

Example 2.6. Let us consider the following curl force field:[4]

$$\mathbf{F}(x,y) = (F_x, F_y) = -\frac{F_0}{a^3}(xy^2, x^3)\,, \tag{2.13}$$

where F_0 and a are positive constants with physical dimensions of force and length, respectively. The curl of this force field is $-\frac{F_0}{a^3}(3x^2 - 2xy)$ (z-component of curl). The PDE (2.12) for this force field is simplified to read

$$x^2\frac{\partial V}{\partial x} - y^2\frac{\partial V}{\partial y} - (3x - 2y)V = 0\,. \tag{2.14}$$

Let us rewrite this PDE in the form of a characteristic equation [Duff, 1956]

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dV}{(3x-2y)V}\,. \tag{2.15}$$

The first equality gives us the characteristic curves

$$\frac{x+y}{xy} = c\,. \tag{2.16}$$

The solution of the PDE is

$$V(x,y) = x^3y^2\,\Phi\left(\frac{x+y}{xy}\right)\,, \tag{2.17}$$

where Φ is an arbitrary differentiable function. Thus,

$$\nabla U(x,y) = \frac{F_0}{a^3\,\Phi\left(\frac{x+y}{xy}\right)}\left(\frac{1}{x^2}, \frac{1}{y^2}\right)\,. \tag{2.18}$$

We see that U is not unique. This is not surprising as $V\,\nabla U$ is invariant under the transformations $(U,V) \mapsto \left(f(U), \frac{V}{f'(U)}\right)$ for any differentiable function f such that $f'(U) \neq 0$. A choice would be $\Phi = a^{-5}$, i.e., $V(x,y) = a^{-5}x^3y^2$. This implies that $\nabla U = F_0 a^2(\frac{1}{x^2}, \frac{1}{y^2})$, and hence

$$U(x,y) = -F_0\,a^2\left(\frac{1}{x} + \frac{1}{y}\right) = -F_0\,a^2\frac{x+y}{xy}\,. \tag{2.19}$$

[3] We are tacitly assuming that the domain is simply connected (in particular, contractible), so that Poincaré's lemma applies and a closed 1-form is exact. However, if one works instead on a non-simply connected subset $\mathcal{U} \subset \mathbb{R}^n$, this is *not* generally sufficient. In that case, even when $\nabla \times \left(\frac{\mathbf{F}}{V}\right) = \mathbf{0}$, one must also check that the period (integral) of the 1-form $\boldsymbol{\alpha} = \frac{\mathbf{F}}{V} \cdot d\mathbf{x}$ around every non-contractible closed loop in $\mathcal{U}$ vanishes. Equivalently, this means that the cohomology class $[\boldsymbol{\alpha}]$ in the first de Rham cohomology group $H^1(\mathcal{U})$ must be zero. Only when both conditions hold—curl-free and all periods vanish—does there exist a globally defined scalar function U satisfying $\nabla U = -\frac{\mathbf{F}}{V}$ [Cantarella et al., 2002, Lee, 2013, Yavari, 2013].

[4] Berry and Shukla [2015] considered a similar example as a force field that cannot be treated using their anisotropic kinetic energy approach.

The mechanical system with a force given by (2.13) admits different solutions depending on the sign of F/a^3. First, we rescale time so that, without without loss of generality the force now reads $\mathbf{F}(x, y) = (xy^2, x^3)$ for $F/a^3 < 0$, or $\mathbf{F}(x, y) = -(xy^2, x^3)$ for $F/a^3 > 0$. In the first case, most solutions blow-up in finite time (see [Goriely, 2001] for a general method). In the second case, the solutions can be periodic for the particular case $x(t) = y(t) \neq 0$, for which the system for both variables is a quartic potential. Otherwise, the solutions for y tend asymptotically to a linear function of t as $t \to \infty$ and $x(t)$ tends to 0 as shown in Fig. 1.

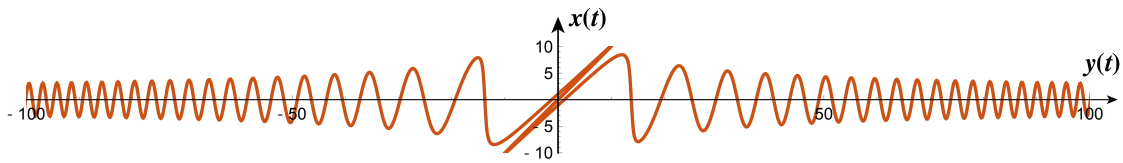

Figure 1: *Typical solutions of (2.13) with $F/a^3 = 1$ shown in the (y, x) plane. Asymptotically, $y(t)$ becomes linear in t and $x(t)$ has decaying oscillations to 0. Two solutions are shown here with initial conditions $(x(0) = \pm 10.01, y(0) = 10, \dot{x}(0) = \dot{y}(0) = 0)$.*

Remark 2.7. Under reasonable regularity assumptions, the first-order linear PDE for $V(x, y)$ obtained in this section always admits local solutions. In particular, since the PDE is of the form $a(x, y)V_{,x} + b(x, y)V_{,y} + c(x, y)V = d(x, y)$ with continuous coefficient functions and $(a(x, y), b(x, y)) \neq (0, 0)$, the method of characteristics applies and guarantees local existence of solutions along characteristic curves [Evans, 1998, John, 1982, Courant and Hilbert, 1962].

2.2 Calculation of the generalized potentials for 3D curl forces

From (2.8) we observe that any force field in dimension three can be additively decomposed into a conservative and a non-conservative part:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_{\mathrm{c}}(\mathbf{x}) + \mathbf{F}_{\mathrm{nc}}(\mathbf{x}), \qquad \mathbf{F}_{\mathrm{c}}(\mathbf{x}) = -\nabla W(\mathbf{x}), \qquad \mathbf{F}_{\mathrm{nc}}(\mathbf{x}) = -V(\mathbf{x})\,\nabla U(\mathbf{x}). \tag{2.20}$$

If we are only interested in the non-conservative part of the force field, i.e., if $\mathbf{F} = \mathbf{F}_{\mathrm{n}}$, the computation of the generalized potentials V and U follows the same procedure as in the two-dimensional case.

Remark 2.8. It should be noted that in the decomposition (2.20), the conservative and non-conservative components are not unique. This is due to the fact that adding a conservative force to a non-conservative force yields a force that is still non-conservative, and subtracting a conservative force from another conservative force results in a force that remains conservative. In other words, $\mathbf{F}(\mathbf{x}) = \tilde{\mathbf{F}}_{\mathrm{c}}(\mathbf{x}) + \tilde{\mathbf{F}}_{\mathrm{nc}}(\mathbf{x})$ is an equivalent decomposition if $\tilde{\mathbf{F}}_{\mathrm{nc}}(\mathbf{x}) - \mathbf{F}_{\mathrm{nc}}(\mathbf{x}) = -(\tilde{\mathbf{F}}_{\mathrm{c}}(\mathbf{x}) - \mathbf{F}_{\mathrm{c}}(\mathbf{x}))$ is a conservative force (see Example 2.9).

In the decomposition (2.8), the three potentials V, U, and W cannot be uniquely determined. Note that $\mathbf{A} = \operatorname{curl}\mathbf{F} = -\nabla V \times \nabla U$. Let us assume that $\nabla V \cdot \nabla U = 0$ (the component of ∇V along ∇U does not contribute to $\operatorname{curl}\mathbf{F}$ and can be assumed to vanish without affecting the decomposition of the curl force into conservative and non-conservative parts). This orthogonality condition can be interpreted as a gauge choice. Under the above two assumptions $\{\nabla V, \nabla U, \nabla V \times \nabla U\}$ is a basis for $\mathbb{R}^3$. The condition $\nabla V \cdot \mathbf{A} = 0$, is a PDE for the unknown scalar field V and can be solved using the method of characteristics. Suppose $\mathbf{x}(s)$ is a curve such that

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{A}(\mathbf{x}(s)). \tag{2.21}$$

Along these curves

$$\frac{df}{ds} = \nabla f \cdot \frac{d\mathbf{x}(s)}{ds} = \nabla f \cdot \mathbf{A}(\mathbf{x}(s)) = 0, \tag{2.22}$$

and hence, $f(\mathbf{x}(s)) = \text{constant}$. Recall that for any three arbitrary vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}. \tag{2.23}$$

Thus

$$\nabla U = \frac{\nabla V \times \operatorname{curl} \mathbf{F}}{\|\nabla V\|^2} \,. \tag{2.24}$$

Therefore

$$\mathbf{F}_{\rm c} = -\nabla W = \mathbf{F} + \frac{V \nabla V \times \operatorname{curl} \mathbf{F}}{\|\nabla V\|^2} \,. \tag{2.25}$$

Example 2.9. Let us consider the following force field

$$\mathbf{F}(x, y, z) = -(yz, 2xz, xy) \,, \tag{2.26}$$

with $\operatorname{curl} \mathbf{F} = (x, 0, -z)$. The condition $\nabla V \cdot \operatorname{curl} \mathbf{F} = 0$ reads

$$x \, \frac{\partial V}{\partial x} - z \, \frac{\partial V}{\partial z} = 0 \,. \tag{2.27}$$

This is a linear first-order PDE. The associated characteristic equations are

$$\frac{dx}{x} = \frac{dz}{-z}, \qquad y = \text{constant} \,. \tag{2.28}$$

Solving these, one obtains the characteristic curves:

$$x = C_1 e^s \,, \qquad z = C_2 e^{-s} \,, \qquad y = C_3 \,. \tag{2.29}$$

Thus, the invariants (quantities constant along characteristics) are

$$xz = C_1 C_2 = \text{constant}, \qquad y = \text{constant} \,. \tag{2.30}$$

Therefore, the general solution is

$$V(x, y, z) = f(xz, y) \,, \tag{2.31}$$

where f is an arbitrary differentiable function. Now from (2.24) we obtain

$$\nabla U = \frac{2}{f_2^2 + (x^2 + z^2) f_1^2} \left(-z f_2, (x^2 + z^2) f_1, -x f_2 \right) \,. \tag{2.32}$$

Therefore

$$\mathbf{F}_{\rm nc} = \frac{2 f(xz, y)}{f_y^2 + (x^2 + z^2) f_{xz}^2} \left(z f_2, -(x^2 + z^2) f_1, x f_2 \right) \,, \qquad \mathbf{F}_{\rm c} = -(yz, 2xz, xy) - \mathbf{F}_{\rm nc} \,. \tag{2.33}$$

For the choice $f(xz, y) = y$, we have

$$\mathbf{F}_{\rm nc} = (2yz, 0, 2xy) \,, \qquad \mathbf{F}_{\rm c} = (-3yz, -3xz, -3xy) \,. \tag{2.34}$$

For the choice $f(xz, y) = xz$, we have

$$\tilde{\mathbf{F}}_{\rm nc} = (0, -2xz, \ 0) \,, \qquad \tilde{\mathbf{F}}_{\rm c} = (-yz, -xz, -xy) \,. \tag{2.35}$$

Note that $\tilde{\mathbf{F}}_{\rm nc}(\mathbf{x}) - \mathbf{F}_{\rm nc}(\mathbf{x}) = -(\tilde{\mathbf{F}}_{\rm c}(\mathbf{x}) - \mathbf{F}_{\rm c}(\mathbf{x})) = (-2yz, -2xz, -2xy)$ is a conservative force, and hence, the above two decompositions are equivalent. In other words, both $\mathbf{F}_{\rm c}$ and $\tilde{\mathbf{F}}_{\rm c}$ are conservative forces and $\operatorname{curl} \tilde{\mathbf{F}}_{\rm nc} = \operatorname{curl} \mathbf{F}_{\rm nc}$. In summary, in (2.33) we have a family of equivalent decompositions of the force field (2.26) into conservative and non-conservative parts, parametrized by the arbitrary function f.

Remark 2.10 (Connection with Helmholtz Decomposition)**.** It is natural to ask whether the decomposition (2.20) is related to the classical Helmholtz decomposition of a vector field into solenoidal (divergence-free) and irrotational (curl-free) components [Helmholtz, 1858, 1867]. The Helmholtz decomposition is also related to the Hodge decomposition of differential forms on Riemannian manifolds [Hodge, 1952, Abraham et al., 2012].

The Helmholtz decomposition, under suitable regularity and decay conditions, states that any sufficiently smooth vector field $\mathbf{F}$ on $\mathbb{R}^3$ that vanishes at infinity can be uniquely written as

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}\,, \tag{2.36}$$

where ϕ is a scalar potential and $\mathbf{A}$ is a vector potential. This decomposition separates the field into a curl-free and a divergence-free part. In contrast, the decomposition (2.20) splits a force field into a conservative part $-\nabla W$ and a curl force of the form $-V\nabla U$, where U, V, and W are generalized scalar potentials. This decomposition is not unique and is defined pointwise rather than through the global elliptic structure that underlies Helmholtz decomposition. Moreover, the curl force $-V\nabla U$ is generally neither divergence-free nor orthogonal to the conservative part, and the decomposition is not orthogonal in L^2. Therefore, the decomposition introduced here is conceptually and structurally distinct from the classical Helmholtz decomposition, and it is tailored to the analysis of non-conservative forces arising in mechanics, rather than being a general analytical tool for arbitrary vector fields.

2.3 Work performed by curl forces in closed and cyclic motions

The *motion of a particle* is a map $\mathbf{x}(t) : [t_1, t_2] \to \mathbb{R}^n$ with associated velocity field $\mathbf{v}(\mathbf{x}) = \dot{\mathbf{x}} = \frac{d}{dt}\mathbf{x}(t)$. The work done by the force field on the particle in this motion is

$$W([t_1,t_2]) = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{x}(t))\cdot\dot{\mathbf{x}}(t)dt = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \int_\Gamma \mathbf{F}^\flat = \int_\Gamma \mathbf{\Omega}\,, \tag{2.37}$$

where $\mathbf{x}_1 = \mathbf{x}(t_1)$, $\mathbf{x}_2 = \mathbf{x}(t_2)$ and Γ is the trajectory of the motion. A *closed motion* is a motion $\mathbf{x}(t) : [t_1, t_2] \to \mathbb{R}^n$ for which $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. A *cyclic motion* is a motion $\mathbf{x}(t) : [t_1, t_2] \to \mathbb{R}^n$ for which $\mathbf{x}(t_1) = \mathbf{x}(t_2)$ and $\dot{\mathbf{x}}(t_1) = \dot{\mathbf{x}}(t_2)$. Clearly, a cyclic motion is closed but the converse is not necessarily true. For a closed motion, $\Gamma \in \mathbb{R}^n$ is a closed curve and encloses a surface $\mathcal{D} \in \mathbb{R}^n$. Using Stokes' theorem the work done on the particle in a closed motion is calculated as

$$W(\Gamma) = \int_\Gamma \mathbf{\Omega} = \int_{\mathcal{D}} d\mathbf{\Omega}\,. \tag{2.38}$$

For a curl force for which $d\mathbf{\Omega} \neq \mathbf{0}$, the work is non-zero only if $\mathcal{D}$ has non-vanishing area. Denoting the reverse cyclic motion corresponding to Γ by $-\Gamma$, it is straightforward to see that $W(-\Gamma) = -W(\Gamma)$. In particular, this implies that curl forces perform zero net work over a cyclic deformation followed by its reverse. This is in contrast with dissipative forces that always perform negative work in such motions.

2.4 Change of the kinetic energy of a particle under curl forces

Carathéodory [Carathéodory, 1909] reformulated thermodynamics by expressing the first and second laws in geometric terms on the manifold $\mathcal{M}$ of thermodynamic states. The first law is represented by the vanishing of a differential 1-form, i.e., $\boldsymbol{\theta} = du - \boldsymbol{\omega}_h - \boldsymbol{\omega}_w = 0$, where u is the internal energy, and $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_w$ are the heat and work 1-forms, respectively [Mrugala, 1978]. An *adiabatic process* is a curve in $\mathcal{M}$ along which $\boldsymbol{\omega}_h$ vanishes. Carathéodory's formulation of the second law asserts that in any neighborhood of a given state, there exist states that are inaccessible via adiabatic processes [Pogliani and Berberan-Santos, 2000, Frankel, 2011]. This inaccessibility condition implies that the distribution defined by $\ker \boldsymbol{\omega}_h$ is not completely integrable, thereby constraining the allowable thermodynamic evolutions. Under suitable conditions, Carathéodory's theorem ensures that $\boldsymbol{\omega}_h$ is locally integrable, leading to the existence of an entropy function s and an integrating factor T such that $\boldsymbol{\omega}_h = T\,ds$ [Cooper, 1967, Boyling, 1972, Buchdahl, 2009].

The kinetic energy of the particle is defined as $K = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$, and hence $\frac{dK}{dt} = m\mathbf{v}\cdot\mathbf{a} = \mathbf{F}\cdot\mathbf{v}$. Thus, $dK = \mathbf{F}\cdot\mathbf{v}dt = \mathbf{F}\cdot d\mathbf{x} = \mathbf{F}^\flat = \mathbf{\Omega}$. This can be rewritten as

$$\boldsymbol{\theta} = dK - \mathbf{\Omega} = 0\,. \tag{2.39}$$

For a particle (or system of particles), the kinetic energy is the internal energy of the system and (2.39) is a statement of the first law of thermodynamics in the Carathéodory's abstract formulation of thermodynamics [Carathéodory, 1909, Mrugala, 1978]. Therefore

$$K(t_2) - K(t_1) = \int_\Gamma \boldsymbol{\Omega} \,. \tag{2.40}$$

For a 2D curl force

$$K(t_2) - K(t_1) = -\int_\Gamma V(\mathbf{x})\, dU(\mathbf{x}) \,. \tag{2.41}$$

When the generalized potentials U and V are functionally independent, the right-hand side is path dependent and change in the kinetic energy is path dependent as well. Similarly, for a 3D curl force

$$K(t_2) - K(t_1) = -\int_\Gamma V(\mathbf{x})\, dU(\mathbf{x}) - \int_\Gamma dW(\mathbf{x}) \,. \tag{2.42}$$

For a potential force

$$K(t_2) - K(t_1) = -\int_\Gamma dU(\mathbf{x}) = U(\mathbf{x}_1) - U(\mathbf{x}_2) \,, \quad \text{or} \qquad K(t_2) + U(\mathbf{x}_2) = K(t_1) + U(\mathbf{x}_1) \,, \tag{2.43}$$

which implies that the total energy $H = K + U$ is conserved. There is no such energy conservation for curl forces.

2.5 The local accessibility property of curl forces

Let $\boldsymbol{\Omega} \in \Lambda^1(\mathbb{R}^n)$ be a 1-form. The first-order partial differential equation

$$\boldsymbol{\Omega} = 0 \,, \tag{2.44}$$

is known as a *Pfaffian equation*, which is the simplest example of an exterior differential system. A p-dimensional integral manifold of (2.44) is an immersion (not necessarily an embedding) $f : \mathcal{D} \to \mathbb{R}^n$ such that $f^*\boldsymbol{\Omega} = 0$. An integral curve of $\boldsymbol{\Omega}$ is a curve $c : I \to \mathbb{R}^n$, where I is an interval on the real line, satisfying $\boldsymbol{\Omega}(c(t)) = 0$ for all $t \in I$. The Pfaffian equation (2.44) is said to have the *local accessibility* property if, for every point $\mathbf{x} \in \mathbb{R}^n$, there exists a neighborhood $U \subset \mathbb{R}^n$ such that for any point $\mathbf{y} \in U$, there is an integral curve of (2.44) connecting $\mathbf{x}$ and $\mathbf{y}$. Conversely, (2.44) has the *local inaccessibility* property if, in every neighborhood U of any point $\mathbf{x} \in \mathbb{R}^n$, there exists at least one point $\mathbf{y} \in U$ that cannot be reached from $\mathbf{x}$ by any integral curve of (2.44).

The rank of the Pfaffian equation (2.44) is the integer r such that $\boldsymbol{\Omega} \wedge (d\boldsymbol{\Omega})^r \neq 0$ and $\boldsymbol{\Omega} \wedge (d\boldsymbol{\Omega})^{r+1} = 0$. Carathéodory's theorem tells us that if the rank of a Pfaffian equation is constant, the Pfaffian equation has the local accessibility property if and only if $r \geq 2$ [Bryant et al., 2013]. Therefore, a Pfaffian equation has the local inaccessibility property only when $r = 0$ or $r = 1$, which correspond to the normal forms $\boldsymbol{\Omega} = d\psi$ and $\boldsymbol{\Omega} = \phi\, d\psi$, respectively. In either case, $\boldsymbol{\Omega} = 0$ implies that $\psi =$ constant, which are hypersurfaces—the integral manifolds of maximum possible dimension $n - 1$.

2.5.1 The local in accessibility property of 2D curl forces

Carathéodory's theorem tells us that any 2D curl force with the representation $\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\, \nabla U(\mathbf{x})$ has the inaccessibility property. This has the following physical interpretation. Consider a particle at position $\mathbf{x}(t) \in \mathbb{R}^2$ at time t moving under the influence of a curl force field $\mathbf{F}(\mathbf{x})$. In every neighborhood of $\mathbf{x}(t)$, there exists at least one point $\mathbf{y}$ such that if the particle moves from $\mathbf{x}$ to $\mathbf{y}$ along any path, the curl force performs nonzero work on the particle.

2.5.2 The local accessibility property of 3D curl forces

Carathéodory's theorem states that any $3D$ curl force with the representation $\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\nabla U(\mathbf{x}) - \nabla W(\mathbf{x})$ such that none of the three potentials vanishes and $\mathbf{F}\cdot\operatorname{curl}\mathbf{F} \neq 0$ has the accessibility property. This property has the following physical interpretation. Consider a particle at position $\mathbf{x}(t) \in \mathbb{R}^3$ at time t moving under the influence of the curl force field. There exists a neighborhood $\mathcal{U}$ of $\mathbf{x}(t)$ such that for any $\mathbf{y} \in \mathcal{U}$, there exists a path Γ connecting $\mathbf{x}(t)$ to $\mathbf{y}$ such that $\int_\Gamma \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = 0$, i.e., the curl force does no work. A $3D$ curl force has the inaccessibility property if it has the representation $\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\nabla U(\mathbf{x})$ or $\mathbf{F}(\mathbf{x}) = -\nabla W(\mathbf{x})$, which is a special case of the former.

Remark 2.11. The accessibility condition introduced in this section is equivalent to the classical Frobenius integrability condition applied to the plane field defined by the kernel of the work 1-form $\mathbf{\Omega}$. In particular, the ability to connect points via admissible curves corresponds to the existence of integral manifolds of this distribution, which is guaranteed locally if and only if $\mathbf{\Omega} \wedge d\mathbf{\Omega} = 0$, or equivalently, if the kernel distribution is involutive under the Lie bracket of vector fields tangent to $\ker \mathbf{\Omega}$.[5] Thus, the structure of accessibility is governed by the classical Frobenius theorem on integrability of distributions Spivak [1979], Lee [2013], Warner [1983].

2.6 The conservative auxiliary force of a curl force

In this section, we show that any curl force, whether in two or three dimensions, has a corresponding conservative auxiliary force. The Hamiltonian associated with this auxiliary force, although not the physical energy, is nevertheless a conserved quantity of the motion.

2.6.1 Two-dimensional curl forces

Any curl force in 2D has the representation (2.6), i.e., $\mathbf{F}(\mathbf{x}) = -V(\mathbf{x})\,\nabla U(\mathbf{x})$, where U and V are the generalized potentials of the force field. We are interested in the dynamics of a particle p under such forces, i.e, the time evolution of its position $\mathbf{x}(t)$, $t \geq 0$. Suppose the initial position and velocity of the particle are given (initial conditions): $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{v}(0) = \mathbf{v}_0$. In particular, the initial momentum of the particle is known: $\mathbf{p}(0) = \mathbf{p}_0 = m\mathbf{v}_0$, where m is the mass of p.

Let us define an auxiliary conservative force field $\bar{\mathbf{F}}(\mathbf{x})$ by rescaling the original force field with the generalized potential V (assuming that $V(\mathbf{x}) \neq 0$):

$$\bar{\mathbf{F}}(\mathbf{x}) = \frac{1}{V(\mathbf{x})}\mathbf{F}(\mathbf{x})\,, \qquad \bar{\mathbf{F}}(\mathbf{x}) = -\nabla U(\mathbf{x})\,. \tag{2.45}$$

The particle p, in general, would have a different dynamics under $\bar{\mathbf{F}}$, which we call the auxiliary dynamics. Let us denote its trajectory by $\bar{\mathbf{x}}(t)$, $t \geq 0$. The velocity field corresponding to the rescaled force field is denoted by $\bar{\mathbf{v}}(t)$ and its momentum by $\bar{\mathbf{p}} = m\bar{\mathbf{v}}$. The initial conditions are independent of the force field, and hence, $\bar{\mathbf{x}}(0) = \mathbf{x}_0$, $\bar{\mathbf{v}}(0) = \mathbf{v}_0$. Also, the initial momentum is independent of the force field as well: $\bar{\mathbf{p}}(0) = \mathbf{p}_0 = m\mathbf{v}_0$. One has the following Hamiltonian corresponding to the conservative auxiliary force field:

$$H(\bar{\mathbf{x}}, \bar{\mathbf{p}}) = \frac{\bar{\mathbf{p}}\cdot\bar{\mathbf{p}}}{2m} + U(\bar{\mathbf{x}})\,. \tag{2.46}$$

We call H the auxiliary Hamiltonian. Hamilton's equations for the auxiliary motion read

$$\begin{cases} -\dfrac{\partial H}{\partial \bar{\mathbf{x}}} = \dot{\bar{\mathbf{p}}} = -\nabla U(\bar{\mathbf{x}}) = \bar{\mathbf{F}}(\bar{\mathbf{x}})\,, \\ \dfrac{\partial H}{\partial \bar{\mathbf{p}}} = \dfrac{\bar{\mathbf{p}}}{m} = \bar{\mathbf{v}}\,. \end{cases} \tag{2.47}$$

[5] At each point p, the kernel of $\mathbf{\Omega}$ is the subspace of tangent vectors annihilated by $\mathbf{\Omega}$, i.e., $\ker \mathbf{\Omega}_p = \{v \in T_pM \mid \mathbf{\Omega}_p(v) = 0\}$. This defines a distribution (i.e., a smoothly varying field of subspaces) on the manifold. A smooth vector field $\mathbf{Y}$ is said to be tangent to $\ker \mathbf{\Omega}$ if it takes values in this distribution, that is, $\mathbf{\Omega}(\mathbf{Y}(p)) = 0$ for all $p \in M$, or equivalently, $\mathbf{\Omega}(\mathbf{Y}) = 0$ everywhere.

Note that $\frac{dH}{dt} = 0$, and hence H is a constant of motion, i.e.,

$$\frac{\bar{\mathbf{p}}(t)\cdot\bar{\mathbf{p}}(t)}{2m} + U(\bar{\mathbf{x}}(t)) = \frac{1}{2}m\,\bar{\mathbf{v}}(t)\cdot\bar{\mathbf{v}}(t) + U(\bar{\mathbf{x}}(t)) = \frac{1}{2}m\mathbf{v}_0\cdot\mathbf{v}_0 + U(\mathbf{x}_0)\,. \tag{2.48}$$

From $\bar{\mathbf{F}} = m\dot{\bar{\mathbf{v}}}$, and knowing that for the original force field $\mathbf{F} = m\dot{\mathbf{v}} = \dot{\mathbf{p}}$ (Newton's law of motion is independent of whether a Hamiltonian exists), one concludes that

$$\dot{\bar{\mathbf{p}}} = \frac{1}{V(\mathbf{x})}\dot{\mathbf{p}}\,. \tag{2.49}$$

Thus

$$\bar{\mathbf{p}}(t) = \mathbf{p}_0 + \int_0^t \frac{\dot{\mathbf{p}}(\xi)}{V(\mathbf{x}(\xi))}\,d\xi = \mathbf{p}_0 - \int_0^t \nabla U(\mathbf{x}(\xi))\,d\xi\,, \tag{2.50}$$

where $\mathbf{p}_0 = \bar{\mathbf{p}}(0)$ and the equation of motion $\dot{\mathbf{p}} = -V\nabla U$ was used in the second equality. Recalling that $\bar{\mathbf{p}}(t) = m\bar{\mathbf{v}}(t) = m\dot{\bar{\mathbf{x}}}(t)$, we have

$$\bar{\mathbf{x}}(t) = \mathbf{x}_0 + \mathbf{v}_0\,t - \frac{1}{m}\int_0^t\int_0^\tau \nabla U(\mathbf{x}(\xi))\,d\xi\,d\tau\,. \tag{2.51}$$

Therefore, the auxiliary Hamiltonian (2.46) is simplified to read

$$\begin{aligned} H = \frac{1}{2m}&\left[|\mathbf{p}_0|^2 - 2\mathbf{p}_0\cdot\int_0^t \nabla U(\mathbf{x}(\xi))\,d\xi + \int_0^t\int_0^t \nabla U(\mathbf{x}(\xi))\cdot\nabla U(\mathbf{x}(\eta))\,d\xi\,d\eta\right] \\ &+ U\left(\mathbf{x}_0 + t\mathbf{v}_0 - \frac{1}{m}\int_0^t\int_0^\tau \nabla U(\mathbf{x}(\xi))\,d\xi\,d\tau\right)\,. \end{aligned} \tag{2.52}$$

Note that H is a conserved quantity of motion under the curl force, although it is not the physical energy. Moreover, due to the fact that it is defined implicitly through integral equations, this conserved quantity is not like the traditional constants of motion that appears in integrability theory. Its existence does not preclude chaos and cannot be used to reduce the dynamics to manifolds of lower dimensions on which this quantity is conserved.

Example 2.12. Let us consider the force field (2.13) of Example 2.6. In this example, the rescaled force field has the following form

$$\bar{\mathbf{F}}(\mathbf{x}) = \frac{1}{V(\mathbf{x})}\mathbf{F}(\mathbf{x}) = -F_0a^2\left(\frac{1}{x^2}, \frac{1}{y^2}\right)\,. \tag{2.53}$$

The auxiliary motion (2.51) is given as

$$\bar{\mathbf{x}}(t) = \mathbf{x}_0 + \mathbf{v}_0\,t - \frac{F_0a^2}{m}\int_0^t\int_0^\tau \left(\frac{1}{x^2(\xi)}, \frac{1}{y^2(\xi)}\right)\,d\xi\,d\tau\,. \tag{2.54}$$

The Hamiltonian (2.52) is simplified to read

$$\begin{aligned} H = \frac{1}{2m}&\left\{p_{x0}^2 + p_{y0}^2 + 2\int_0^t \left[\frac{p_{x0}}{x^2(\xi)} + \frac{p_{y0}}{y^2(\xi)}\right]d\xi - F_0^2a^4\int_0^t\int_0^t \left[\frac{1}{x^2(\xi)\,x^2(\eta)} + \frac{1}{y^2(\xi)\,y^2(\eta)}\right]d\xi\,d\eta\right\} \\ &+ U\left(\mathbf{x}_0 + \mathbf{v}_0\,t + \frac{F_0a^2}{m}\int_0^t\int_0^\tau \left(\frac{1}{x^2(\xi)}, \frac{1}{y^2(\xi)}\right)\,d\xi\,d\tau\right)\,, \end{aligned} \tag{2.55}$$

where $\mathbf{p}_0 = (p_{x0}, p_{y0})$ and U is given in (2.19). H is a conserved quantity for the motion under the curl force (2.13).

2.6.2 Three-dimensional curl forces

Any curl force in 3D has the representation (2.8), and hence, the rescaled force field

$$\bar{\mathbf{F}}(\mathbf{x}) = \frac{\mathbf{F}(\mathbf{x}) + \nabla W(\mathbf{x})}{V(\mathbf{x})}\,, \tag{2.56}$$

is conservative, i.e., $\bar{\mathbf{F}}(\mathbf{x}) = -\nabla U(\mathbf{x})$. Let us denote the velocity field corresponding to the auxiliary force field by $\bar{\mathbf{v}}(\mathbf{x})$ and its momentum by $\bar{\mathbf{p}} = m\bar{\mathbf{v}}$. One still has the Hamiltonian (2.46) corresponding to the auxiliary force. Hamilton's equations are given in (2.47). The relations between the original and auxiliary dynamics are still described by Eqs. (2.49)-(2.51). The auxiliary motion still has the form (2.51). Therefore, the auxiliary Hamiltonian (2.46) simplifies to read:

$$\begin{aligned} H(\mathbf{x},\mathbf{p}) &= \frac{1}{2m}\left[\mathbf{p}_0\cdot\mathbf{p}_0 + 2\mathbf{p}_0\cdot\int_0^t \frac{\dot{\mathbf{p}}(\xi)}{V(\mathbf{x}(\xi))}\,d\xi + \int_0^t \frac{\dot{\mathbf{p}}(\xi)}{V(\mathbf{x}(\xi))}\,d\xi\cdot\int_0^t \frac{\dot{\mathbf{p}}(\eta)}{V(\mathbf{x}(\eta))}\,d\eta\right] + U(\bar{\mathbf{x}}) \\ &= \frac{1}{2m}\left[|\mathbf{p}_0|^2 + 2\mathbf{p}_0\cdot\int_0^t \frac{\dot{\mathbf{p}}(\xi)}{V(\mathbf{x}(\xi))}\,d\xi + \int_0^t\int_0^t \frac{\dot{\mathbf{p}}(\xi)\cdot\dot{\mathbf{p}}(\eta)}{V(\mathbf{x}(\xi))\,V(\mathbf{x}(\eta))}\,d\xi\,d\eta\right] + U(\bar{\mathbf{x}})\,. \end{aligned} \tag{2.57}$$

Using the equations of motion $\dot{\mathbf{p}} = -V\nabla U - \nabla W$, Hamiltonian is simplified to read

$$\begin{aligned} H = \frac{1}{2m}\Bigg\{ &|\mathbf{p}_0|^2 - 2\mathbf{p}_0\cdot\int_0^t \left[\nabla U(\mathbf{x}(\xi)) + \frac{\nabla W(\mathbf{x}(\xi))}{V(\mathbf{x}(\xi))}\right] d\xi \\ &+ \int_0^t\int_0^t \left[\nabla U(\mathbf{x}(\xi)) + \frac{\nabla W(\mathbf{x}(\xi))}{V(\mathbf{x}(\xi))}\right]\cdot\left[\nabla U(\mathbf{x}(\eta)) + \frac{\nabla W(\mathbf{x}(\eta))}{V(\mathbf{x}(\eta))}\right] d\xi\,d\eta\Bigg\} \\ &+ U\left(\mathbf{x}_0 + t\mathbf{v}_0 - \frac{1}{m}\int_0^t\int_0^\tau \nabla U(\mathbf{x}(\xi))\,d\xi\,d\tau\right)\,. \end{aligned} \tag{2.58}$$

This is a conserved quantity of motion under an arbitrary three-dimensional curl force, although it does not represent the physical energy.

3 Conclusions

In this paper, we investigated the dynamics of particles under curl forces, a class of non-conservative, non-dissipative forces. We first pointed out that the fundamental quantity associated with a curl force is its work 1-form, which turns out to be the flat force field, that is, the 1-form representation (or proxy) of the force vector field. Using the Darboux classification of differential 1-forms, we established that any two-dimensional curl force has at most two generalized potentials, while in three dimensions, it has at most three. This classification provides a systematic framework for understanding the structure of curl forces and their associated work 1-forms. It was shown that one feature distinguishing curl forces from potential forces is their ability to perform non-zero work over certain classes of closed paths. We also discussed the accessibility property of curl forces in dimensions two and three. Furthermore, we demonstrated that for any given curl force field, a corresponding conservative force field can be constructed. This conservative auxiliary force enabled the definition of an auxiliary Hamiltonian for curl forces, which is a nonlocal functional of motion and is a conserved quantity of motion. Our formulation offers a new perspective on the representation of curl forces in dimensions two and three and their underlying mathematical properties.

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