# Trouble with linear elastic theory of strength

## To Deform, or to Rupture?

**The toughest hydrogel in the world.** We reported an exceptionally tough hydrogel: Jeong-Yun Sun, Xuanhe Zhao, Widusha R.K. Illeperuma, Kyu Hwan Oh, David J. Mooney, Joost J. Vlassak, Zhigang Suo. Highly stretchable and tough hydrogels. Nature 489, 133-136 (2012). A hydrogel is a three-dimensional polymer network swollen with water. Familiar examples include jello and contact lenses. Our hydrogel achieved a toughness of ~9,000 J/m$^2$. This statement raises several questions.

What is toughness? Toughness is the ability of a material to resist the growth of crack. Understanding toughness is a main object of this course.

What does the value 9,000 J/m$^2$ mean? Toughness of a material is determined by an experimental measurement. We will talk about how to measure toughness later in the course. For now, you can have some feel for orders of magnitude. Jello and tofu have toughness ~10 J/m$^2$. Contact lenses have toughness ~100 J/m$^2$. Cartilage has toughness ~1,000 J/m$^2$. Natural rubber has toughness ~10,000 J/m$^2$. Our tough gel contains about 90% water, yet its toughness approaches that of the natural rubber.

How can our hydrogel be so tough? For now let's have a qualitative picture. A window glass is brittle, but a steal is tough. That is, the ability of a window glass to resist the growth of a crack is much less than the ability of a steel to resist the growth of a crack. This is everyday experience, but why? The answer to such question sooner or later leads to an atomistic picture.

**Atomic processes associated with fracture**. You prepare a sheet of glass. To study the growth of the crack, you cut a crack into the glass with a diamond saw. You then pull the glass to cause the crack to grow. The crack grows by breaking atomic bonds. The tip of the crack concentrates stress, so that the atomic bonds at the tip of crack break. As the tip the crack advances, a plane of atomic bonds unzips. Atomic bonds off the plane of the crack remain elastic, and do not participate in resisting the growth of the crack. The elasticity is visible: after fracture, the two halves the glass fit together nicely.

Now you prepare a sheet of steel with a pre-cut crack. You then pull the steel to cause the crack to grow. The crack grows by breaking atomic bonds, but something else happens. Atomic bonds off the plane of the crack no longer remain elastic: they change neighbors. That is, the steel off the crack plane undergoes *plastic deformation*. The amount of material involved in plastic deformation is much, much more than the two planes of atoms on the surfaces of the crack. The plastic deformation enables the steel to resist the growth of the crack much more than breaking a plane of atomic bonds. The plasticity is visible: after fracture, the two halves the steel do not fit together nicely.

Our hydrogel is tough because the growth of a crack does more than breaking a single layer of polymer chains. The polymer off the plane of the crack undergoes deformation similar to plastic deformation in the steel. We will make this picture precise as we go along in this course. But if you absolutely cannot wait to know how our gel works, I'd be delighted if you read the paper now.

**Demonstration of stretchability and toughness.** Our paper is titled highly stretchable and tough hydrogels. Our hydrogels can be stretched more than 20 times its initial length. Here I will demonstrate stretchability and toughness using rubber band.

Demonstrate in class the effect of a crack. Use a wide rubber band, so that it is easy to see and easy to introduce a crack. Show that the rubber band can be stretched about six times its original length. The large deformation of the rubber may hide its *brittleness*. The rubber band is *notch-sensitive*. Use scissors to cut a crack into an edge of the rubber band, pull the band, and show that the band ruptures at a stretch less than three. Before the rubber band ruptures, the tip of the crack blunts. Pass the scissors and some rubber bands around. Invite every student to try.

Do the same experiment using a long party balloon. Stretch the balloon without precut. Introduce a cut and stretch again. Show that the stretch at rupture is insensitive to the crack.

Demonstrate the effect of cyclic load. Fatigue.

**Fracture of glass**. Hydrogel is a rather complex material. Let us begin with a body made of a silica glass. This body deforms elastically: if a load does not cause fracture, upon the removal of the load, the body recovers its shape. The body can sustain only small strains, typically less than 1%.

What is the magnitude of the load that will cause the body to fracture? A procedure you have been taught before probably goes as follows. You first determine the *maximum stress in the body*. You then determine the *strength of the material*. The body is supposed to fracture when the maximum stress in the body reaches the strength of the material.

I'll first review this procedure, so that you and I agree exactly what this procedure is. I'll then explain why this procedure does not work in practice.

## Stress

**Theory of linear elasticity.** I assume that you know the rudiments of the theory of linear elasticity. You know, for example, that the **state of stress** of a material particle is described by a **tensor**, and that the state of stress in a body is described by a **field**. Furthermore, the field of stress in the body can be determined by solving a boundary-value problem. The theory of elasticity can be found in many textbooks. You can find a list of the governing equations of elasticity in my notes (http://imechanica.org/node/205).

For convenience, let us list the governing equations of linear elasticity here. The balance of forces requires that the stress tensor $\sigma_{ij}\left(x_1,x_2,x_3\right)$ should satisfy

$$\sigma_{ij,j} = 0 .$$

The geometry of deformation relates the displacement vector $u_i(x_1, x_2, x_3)$ to the strain tensor $\varepsilon_{ij}(x_1, x_2, x_3)$:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) .$$

The theory relates stress tensor and the strain tensor by the model of elasticity:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} ,$$

where $C_{ijkl}$ is the stiffness tensor.

I'll not spend much time on these equations in class. I will, however, slow down whenever features of the theory of elasticity play significant roles in developing fracture mechanics.

**On solving boundary-value problems**. A few quick notes about solving boundary-value problems in elasticity:

- Governing equations, as listed above.
- Boundary conditions. Prescribe traction. Prescribe displacement.
- Approximate analytical solutions. Beams and plates.
- Exact analytical solutions. Few problems can be solved exactly. Timoshenko and Goodier describe some.
- Handbooks contain useful solutions obtained by the above methods.
- Numerical solutions. Finite element method. Commercial software such as ABAQUS. A local community of users.

**Maximum stress in a body.** Imagine a body (e.g., a plate with a hole) subject to an applied stress, $\sigma_{\text{appl}}$. At each material particle, the state of stress is a tensor, with 6 components. In the body, the state of stress varies form one material particle to another. Thus, the state of stress in the body is described by a field. This field is determined by solving the boundary-value problem. Solving boundary-value problems is a big task by itself, but is not the subject of this course. Let's say we already have the solution. That is, we know all six components of stress at every material particle in the body, $\sigma_{11}(x_1, x_2, x_3)$, $\sigma_{12}(x_1, x_2, x_3)$…

What do we do with this massive amount of data? We are interested in predicting the conditions of fracture of the body. For example, if the applied stress $\sigma_{\text{appl}}$ is small enough, the body will not fracture. How small is small enough?

Here is a procedure taught in the theory of elasticity. From the field of stress we determine the maximum stress in the body, $\sigma_{\max}$. The maximum component of stress at each material particle is determined by a principal stress, determined by solving an eigenvalue problem. Then we look for the largest value of the principal stress by comparing all material particles in the body. Today, all this procedure is embodied in commercial software such as ANAQUS. So you know the maximum component of stress in the body, $\sigma_{\max}$.

**Stress concentration factor**. The equations in elasticity are linear, so that the maximum stress in the body is proportional to the applied stress. We write

$$\sigma_{\max} = C\sigma_{appl},$$

where $C$ is a dimensionless number. The basic phenomenon that the stress is higher at some material particles in a body than others is known as stress concentration. The number $C$ is known as the stress concentration factor.

When the hole is circular, and the plate is much larger than the hole, the boundary-value problem is solved analytically in Timoshenko and Goodier. The maximum stress occurs at the surface of the hole. The stress concentration factor is

$$\frac{\sigma_{\max}}{\sigma_{appl}} = 3.$$

**Two significant features of linear elasticity**. First, because the governing equations of elasticity are linear, the stress concentration factor is independent of the applied stress. Second, because the governing equations of elasticity contain no length, the stress concentration factor depends on ratios of the lengths in boundary conditions.

For a circular hole in an infinite plate, the only length in the boundary conditions is the radius of the hole. Consequently, the stress concentration factor is a dimensionless number, independent of any parameter. In particular, the stress concentration factor of the circular hole is independent of the radius of the hole.

**Elliptic hole**. C.E. Inglis, Stress in a plate due to the presence of cracks and sharp corners. Proceedings of the Institute of Naval Architects, 14 March 1913. (http://imechanica.org/node/7457). As another example of boundary-value problems, consider an elliptic hole in an infinite plate subject to a remote stress $\sigma_{\mathrm{appl}}$. The problem can still be solved analytically. The maximum stress $\sigma_{\max}$ in the plate is given by

$$\frac{\sigma_{\max}}{\sigma_{appl}} = 1 + 2\frac{a}{b},$$

where $a$ and $b$ are semi-axes of the ellipse. The stress concentration factor depends on the shape of the hole, characterized by the ratio $a/b$. When $a = b$, the hole is circular, and the stress concentration factor is 3. When the ellipse is very elongated, $a \gg b$, the stress concentration factor is very large.

**Stress concentration at the tip of a sharp flaw**. The problem of the elliptic hole would have been just another unremarkable boundary-value problem were not for an inspired way of using its solution. You might say that 100 years ago it was so difficult to solve boundary-value problems that, whenever a particular problem was solved, you would try to make the most out of the solution. Here is the inspired, perhaps dubious, way of using the solution of the elliptic hole. Do some math and convince yourself that the radius of curvature at the tip of the ellipse is $\rho = b^2 / a$. You can express the above formula in terms $a$ and $\rho$, namely,

$$\frac{\sigma_{\max}}{\sigma_{appl}} \approx 2\left(\frac{a}{\rho}\right)^{1/2}.$$

You then use this formula to estimate the stress concentration factor for a flaw of some other shape, where $a$ is interpreted as the "overall size" of the flaw, and $\rho$ is the radius at the root of the flaw.

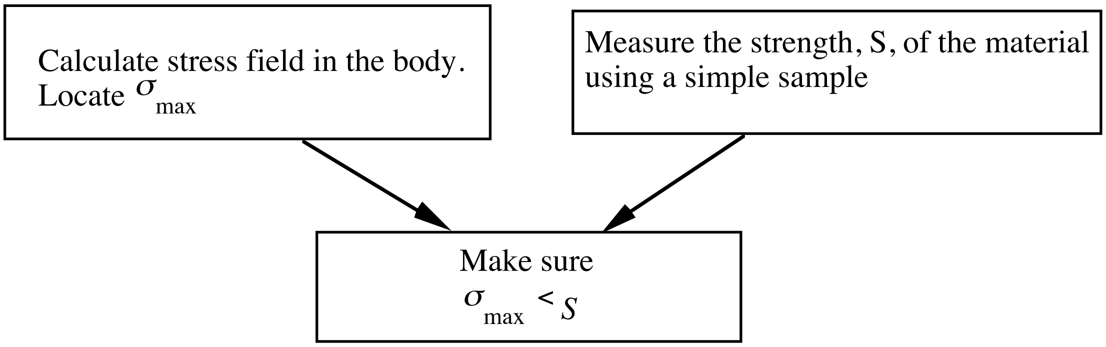
## Strength

**Strength of a material.** Now we have calculated the maximum stress $\sigma_{\max}$ in the body. Will the body sustain this stress? The theory of elasticity will *not* answer this question. Nothing, absolutely nothing, in the equations of elasticity will answer this question. You need to find the strength of the material from somewhere else. For example, you can determine the strength of the material experimentally. You pull a sample until it breaks. You record the stress that breaks the sample. You call this stress the strength of the material, $S$. Because it is determined experimentally, we will call it the experimental strength. The experimental values for glass are on the order $S \sim 100$ MPa. Draw a stress strain diagram. Define stress, strain, stiffness, and strength. If we wish to emphasize that the strength is measured experimentally, we call it experimental strength, $S_{\exp}$.

**Design for strength based on linear elasticity**. What is the maximum load that can be sustained by a body? We can now summarize the procedure as follows.

1. Calculate the stress field by solving a boundary-value problem. Find the location and the direction of the maximum stress $\sigma_{\max}$ in the body.
2. Assume that the material has a definite strength. That is, the same material has the same strength, independent of the shape of the body. Measure the strength $S$ using a simple sample of the material, such as a tensile bar.

You then Make sure that the maximum stress in the body is below the strength of the material.



**Linear elastic theory of strength does not work in practice**

1. The maximum stress in a body is sensitive to the shape of the flaw.
2. The shape of the flaw in the body is seldom known in practice. Do you know the shapes of all the flaws in the screen of your smart phone?
3. The procedure assumes that the body is linearly elastic everywhere, which is never true.
4. The procedure assumes that the strength of a material is independent of the sample used in experiment. In reality, strengths measured from different samples are different, because each sample has different flaws.

That is, the procedure is hit in both ways: the maximum stress is impossible to calculate, and the strength is impossible to measure.

## Experimental Strength vs. Theoretical Strength

**Theoretical strength.** For a body to fracture, the stress must break atomic bonds. The theoretical strength of a material can be calculated by atomistic simulation. A rough estimate is

$$S_{th} = \frac{E}{10}.$$

For a glass, Young's modulus is 70 GPa, so that the estimated theoretical strength is 7 GPa. This value is about two orders of magnitude higher than the strength measured in a bulk sample.

**One exceptional case. Experimental strength for optic fibers.** When a silica fiber develops etch pits, the stress concentration reduces the measured strength to

$$S_{\exp} = \frac{S_{th}}{C},$$

where $C$ is the stress concentration factor. For an etch pit comparable to a hemisphere, $C = 2 \sim 3$. See C.R. Kurkjian and U.C. Paek, "Single-valued strength of perfect silica fibers," Appl. Phys. Lett. 42, 251-253 (1983).

**Experimental strength for bulk samples.** Now we have described two observations:

- For a typical macroscopic sample, the experimental strength is about 2 orders of magnitude lower than the theoretical strength.
- The experimental strength varies from one sample to another.

Both observations can be explained by a single idea: the strength of a sample depends on flaws in the sample. Specifically, let us apply the Inglis formula $\sigma_{\max} / \sigma_{appl} \approx 2(a/\rho)^{1/2}$ to a deep, sharp flaw (i.e., large $a/\rho$ ratio). Take atomic dimension $\rho = 10^{-10}\mathrm{m}$, and flaw size $a = 10^{-6}\mathrm{m}$. We get $\sigma_{\max} / \sigma_{appl} \approx 200$. It is easy to understand the discrepancy between the theoretical strength and the experimental strength.

**Body, flaws, and atoms**. This little calculation suggests that perhaps there isn't any deep mystery about strength after all. All we really need to do is to calculate stress carefully. But to do so is really hard. The calculation involves at least three things of very different sizes: body, flaws, and atoms. In today's language, such a problem is a multiscale problem.

In the above estimate, we have made several assumptions. In closer examination of these assumptions, we may convince ourselves that we've probably get the essential idea right: the strength of a body of glass can be knocked down by several orders of magnitude by tiny flaws. However, we cannot be sure that this procedure can give us values of strength reliable enough for engineering design.

Or maybe you would say, "Let us calculate stress more accurately by using more powerful computers, and more detailed atomistic model". Perhaps this brute-force approach will become viable in future. But even if you can calculate the stress accurately, you still need to know flaws in the body. Maybe someday we will invent a method to detect all the flaws in a sheet of glass.

We need a viable method to design for safety now, not to wait for the power of computation and flaw detection.

## Historical Notes

**The confrontation between Timoshenko and Swain**. When I was a graduate student, a large portrait of George Swain could be found on the third floor of Pierce Hall. He was a professor of Engineering at Harvard from 1881 to 1927. I was told of a confrontation between him and Timoshenko. Searching on the Internet, I found quite a few entries about the confrontation, some quite inflammatory. (Google yourself for Swain Timoshenko.) The issue concerned the stress concentration factor. It was said that Swain did not believe that the stress concentration factor of a circular hole is independent of the radius of the hole. He would say that a very small hole should have negligible effect on strength. See a description by V.V. Meleshko (Selected topics in the history of the two-dimensional biharmonic problem. Appl. Mech. Rev. 56, 33-85, 2003).

It is unimportant for us who said what. Let's focus on the effect of the size of the hole on the strength of a body. Indeed, the stress concentration factor of 3 is a result of linear elasticity. The result is correct so long as the assumption of linear elasticity is correct. In particular, the body has to be linearly elastic. This assumption can be violated, for example, by metals undergoing plastic deformation. Even for a brittle solid such as a silica glass, when the hole approaches atomic dimension, linear elastic assumption breaks down. It is entirely possible that a small enough hole will not reduce the breaking stress of a body. We will return to this question later in the course. But already you can tell that you should not trust linear elasticity.

**Charles Edward Inglis (1875-1952)**. J.F. Baker, Obituary Notices of Fellows of the Royal Society 8, 444-457 (1953).
http://www.jstor.org/stable/769222

> Inglis was devoted to young people and so remained young himself, but at a dinner given in 1950 by the Cambridge University Engineers Association, when a large company of his former pupils met to do him honour, he was able to say, 'I know the young are apt to pity old age, but there, as in many other ways, they are all wrong. Spring is glorious, and full of promise; Summer is garish and a bit vulgar; Autumn is, to my mind, the best part of the year. In the Autumn of life one has acquired a sense of values—one has learnt to appreciate a lot one did not discern in the Spring time.'

**Galileo, Discourses and Mathematical Demonstrations Relating to Two New Science (1638)**. This was Galileo's final book. The two new sciences were the motion of rigid bodies and the theory of strength. Galileo was credited/blamed for the idea that the strength of a material is a constant, independent of the size of the body.