

Finite Element Method

Many textbooks on FEM exist, some of which are on the shelves of the library. See also the recommendations by users of iMechanica: <http://imechanica.org/node/3921>.

FEM has become such a robust tool that today far more people are users than developers. You do not need sophisticated knowledge of FEM to be an effective and creative user. You may, however, have your own reasons to take a full course on FEM.

This course will give a brief introduction to FEM, so that the student gains an impression of the basics. Materials to aid study include

- This set of notes (<http://www.imechanica.org/node/324>).
- A set of homework on theoretical aspects of FEM (<http://imechanica.org/node/329>).
- A tutorial on using ABAQUS, along with practices in the computer lab.
- A project created by each student (<http://imechanica.org/node/384>).

Divergence theorem. This is a theorem in calculus. Let $f(x_1, x_2, x_3)$ be a function defined in a volume in the space (x_1, x_2, x_3) , and n_i be the unit vector normal to the surface enclosing the volume. The divergence theorem states that

$$\int \frac{\partial f}{\partial x_i} dV = \int f n_i dA.$$

The integral on the left-hand side extends over the volume, and the integral on the right-hand side extends over the surface enclosing the volume.

Weak statement. Consider a body that occupies a volume in the space. Recall the equilibrium equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0, \quad (a1)$$

$$\sigma_{ij} n_j = t_i, \quad (a2)$$

where (a1) holds at every point in the volume, and (a2) holds at every point on the surface.

Instead of requiring that these equations hold for every point in volume and on surface, it is equivalent that we require that

$$\int \left(\frac{\partial \sigma_{ij}}{\partial x_j} + b_i \right) w_i dV + \int (t_i - \sigma_{ij} n_j) w_i dA = 0 \quad (b)$$

hold for arbitrary function w_i . The first integral extends over the volume of the body, and the second integral extends over the surface of the body.

Using the divergence theorem, we obtain that

$$\begin{aligned} \int \frac{\partial \sigma_{ij}}{\partial x_j} w_i dV &= \int \frac{\partial (\sigma_{ij} w_i)}{\partial x_j} dV - \int \sigma_{ij} \frac{\partial w_i}{\partial x_j} dV \\ &= \int \sigma_{ij} n_j w_i dA - \int \sigma_{ij} \frac{\partial w_i}{\partial x_j} dV \end{aligned}$$

Inserting into (b), we obtain that

$$\int \sigma_{ij} \frac{\partial w_i}{\partial x_j} dV = \int b_i w_i dV + \int t_i w_i dA \quad (c)$$

holds for arbitrary function w_i . The function w_i is called the test function.

The statement (c) is known as the weak statement. The equilibrium equations (a1) and (a2) are known collectively as the strong statement. In the above, we start with the strong

statement and arrive at the weak statement. Observe that the converse is also true: if (c) holds for all functions w_i , (a1) and (a2) must hold. The weak statement (c) is the basis for the finite element method.

In many boundary value problems, the traction is only prescribed on part of the surface of the body, S_t . On the remainder of the surface, S_u , the displacement is prescribed. To make the weak statement (c) reproduce this boundary condition, we need to choose the test function such that $\mathbf{w} = 0$ on S_u .

Virtual displacement and virtual work. Words have been used to describe the weak statement (c). These words, of course, add no substance to the statement, but may make you feel good about the statement. Let's get over with these words now.

- *Virtual displacement.* The function w_i is sometimes called the virtual displacement. From the above derivation, it is clear that w_i can be any function: it has absolutely nothing to do with displacement. The virtual displacement is usually written as δu_i . In this context, δu_i is just another way to write an arbitrary function, and does not represent a small change in actual displacement, or small change of anything.
- *External virtual work.* If you call w_i the virtual displacement, the right-hand side of (c) looks like the work done by the body force and traction. You might as well call the expression $\int b_i \delta u_i dV + \int t_i \delta u_i dA$ the external virtual work.
- *Virtual strain.* You might as well call the expression $(w_{i,j} + w_{j,i})/2$ virtual strain. The virtual strain is often written as $\delta \varepsilon_{ij} = (\delta u_{i,j} + \delta u_{j,i})/2$.
- *Internal virtual work.* Because $\sigma_{ij} = \sigma_{ji}$, we note that $\sigma_{ij} w_{i,j} = \sigma_{ij} (w_{i,j} + w_{j,i})/2$. In the virtual notation, the left-hand side looks like $\int \sigma_{ij} \delta \varepsilon_{ij}$, and is called the internal virtual work.
- *Principle of virtual work.* Statement (c) is sometimes called the principle virtual work.

If you dislike anything virtual or fake, you can ignore this list of words without missing anything real.

From all functions to a subset of functions. To enforce the weak statement for *all functions* is difficult. Instead, we enforce the weak statement for *a subset of functions*. This relaxed enforcement approximately enforces the equilibrium equations (a1) and (a2). Let us see an implementation of this idea, known as Galerkin's method.

Let $\mathbf{h}^1(x_1, x_2, x_3)$, $\mathbf{h}^2(x_1, x_2, x_3)$, ..., $\mathbf{h}^n(x_1, x_2, x_3)$ be a set of known functions. The superscripts indicate different functions, rather than powers. Use a linear combination to represent the test function w_i , namely,

$$\mathbf{w}(x_1, x_2, x_3) = c_1 \mathbf{h}^1(x_1, x_2, x_3) + c_2 \mathbf{h}^2(x_1, x_2, x_3) + \dots + c_n \mathbf{h}^n(x_1, x_2, x_3),$$

where c_1, c_2, \dots, c_n are a set of numbers. Inserting this test function into the weak statement (c), we obtain that

$$\sum_{\gamma=1}^n c_{\gamma} \left(\int \sigma_{ij} \frac{\partial h_i^{\gamma}}{\partial x_j} dV - \int b_i h_i^{\gamma} dV - \int t_i h_i^{\gamma} dA \right) = 0.$$

We enforce this weak statement for any choice of the set of numbers c_1, c_2, \dots, c_n . This statement is equivalent to

$$\int \sigma_{ij} \frac{\partial h_i^\gamma}{\partial x_j} dV = \int b_i h_i^\gamma dV + \int t_i h_i^\gamma dA, \quad \gamma = 1, 2, \dots, n.$$

This is a set of n equations.

Use another linear combination to represent the displacement field:

$$\mathbf{u}(x_1, x_2, x_3) = a_1 \mathbf{h}^1(x_1, x_2, x_3) + a_2 \mathbf{h}^2(x_1, x_2, x_3) + \dots + a_n \mathbf{h}^n(x_1, x_2, x_3),$$

where a_1, a_2, \dots, a_n are numbers. Assume that the material is linearly elastic, so that the stress relates to the displacement as

$$\sigma_{ij} = c_{ijpq} \frac{\partial u_p(x_1, x_2, x_3)}{\partial x_q}.$$

Inserting the expression for the displacement, we obtain that

$$\sigma_{ij} = c_{ijpq} \sum_{\alpha=1}^n a_\alpha \frac{\partial h_p^\alpha(x_1, x_2, x_3)}{\partial x_q}.$$

The weak statement becomes

$$\sum_{\alpha=1}^n a_\alpha \int c_{ijpq} \frac{\partial h_p^\alpha}{\partial x_q} \frac{\partial h_i^\gamma}{\partial x_j} dV = \int b_i h_i^\gamma dV + \int t_i h_i^\gamma dA, \quad \gamma = 1, 2, \dots, n.$$

This is a set of linear algebraic equations for a_1, a_2, \dots, a_n . Once we solve a_1, a_2, \dots, a_n from this set of equations, we obtain the displacement field.

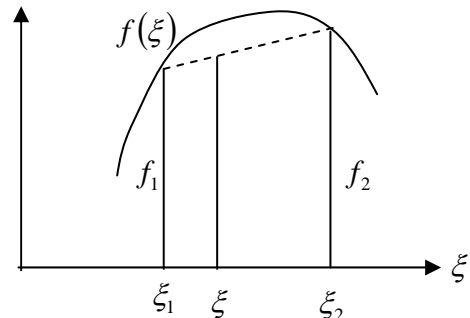
Finite element method in a nutshell. To implement the above procedure in practice, we'll need a practical way to prescribe the known functions $\mathbf{h}^1(x_1, x_2, x_3)$, $\mathbf{h}^2(x_1, x_2, x_3)$, ..., $\mathbf{h}^n(x_1, x_2, x_3)$. A special way to prescribe these known functions is the finite element method, as outlined below.

1. Divide the body into small parts, the elements.
2. Interpolate the test function by its nodal values.
3. Enforce the weak statement for arbitrary nodal values of the test function, leading to a set of algebraic equations.
4. Interpolate the displacement field by its nodal values.
5. Use the stress-strain relations to express the stress in terms of the nodal displacements.
6. Determine the nodal displacement by solving the linear algebraic equations.

When we divide the body into very small elements, the nodal displacements approach the displacement field. In practice, the element sizes are finite, not infinitesimal. Thus the name finite element method. The next few paragraphs walk you through the steps to implement this idea, using the plane strain problems and four-node elements.

Interpolation. Let us first look at interpolation of a function of one variable, $f(\xi)$. Let ξ_1 and ξ_2 be the two nodes. The element is the segment between the two nodes. Let f_1 and f_2 be the values of the function at the two nodes, namely, the nodal values of the function. We can approximate the function at an arbitrary point in the element by

$$f(\xi) = \frac{\xi_2 - \xi}{\xi_2 - \xi_1} f_1 + \frac{\xi - \xi_1}{\xi_2 - \xi_1} f_2.$$



This interpolation is linear in ξ .

Write the interpolation in the form

$$f(\xi) = N_1(\xi)f_1 + N_2(\xi)f_2.$$

The coefficients N_1, N_2 , are functions of ξ , and are known as the **shape functions**. They must satisfy the following requirements:

$$N_1(\xi_1) = 1, N_1(\xi_2) = 0, N_2(\xi_2) = 1, N_2(\xi_1) = 0.$$

If, in addition, we stipulate that the shape functions are linear in ξ , we can uniquely determine the shape functions, as given above.

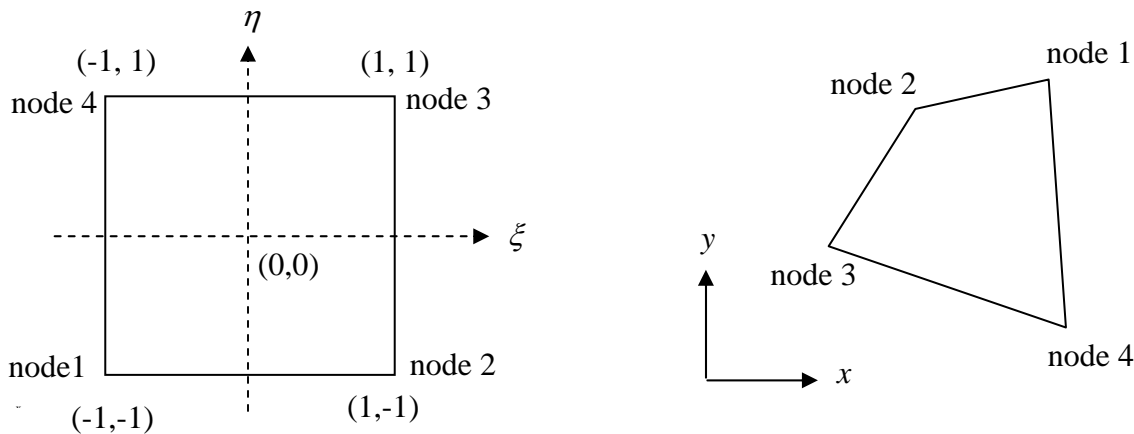
Map a square to a quadrilateral. We would like to solve a plane problem on the (x, y) plane. We divide the body into quadrilateral elements. To represent a body of a general shape, we must consider quadrilaterals of arbitrary shapes. To ease the integration, we map a unit square on a different plane, (ξ, η) , to the quadrilateral in the (x, y) plane.

Now consider a quadrilateral element. Label its four nodes, *counterclockwise*, as 1, 2, 3, 4. On the (x, y) plane, the four nodes have coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. The following functions map a point in the (ξ, η) plane to a point in the (x, y) plane:

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

The functions are linear in the nodal coordinates. The shape functions, N_1, N_2, N_3, N_4 , are functions of ξ and η . They are so constructed that the four corners of the square on the (ξ, η) plane map to the four nodes of the quadrilateral on the (x, y) plane.



Let us determine the shape functions. For example, on the (ξ, η) plane, N_1 should be 1 at node 1, and vanish at the other three nodes. The simplest function that fulfill the second requirement is

$$N_1(\xi, \eta) = c(1 - \xi)(1 - \eta),$$

where c is a constant. To fulfill the first requirement, $N_1(-1, -1) = 1$, we set the constant to be $c = 1/4$. Thus,

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$

We can similarly write out the other three shape functions:

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)$$

Interpolate displacement field. Let the displacements at the four nodes of the quadrilateral be (u_1, v_1) , (u_2, v_2) , (u_3, v_3) , (u_4, v_4) . Interpolate the displacement (u, v) of a point inside the quadrilateral as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

We have used the same shape functions to interpolate the coordinates and the displacements. This is **isometric interpolation**.

List the displacements of the four nodes of the quadrilateral by a column:

$$\mathbf{q} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Write

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}.$$

In the matrix form, the displacement vector at a point in the element is an interpolation of the nodal displacements:

$$\mathbf{u} = \mathbf{Nq}.$$

Express strains in terms of nodal displacements. To calculate the strains, we need to calculate displacement gradients, such as $\partial u / \partial x$. However, in the above interpolations, the displacement is given as a function $u(\xi, \eta)$, so that

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{1-\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & -\frac{1+\eta}{4} \\ -\frac{1-\xi}{4} & -\frac{1+\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

We need to convert the gradient on the (ξ, η) to that on the (x, y) plane.

Using the chain rule of differentiation, we obtain that

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

This equation relates the gradients in the two planes. The two-by-two matrix is known as the Jacobian matrix \mathbf{J} of the coordinate transformation. The elements of the Jacobian matrix are readily calculated; for example

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{1}{4}(1-\eta)(-x_1 + x_2) + \frac{1}{4}(1+\eta)(x_3 - x_4)$$

The determinant of the Jacobian matrix is

$$\det \mathbf{J} = J_{11}J_{22} - J_{12}J_{21}$$

We invert the matrix equation by using Cramer's rule, and obtain that

$$\frac{\partial u}{\partial x} = \frac{1}{\det \mathbf{J}} \begin{vmatrix} \frac{\partial u}{\partial \xi} & J_{12} \\ \frac{\partial u}{\partial \eta} & J_{22} \end{vmatrix} = \frac{1}{\det \mathbf{J}} \left(J_{22} \frac{\partial u}{\partial \xi} - J_{12} \frac{\partial u}{\partial \eta} \right),$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det \mathbf{J}} \begin{vmatrix} J_{11} & \frac{\partial u}{\partial \xi} \\ J_{21} & \frac{\partial u}{\partial \eta} \end{vmatrix} = \frac{1}{\det \mathbf{J}} \left(J_{11} \frac{\partial u}{\partial \eta} - J_{21} \frac{\partial u}{\partial \xi} \right).$$

These express the displacement gradients as linear combinations of the nodal displacements:

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-J_{22}(1-\eta) + J_{12}(1-\xi)}{4 \det \mathbf{J}} & \frac{J_{22}(1-\eta) + J_{12}(1+\xi)}{4 \det \mathbf{J}} & \frac{J_{22}(1+\eta) - J_{12}(1+\xi)}{4 \det \mathbf{J}} & \frac{-J_{22}(1+\eta) - J_{12}(1-\xi)}{4 \det \mathbf{J}} \\ \frac{-J_{11}(1-\xi) + J_{21}(1-\eta)}{4 \det \mathbf{J}} & \frac{-J_{11}(1+\xi) - J_{21}(1-\eta)}{4 \det \mathbf{J}} & \frac{J_{11}(1+\xi) - J_{21}(1+\eta)}{4 \det \mathbf{J}} & \frac{J_{11}(1-\xi) + J_{21}(1+\eta)}{4 \det \mathbf{J}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

We can also write similar expressions for $\partial v / \partial x$ and $\partial v / \partial y$.

Recall that the strain column is

$$\begin{bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{bmatrix}.$$

Consequently, the strain column of a point in the element is linear in the nodal displacement column. We write

$$\begin{bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{bmatrix} = \mathbf{B} \mathbf{q}.$$

The entries to the matrix \mathbf{B} can be worked out readily by combing the above relations. For example,

$$B_{11} = \frac{1}{4 \det \mathbf{J}} [-J_{22}(1-\eta) + J_{12}(1-\xi)].$$

Express stresses in terms of the nodal displacements. The stress-strain relation under the plane strain conditions is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{bmatrix}.$$

The stress column is also linear in the displacement column:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \mathbf{DBq}.$$

Virtual displacements and virtual strains. List the virtual displacements of the four nodes of the quadrilateral by a column:

$$\delta \mathbf{q} = \begin{bmatrix} \delta u_1 \\ \delta v_1 \\ \delta u_2 \\ \delta v_2 \\ \delta u_3 \\ \delta v_3 \\ \delta u_4 \\ \delta v_4 \end{bmatrix}$$

The virtual displacement field in the elements is interpolated in the same way as the actual displacement field:

$$\delta \mathbf{u} = \mathbf{N} \delta \mathbf{q}$$

Similarly, the virtual strain field is

$$\begin{bmatrix} \partial(\delta u) / \partial x \\ \partial(\delta v) / \partial y \\ \partial(\delta u) / \partial y + \partial(\delta v) / \partial x \end{bmatrix} = \mathbf{B} \delta \mathbf{q}.$$

Assemble equilibrium equations. Recall the weak statement:

$$\int \sigma_{ij} \frac{\partial \zeta_i}{\partial x_j} dV = \int b_i \zeta_i dV + \int t_i \zeta_i dA$$

Inserting the above expressions, we obtain that

$$\sum \delta \mathbf{q}^T \mathbf{k} \mathbf{q} = \sum \delta \mathbf{q}^T \mathbf{f}.$$

The sum is taken over all the elements. The element stiffness matrix is an integral over the volume of the element:

$$\mathbf{k} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} dV.$$

The force column \mathbf{f} is the sum of two contributions. The body force term integrates over the volume of an element, giving

$$\mathbf{f}_b = \int \mathbf{N}^T \mathbf{b} dV.$$

The surface traction term integrates over the surface area of an element on which traction is prescribed, giving

$$\mathbf{f}_t = \int \mathbf{N}^T \mathbf{t} dA.$$

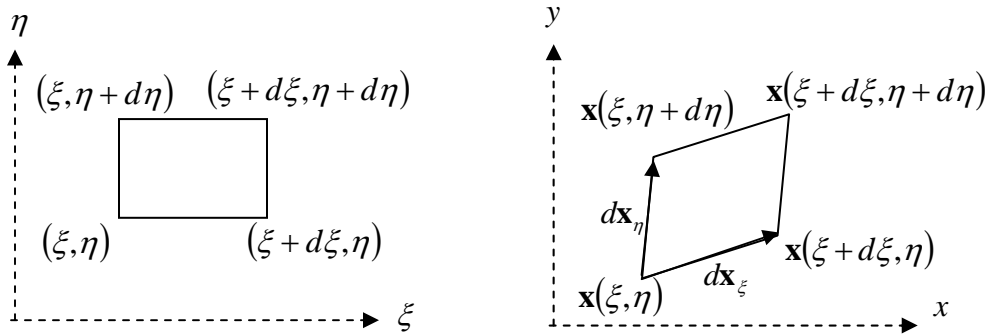
List all the nodal displacements in the body by the column \mathbf{Q} . The weak statement becomes

$$\delta \mathbf{Q}^T \mathbf{K} \mathbf{Q} = \delta \mathbf{Q}^T \mathbf{F}$$

for virtual nodal displacement column, $\delta \mathbf{Q}$. Consequently, the displacement column \mathbf{Q} satisfies $\mathbf{K} \mathbf{Q} = \mathbf{F}$.

After enforcing the prescribed displacements, the computer solves the linear algebraic equation.

Differential volume and area. To calculate the virtual work, we will need to integrate over the volume of the body. The body is now divided into elements. So we will integrate over each element, and then sum over all elements. For the plane problems, the differential volume dV is the differential area dA on the (x, y) plane times the thickness h of the element. We have expressed various fields as functions of (ξ, η) . We need to carry out the integral on the (ξ, η) plane.



Consider a rectangular infinitesimal element on the (ξ, η) plane, defined by four points (ξ, η) , $(\xi + d\xi, \eta)$, $(\xi, \eta + d\eta)$ and $(\xi + d\xi, \eta + d\eta)$. This infinitesimal element maps to a quadrilateral infinitesimal element on the (x, y) plane. The point (ξ, η) maps to a vector $\mathbf{x}(\xi, \eta)$ on the (x, y) plane. The point $(\xi + d\xi, \eta)$ maps to point $\mathbf{x}(\xi + \xi, \eta)$. Consequently, one side of the quadrilateral is the vector

$$d\mathbf{x}_\xi = \mathbf{x}(\xi + d\xi, \eta) - \mathbf{x}(\xi, \eta) = \frac{\partial \mathbf{x}}{\partial \xi} d\xi.$$

Another side of the quadrilateral is the vector

$$d\mathbf{x}_\eta = \mathbf{x}(\xi, \eta + d\eta) - \mathbf{x}(\xi, \eta) = \frac{\partial \mathbf{x}}{\partial \eta} d\eta.$$

Recall that the cross product of any two vectors is a vector, whose magnitude is the area of the parallelogram defined by the two vectors. Consequently, the area of the quadrilateral infinitesimal element is

$$dA = |d\mathbf{x}_\xi \times d\mathbf{x}_\eta|.$$

Calculate the cross product, and we obtain that

$$\mathbf{dx}_\xi \times \mathbf{dx}_\eta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \mathbf{x}}{\partial \xi} d\xi & \frac{\partial \mathbf{y}}{\partial \xi} d\xi & 0 \\ \frac{\partial \mathbf{x}}{\partial \eta} d\eta & \frac{\partial \mathbf{y}}{\partial \eta} d\eta & 0 \end{vmatrix} = \mathbf{k} \left(\frac{\partial \mathbf{x}}{\partial \xi} \frac{\partial \mathbf{y}}{\partial \eta} - \frac{\partial \mathbf{x}}{\partial \eta} \frac{\partial \mathbf{y}}{\partial \xi} \right) d\xi d\eta.$$

Consequently, the area of the differential element is

$$dA = (\det \mathbf{J}) d\xi d\eta.$$

In calculating the contribution due to the traction, we will integrate over the surface area of the body. For the plane elasticity problems, the differential surface area becomes the differential line length dL times the element thickness h . Say the line is the edge of a quadrilateral element between node 2 and 3, namely, $\xi = 1$. The differential line length is

$$dL = |\mathbf{dx}_\eta| = \sqrt{\left(\frac{\partial \mathbf{x}}{\partial \eta}\right)^2 + \left(\frac{\partial \mathbf{y}}{\partial \eta}\right)^2} d\eta,$$

or

$$dL = \frac{l_{23}}{2} d\eta,$$

where l_{23} is the distance between node 2 and 3.

Thus, the volume integrals become

$$\mathbf{k} = \int_{-1}^1 \int_{-1}^1 h \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta$$

$$\mathbf{f}_b = \int_{-1}^1 \int_{-1}^1 h \mathbf{N}^T \mathbf{b} \det \mathbf{J} d\xi d\eta$$

The surface integral becomes

$$\mathbf{f}_t = \frac{l_{23}}{2} \int_{-1}^1 h \mathbf{N}^T \mathbf{t} d\eta$$

Only nodes on the edge of the element, $\xi = 1$, contribute to the force.

Gaussian quadrature. Consider the integral

$$I = \int_{-1}^1 f(\xi) d\xi.$$

A generic way to construct a numerical integration method is as follows. Select a set of n points $\xi_1, \xi_2, \dots, \xi_n$ in the interval $(-1, 1)$. Evaluate the function at these points: $f(\xi_1), f(\xi_2), \dots, f(\xi_n)$.

The integral is a linear combination of the function values:

$$I = \int_{-1}^1 f(\xi) d\xi = w_1 f(\xi_1) + w_2 f(\xi_2) + \dots + w_n f(\xi_n).$$

The coefficients are known as the weights. The weights are selected so that the above formula gives exact results for some known functions.

Evaluating function values is costly. The Gaussian quadrature selects the points $\xi_1, \xi_2, \dots, \xi_n$ and the weights such that the above formula is exact for polynomials of as large a

degree as possible. For n points and n weights, the formula is exact for polynomial of degree $2n-1$, for it has $2n$ terms.

As an example, consider the two-point Gaussian quadrature:

$$I = \int_{-1}^1 f(\xi) d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2).$$

We need to determine four numbers: ξ_1, ξ_2, w_1, w_2 . A polynomial of degree 3 is a linear combination of monomials $1, x, x^2$ and x^3 . The integral I is a number linear in the function f . To ensure the integral is exact for *every* polynomial of degree up to 3, all we need to do is to ensure that the formula is exact for the four monomials. Thus,

$$\begin{aligned} 2 &= w_1 + w_2 \\ 0 &= w_1 \xi_1 + w_2 \xi_2 \\ \frac{2}{3} &= w_1 \xi_1^2 + w_2 \xi_2^2 \\ 0 &= w_1 \xi_1^3 + w_2 \xi_2^3 \end{aligned}$$

This is a set of nonlinear algebraic equations. The solution is

$$\begin{aligned} w_1 &= 1, \xi_1 = -1/\sqrt{3} \\ w_2 &= 1, \xi_2 = +1/\sqrt{3} \end{aligned}$$

Many mathematics handbooks list integration points and weights for Gaussian quadrature.

A two dimensional integral is evaluated as

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \int_{-1}^1 \sum_{i=1}^n w_i f(\xi_i, \eta) d\eta = \sum_{j=1}^n \sum_{i=1}^n w_i w_j f(\xi_i, \eta_j).$$

For example, the 2-point quadrature in one dimension becomes 4-point quadrature in two dimensions:

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

We have walked through the steps of implementing the finite element method. The following extensions are straightforward and have little educational value. These options are available in ABAQUS, and are important in applications. We simply list them, with brief comments.

Element type. You do not have to divide the plane into quadrilaterals. For example, triangular elements may be more versatile in modeling odd-shaped bodies.

You do not have to restrict the nodes to the corners of an element. Some elements have nodes on the sides of the elements, inside the elements, as well as at the corners of the elements. If an element contains more nodes, the displacements in the element vary as a higher order polynomial of the coordinates. This represents a larger family of displacement fields. To divide a body into elements, you can increase the number of nodes by using either a large number of low order elements, or a small number of high order elements.

Axisymmetric problems. In practice, you would like to avoid as much as possible three-dimensional problems. Such problems dramatically increase the size of computation job. Even if the computer is not tired of the job size, you will, for it will take a long time for you to go through the output to extract useful information. When the deformation field is axisymmetric, all the fields are function of the axial coordinate z and the radial coordinate r . Consequently,

both input and output are made in the plane (r, z) . The amount of work is comparable to solving the plane elasticity problems.

Three-dimensional problems. If you have to solve a three-dimensional model, ABAQUS provides such an option.

Other fields. Finite element method has been developed for other fields, such as temperature field and electrostatic field. ABAQUS provides many options.